CRANBROOK

MATHEMATICS EXTENSION 2

2008

HIGHER SCHOOL CERTIFICATE

TRIAL EXAMINATION

General Instructions

- Reading time – 5 minutes
- Writing time – 3 hours
- All eight questions should be attempted
- Total marks available - 120
- All questions are worth 15 marks
- An approved calculator may be used
- All relevant working should be shown for each question
- Answer each question in a separate 8 page booklet.
- Standard integrals sheet at back of examination.

Question 1 (15 marks) Marked by SKB **Marks**

(a) Find $\int x \tan x^2 \, dx$. **2**

(b) Use the substitution $u = \sqrt{x}$ to evaluate $\int_4^9 \frac{x}{\sqrt{x}(1+x)} \, dx$. **3**

(c) Use the completion of squares method to find $\int \frac{-2}{\sqrt{3+2x-x^2}} \, dx$. **2**

(d) (i) Find the real numbers a, b and c such that **2**

$$\frac{2x^2+2x+5}{(x^2+2)(1-x)} \equiv \frac{ax+b}{x^2+2} + \frac{c}{1-x}.$$

(ii) Hence find $\int \frac{2x^2+2x+5}{(x^2+2)(1-x)} \, dx$. **2**

(e) Use integration by parts to evaluate $\int_1^5 \frac{\ln x}{\sqrt{x}} \, dx$. **4**

Question 2 (15 marks) Marked by SKB **Marks**

(a) Evaluate $\int_{-1}^{1} \frac{\tan^{-1} x}{1+x^4}\, dx$ **2**

(b) Evaluate $\int_{0}^{1} \sqrt{4-x^2}\, dx$ **4**

(c) By using the result $\int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx$

evaluate $\int_0^{2\pi} \frac{x \cos x}{1+\sin^2 x}\, dx$ **3**

(d) Let $I_n = \int_0^1 x\left(x^2-1\right)^n dx$ for $n = 0,1,2,\ldots$

(i) Use integration by parts to show that **3**

$$I_n = \frac{-n}{n+1} I_{n-1} \quad \text{for } n \geq 1.$$

(ii) Hence or otherwise show that **2**

$$I_n = \frac{(-1)^n}{2(n+1)} \quad \text{for } n \geq 0.$$

(iii) Explain why $I_{2n} > I_{2n+1}$ for $n \geq 0$ **1**

Question 3 (15 marks) Marked by JSH **Marks**

(a) Let $z = 3 - i$ and $w = 2 + 4i$.
Find the following in the form $x + yi$.

(i) $z\overline{w}$ **1**

(ii) $\dfrac{z}{w}$ **1**

(b) (i) Express $1 + i$ in modulus-argument form. **2**

(ii) Hence, find the values of *n*, for which **3**

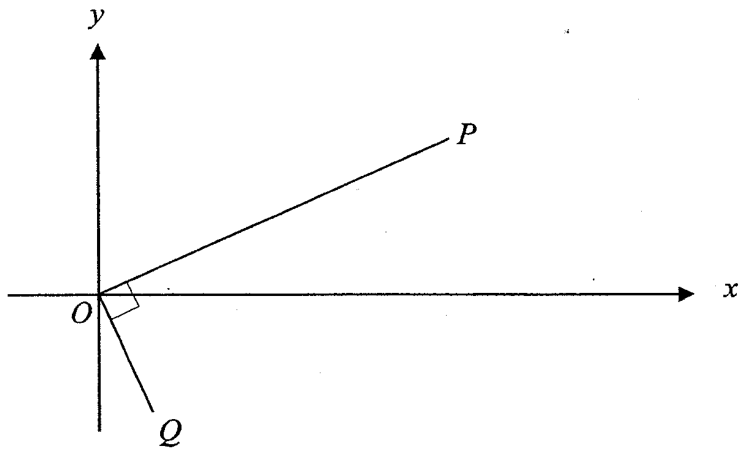
$$(1+i)^n + (1-i)^n = 0$$

where *n* is a positive integer.

