CRANBROOK

MATHEMATICS EXTENSION 2

2006

HIGHER SCHOOL CERTIFICATE

TRIAL EXAMINATION

General Instructions

- Reading time – 5 minutes
- Writing time – 3 hours
- All eight questions should be attempted
- Total marks available - 120
- All questions are worth 15 marks
- An approved calculator may be used
- All relevant working should be shown for each question
- Answer each question in a separate 8 page booklet.

Question 1 (15 marks) Marked by SKB **Marks**

(a) (i) Find $\int \frac{dx}{x^2 + 8x + 11}$ 3

(ii) Find $\int \frac{\cos 3x \, dx}{1 + \sin 3x}$. 2

(iii) Find $\int \frac{5x + 4}{\sqrt{4 + 3x - 5x^2}} \, dx$ 3

(b) Evaluate $\int_{-1}^{1} \frac{\sin^{-1} x}{1 + x^2} \, dx$ 2

(c) Show that $\int_0^{\frac{\pi}{4}} (\tan^3 x + \tan x) dx = \frac{1}{2}$. 3

Hence show that $\int_0^{\frac{\pi}{4}} \tan^3 x \; dx = \frac{1}{2}(1 - \ln 2)$ 2

Question 2 (15 marks) Marked by SKB **Marks**

(a) (i) Find $\int \frac{\sin^{-1} 2x}{\sqrt{1-4x^2}}\, dx$. **1**

(ii) Find $\int \frac{\cos^5 x}{\sin^2 x}\, dx$. **2**

(b) Evaluate $\int_0^2 x^3 e^{x^2}\, dx$. **2**

(c) By expressing $\frac{48}{x^3+64}$ as partial fractions, find $\int \frac{48}{x^3+64}\, dx$. **5**

(d) Let $I_n = \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \cot^n x\, dx$ where n is a positive integer.

(i) Find the value of I_1. **1**

(ii) Show $I_n + I_{n-2} = \frac{1}{n-1}$ for $n \geq 2$. **3**

(iii) Hence evaluate I_5. **1**

Question 3 (15 marks) Marked by HRK **Marks**

(a) (i) If $z = \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}i$, find z^4, in the form $x + iy$. 2

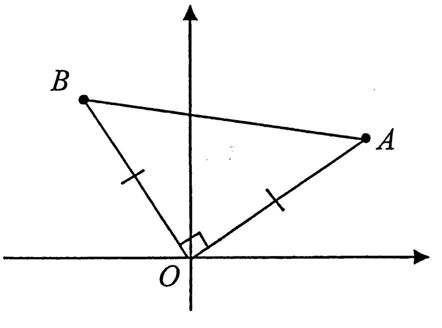
(ii) Hence or otherwise, find z^{13} in the form $x + iy$. 2

(b) Find all the complex numbers $z = a + ib$, where a and b are real, such that $|z|^2 - 7 = 2i(z + 2)$. 4

(c) (i) Express $z = 4 + 4\sqrt{3}i$ in modulus-argument form. 1

(ii) Hence find the three values of $z^{\frac{1}{3}}$ in modulus-argument form. 3

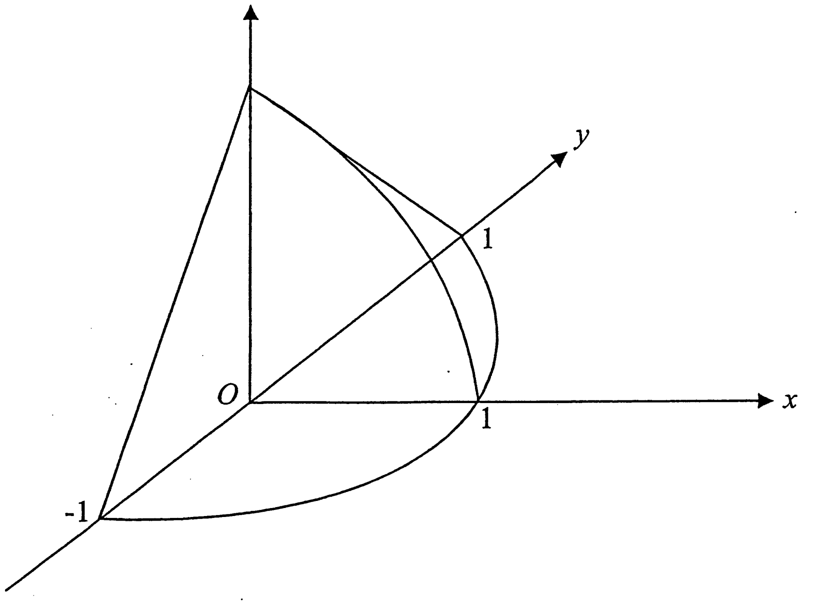
(d) The Argand diagram shows the points A and B, which represent the complex numbers z_1 and z_2 respectively. Given that ΔBOA is a right-angled, isosceles triangle, show that $(z_1 + z_2)^2 = 2z_1 z_2$. 3



Question 4 (15 marks) Marked by SKB **Marks**

(a) The area bounded by the curve $y = x^3 + 1$ and the x and y axes is rotated about the y-axis. By taking slices perpendicular to the y-axis find the volume of the solid generated. 4

(b)



The base of a solid is the semi-circular region in the x - y plane with the straight edge running from the point $(0,-1)$ to the point $(0,1)$ and the point $(1,0)$ on the curved edge of the semicircle.
Each cross-section perpendicular to the x-axis is an isosceles triangle with each of the two equal sidelengths three quarters the length of the third side.