(c) Sketch the region in the Argand diagram where the inequalities **2**

$$|z-1| \le 1 \text{ and } \frac{\pi}{4} \le \arg(z-1) \le \frac{\pi}{2} \text{ both hold.}$$

(d)



In the Argand diagram above, point *P* corresponds to the complex number *z*. **1**
The triangle *OPQ* is a right-angled triangle and $OP = 3OQ$.
What is the complex number that corresponds to point *Q*?

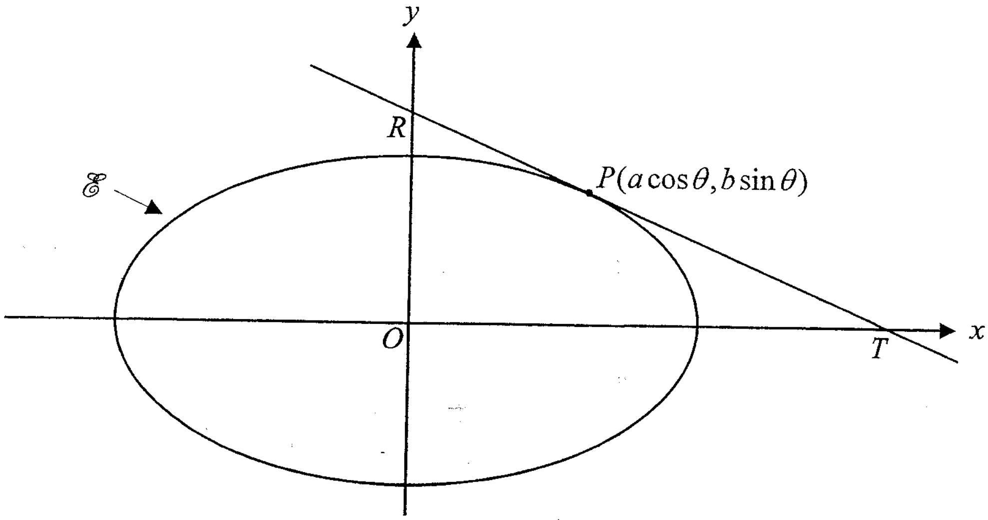
(e) (i) Find all the solutions to the equation $z^6 = 1$ in the form $x + yi$. **2**

(ii) If ω is a non-real solution to the equation $z^6 = 1$, show that $\omega^4 + \omega^2 = -1$. **2**

(iii) By choosing one particular value of ω, explain with the aid of a diagram why $\omega^4 + \omega^2 = -1$. **1**

Question 4 (15 marks) Marked by JSH **Marks**

(a)



The ellipse E with equation $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ shown in the diagram above, has a tangent at the point $P(a\cos\theta, b\sin\theta)$. The tangent cuts the x-axis at T and the y-axis at R.

(i) Show that the equation of the tangent at the point P is **2**

$$\frac{x\cos\theta}{a}+\frac{y\sin\theta}{b}=1.$$

(ii) If T is the point of intersection between the tangent at point P and one of the directrices of the ellipse, show that $\cos\theta = e$. **3**

(iii) Hence find the angle that the focal chord through P makes with the x-axis. **1**

(iv) Using similar triangles or otherwise, show that $RP = e^2 RT$. **3**

(b) $P(ct, \frac{c}{t}), t \neq 1$ lies on the hyperbola $xy = c^2$. The tangent and normal at P meet the line $y = x$ at T and N respectively. If O is the origin show that $OT.ON = 4c^2$. Include a labelled diagram with your answer. **6**