(i) Show that the area of the triangular cross-section at $x = a$ is $\frac{\sqrt{5}}{2}(1-a^2)$. 3

(ii) Hence find the volume of the solid. 2

(c) The region S is enclosed by the line $x + y = a$; $a > 0$, the curve $y = x^3 - ax^2$ and the y-axis.

(i) Sketch the region S; clearly labelling its intercepts with the axes. 2

(ii) The region S is rotated around the line $x = a$ to form a solid. Use the method of cylindrical shells to find the volume of this solid. 4

Question 5 (15 marks) Marked by HRK **Marks**

(a) The equation $x^3 - 3x - 2 = 0$ has roots α, β and γ.

(i) Find the equation with roots α^3, β^3 and γ^3. **2**

(ii) Hence or otherwise, find $\alpha^7\beta\gamma + \alpha\beta^7\gamma + \alpha\beta\gamma^7$. **2**

(b) (i) Find the equation of the normal to $x^2 - xy - y^2 = 1$ at the point $(2,1)$. **2**

(ii) Find the coordinates of the other point of intersection where the normal intersects with the curve. **2**

(c) Using mathematical induction, prove that **3**

$$\sum_{r=1}^{n} r^3 < n^2(n+1)^2 \text{ for } n = 1,2,3\ldots$$

(d) The ellipse with equation $\dfrac{x^2}{9} + \dfrac{y^2}{4} = 1$ and the hyperbola with equation $\dfrac{x^2}{A^2} - \dfrac{y^2}{B^2} = 1$, have the same directrices. **4**

Show that $A^2 = \dfrac{9}{5}\sqrt{5(A^2 + B^2)}$.

Question 6 (15 marks) Marked by HRK **Marks**

(a) The equation $ax^4 + bx^3 + cx^2 + dx + e = 0$ has a quadruple root α.

(i) Find α in terms of a and b. 2

(ii) Hence, show $\left(1+\dfrac{b}{4a}\right)^4 = \dfrac{a+b+c+d+e}{a}$. 2

(b) Find the equation of the normal to the hyperbola $\dfrac{x^2}{6}-\dfrac{y^2}{8}=1$ at the point (3,2). 3

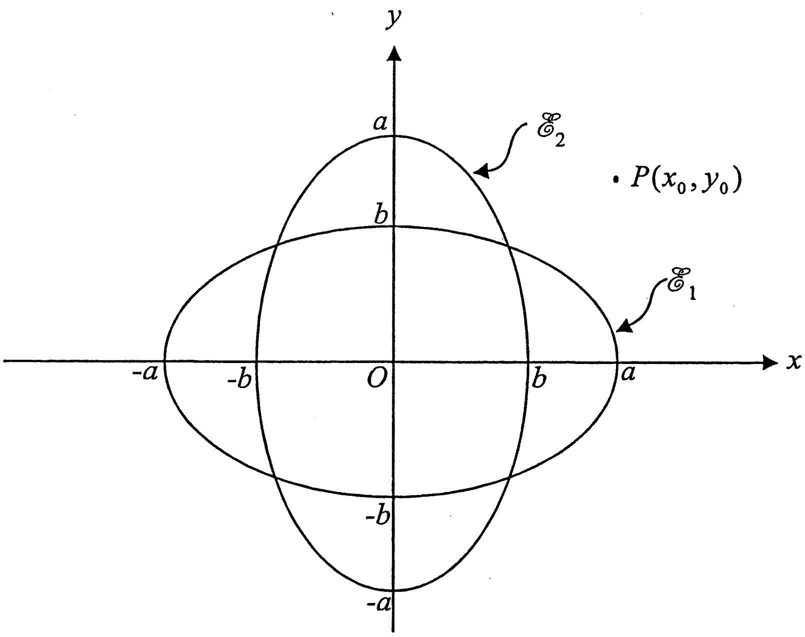
(c) $P(a\cos\theta, b\sin\theta)$ lies on the ellipse $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}=1$. The tangent at P meets the tangents at the ends of the major axis at Q and R. Show that QR subtends a right angle at either focus. Include a neat diagram with your answer. 8

Question 7 (15 marks) Marked by HRK **Marks**

(a) For what values of k does the equation $x^3 - 3x^2 - 24x + k = 0$ have one real root? **4**

(b) A polynomial $P(x)$ gives remainders -2 and 1 when divided by $2x-1$ and $x-2$ respectively. What is the remainder when $P(x)$ is divided by $2x^2 - 5x + 2$? **3**

(c)



The ellipse E_1 has equation $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$ and the ellipse E_2 has equation $\dfrac{x^2}{b^2} + \dfrac{y^2}{a^2} = 1$. The point $P(x_0, y_0)$ lies outside both E_1 and E_2.