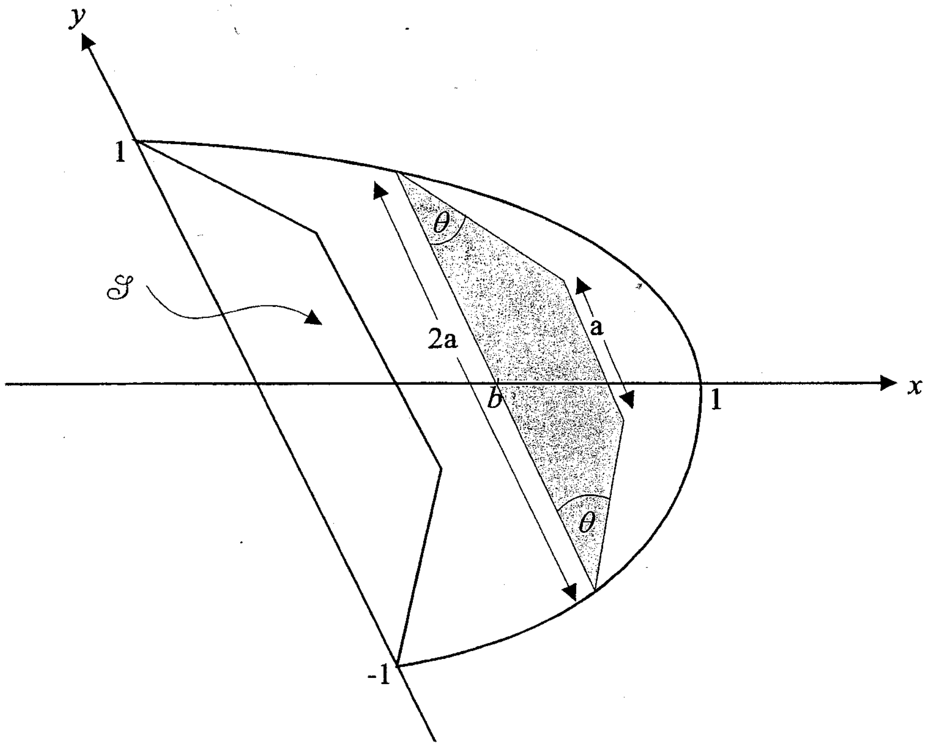
Question 5 (15 marks) Marked by SKB **Marks**

(a) The region bounded by the curve $x = y^2$ and the line $x = 4$ is rotated about the line $y = 2$. Find the volume generated when:

(i) Slices of thickness Δx are taken perpendicular to the x-axis in this region to create hollow cylindrical discs. **4**

(ii) Slices of thickness Δy are taken perpendicular to the y-axis in this region to create thin cylindrical shells. **4**

(b)



A solid S has a semi-circular base in the x-y plane with its diameter along the y-axis.
Each cross-section of the solid running perpendicular to the x-y plane is a regular trapezium with its base sidelength twice that of its parallel sidelength. The angle between the base sidelength and the sides of the trapezium is θ.
A typical cross-section taken at $x = b$ is shown in the diagram.

(i) Show that if $\theta = 45^\circ$, the area of the trapezium at $x = b$ is $\dfrac{3a^2}{4}$. **1**

(ii) Find the volume of the solid S when $\theta = 45^\circ$. **2**

(iii) Find the volume of the solid, $\mathscr{D}$, generated when the semi-circle is rotated through an angle of 90° about the y-axis. **1**

(iv) Find the values of θ for which the volume of S found in part (ii) is greater than the volume of $\mathscr{D}$. **3**

Question 6 (15 marks) Marked by JSH **Marks**

(a) If $1-i$ is a zero of $P(x)=x^3+ax^2+bx+6$, where $a,b \in$ Real

(i) Evaluate a and b **4**

(ii) Hence fully factorise $P(x)$ over the complex field. **1**

(b) (i) Use De Moivre's theorem to express $\tan 5\theta$ in terms of powers of $\tan\theta$. **3**

(ii) Hence show $x^5-5x^4-10x^3+10x^2+5x-1=0$ has roots $1,\ \tan\frac{\pi}{20},\ \tan\frac{9\pi}{20},-\tan\frac{3\pi}{20}$ and $-\tan\frac{7\pi}{20}$. **2**

(iii) By solving $x^5-5x^4-10x^3+10x^2+5x-1=0$ another way, show that **5**

$$\tan\frac{9\pi}{20}+\tan\frac{\pi}{20}=2+2\sqrt{5} \text{ and } \tan\frac{7\pi}{20}+\tan\frac{3\pi}{20}=2\sqrt{5}-2.$$

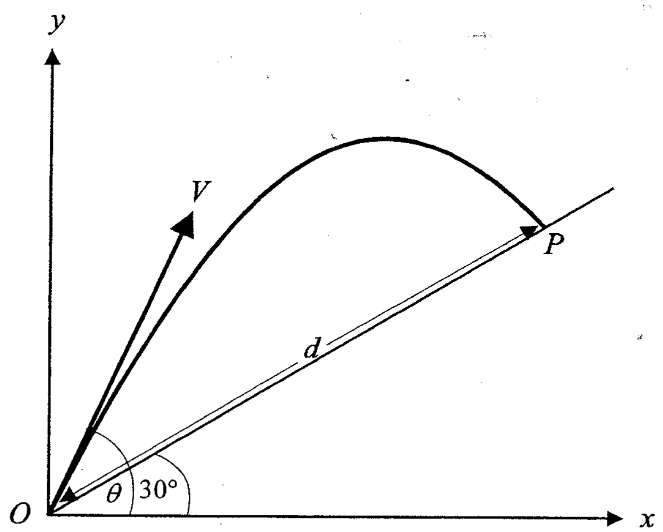
Question 7 (15 marks) Marked by JSH **Marks**

(a) AB is a chord of a circle. X is a point on AB produced. XT is a tangent from X to the circle.

(i) Prove that ΔXAT is similar to ΔXTB. 2

(ii) Deduce that $XT^2 = XA.XB$ 2

(b)



The diagram above shows the path of a particle which has been projected from point 0 at an angle of θ to the horizontal. The speed at which the particle was projected was $\sqrt{g}$ m/sec where g is the acceleration due to gravity. The particle lands at point P which lies on a plane inclined at an angle of 30° to the horizontal. The base of this inclined plane is at O and point P lies d metres from O. The position of the particle at time t seconds is given by

$$x = \sqrt{g}\,t\cos\theta$$

$$\text{and } y = \sqrt{g}\,t\sin\theta - \frac{1}{2}gt^2$$

(i) Show that the path of trajectory of the particle is given by 1

$$y = x\tan\theta - \frac{x^2\sec^2\theta}{2}.$$

(ii) If there is only one path of trajectory for the particle to land at point P, find θ for that path. 4

Marks

(c) Find the general solutions to the equation **6**

$$\cos 4\theta + \cos 2\theta = \sqrt{2}\cos^2\theta + \frac{1}{\sqrt{2}}\sin 2\theta .$$

Question 8 (15 marks) Marked by SKB **Marks**

(a) (i) If a > 0, b > 0 and c > 0, show that $a^2 + b^2 \geq 2ab$ and hence deduce that $a^2 + b^2 + c^2 \geq ab + bc + ca$. **2**

(ii) If $a + b + c = 9$, show that $ab + bc + ca \leq 27$ and **3**

$$\frac{1}{a} + \frac{1}{b} + \frac{1}{c} \leq \frac{27}{abc}.$$

(b) If $U_1 = 1, U_2 = 5$ and $U_n = 5U_{n-1} - 6U_{n-2}$ for $n \geq 3$, prove by mathematical induction that $U_n = 3^n - 2^n$ for $n \geq 1$. **5**

(c) The lines $y = 0$, $3x - 4y + 3 = 0$ and $3x + 4y - 15 = 0$ are the sides of a triangle. Find the co-ordinates of the centre of the circle inscribed in the triangle. Hence or otherwise write down the equation of the circle. **5**