(i) Find all the points of intersection of E_1 with E_2. **2**

(ii) The chord of contact to E_1 from the point $P(x_0, y_0)$ has equation

$$\frac{xx_0}{a^2} + \frac{yy_0}{b^2} = 1.$$

This chord of contact intersects with the chord of contact to E_2 from the point $P(x_0, y_0)$ at the point $Q(x_2, y_2)$. **3**

Find the coordinates of the point Q.

(iii) Using your answer to part (i) or otherwise show that Q cannot lie outside both E_1 and E_2. **3**

Question 8 (15 marks) Marked by SKB **Marks**

(a) The real numbers $a > 0$, $b > 0$ and $c > 0$ are such that $\frac{1}{a}$, $\frac{1}{b}$ and $\frac{1}{c}$ are in geometric progression.

(i) Using the fact that $a^2 + b^2 \geq 2ab$, show that $a^2 + c^2 \geq ab + bc$. **2**

(ii) Show $\frac{1}{a^2} + \frac{1}{c^2} \geq \frac{2}{b^2}$. **1**

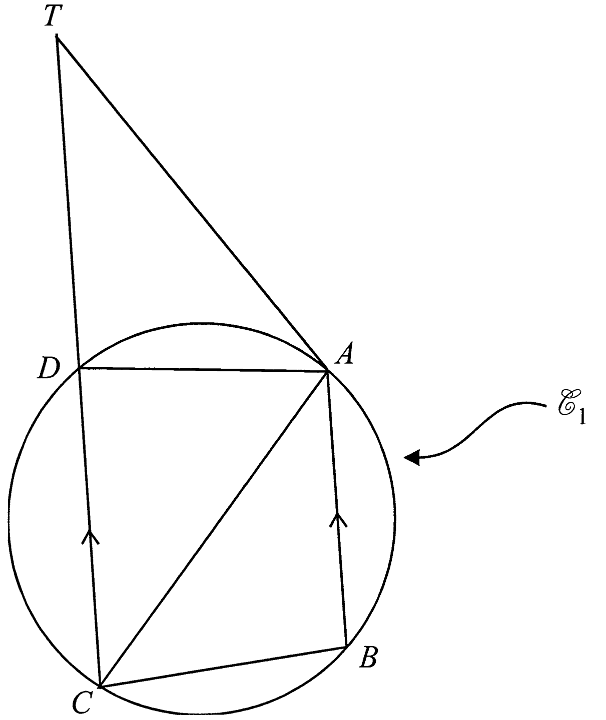
(b) Consider the function $f(x) = x - \ln(x^2 + 1)$ for $x \geq 0$.

(i) Show that $x > \ln(x^2 + 1)$ for $x > 0$. **2**

(ii) By evaluating $\int_0^1 x \, dx$ and $\int_0^1 \ln(x^2 + 1) \, dx$, show that $5 > 2\ln 2 + \pi$. **3**

Question 8 continues on the next page

(c)



The points A, B, C and D lie on the circle $\mathscr{C}_1$. From the exterior point T, a tangent is drawn to point A on $\mathscr{C}_1$. The line CT passes through D and TC is parallel to AB.

(i) Prove that ΔADT is similar to ΔABC. **3**

(ii) The line BA is produced through A to point M, which lies on a second circle $\mathscr{C}_2$. The points A, D, T also lie on $\mathscr{C}_2$ and the line DM crosses AT at O.

(1) Show that ΔOMA is isosceles. **1**

(2) Show that $TM = BC$ **3**

STANDARD INTEGRALS

$$\int x^n \, dx = \frac{1}{n+1} x^{n+1}, \quad n \neq -1; \quad x \neq 0, \text{ if } n < 0$$

$$\int \frac{1}{x} \, dx = \ln x, \quad x > 0$$

$$\int e^{ax} \, dx = \frac{1}{a} e^{ax}, \quad a \neq 0$$

$$\int \cos ax \, dx = \frac{1}{a} \sin ax, \quad a \neq 0$$

$$\int \sin ax \, dx = -\frac{1}{a} \cos ax, \quad a \neq 0$$

$$\int \sec^2 ax \, dx = \frac{1}{a} \tan ax, \quad a \neq 0$$

$$\int \sec ax \tan ax \, dx = \frac{1}{a} \sec ax, \quad a \neq 0$$

$$\int \frac{1}{a^2 + x^2} \, dx = \frac{1}{a} \tan^{-1} \frac{x}{a}, \quad a \neq 0$$

$$\int \frac{1}{\sqrt{a^2 - x^2}} \, dx = \sin^{-1} \frac{x}{a}, \quad a > 0, \quad -a < x < a$$

$$\int \frac{1}{\sqrt{x^2 - a^2}} \, dx = \ln\left(x + \sqrt{x^2 - a^2}\right), \quad x > a > 0$$

$$\int \frac{1}{\sqrt{x^2 + a^2}} \, dx = \ln\left(x + \sqrt{x^2 + a^2}\right)$$

NOTE : $\ln x = \log_e x, \quad x > 0$