Centre Number

Student Number

CATHOLIC SECONDARY SCHOOLS
ASSOCIATION OF NEW SOUTH WALES

2003
TRIAL HIGHER SCHOOL CERTIFICATE EXAMINATION

Mathematics Extension 2

Morning Session
Monday 11 August 2003

General Instructions

- Reading time – 5 minutes
- Working time – 3 hours
- Write using blue or black pen
- Board-approved calculators may be used
- A table of standard integrals is provided separately
- All necessary working should be shown in every question

Total marks **(120)**

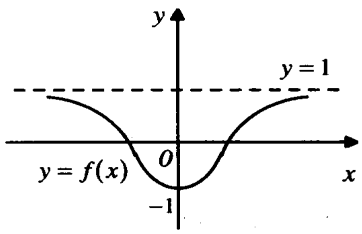
- Attempt Questions 1 – 8
- All questions are of equal value

Examiners: Graham Arnold, Terra Sancta College, Nirimba & Denise Arnold, Patrician Brothers College, Blacktown.

Marks

Question 1   Begin a new page

(a) The diagram shows the graph of $y=f(x)$ where $f(x)=1-2e^{-x^2}$.



(i) Find the values of the x intercepts. **1**

(ii) On separate diagrams sketch the graphs of **5**
$y=\{f(x)\}^2$, $y^2=f(x)$, $y=\cos^{-1} f(x)$,
in each case showing the intercepts on the axes and the equations of any asymptotes.

(b) Consider the function $f(x)=\dfrac{x}{1-x^2}$.

(i) Show that the function is increasing for all values of x in its domain. **1**

(ii) Sketch the graph of $y=f(x)$ showing the intercepts on the axes and the equations of any asymptotes. **2**

(iii) Find the values of k such that the equation $\dfrac{x}{1-x^2}=kx$ has three distinct real roots. **2**

(c) Consider the curve defined by $2x^2+xy-y^2=0$. At the point $(2,4)$ on the curve, find the values of $\dfrac{dy}{dx}$ and $\dfrac{d^2y}{dx^2}$. **4**

Question 2   Begin a new page

(a)(i) Find $\displaystyle\int \frac{\cos 2x}{\cos^2 x}\,dx$. **2**

(ii) Find $\displaystyle\int \frac{x^3}{1+x^2}\,dx$. **2**

(b) Use the substitution $u=1+e^x$ to find $\displaystyle\int \frac{e^{2x}}{\sqrt{1+e^x}}\,dx$. **3**

(c) Use integration by parts to evaluate $\displaystyle\int_1^e \frac{\ln x}{x^2}\,dx$. **3**

(d)(i) Use the substitution $t=\tan\frac{x}{2}$ to evaluate $\displaystyle\int_{\frac{3\pi}{2}}^{\frac{5\pi}{2}} \frac{1}{2+\cos x}\,dx$. **3**

(ii) Hence use the substitution $u=4\pi-x$ to evaluate $\displaystyle\int_{\frac{3\pi}{2}}^{\frac{5\pi}{2}} \frac{x}{2+\cos x}\,dx$. **2**

Marks

Question 3 **Begin a new page**

(a) If x is real and $(x+i)^4$ is imaginary, find the possible values of x in surd form. **3**

(b) z and w are two complex numbers such that $|z|=4$, $\arg z = \frac{5\pi}{6}$, $|w|=2$, $\arg w = \frac{\pi}{3}$.

(i) Express each of z and w in the form $a+ib$, where a and b are real. **2**

(ii) In an Argand diagram the points P and Q represent the complex numbers z and w respectively. Find the distance PQ in simplest exact form. **2**

(c)(i) Express $\sqrt{3}+i$ in modulus / argument form. **1**

(ii) On an Argand diagram sketch the locus of the point P representing the complex number z such that $\left|z-\left(\sqrt{3}+i\right)\right|=1$, and find the set of possible values of $|z|$ and $\arg z$. **3**

(d) In an Argand diagram the points P, Q and R represent the complex numbers z_1, z_2 and $z_2+i\left(z_2-z_1\right)$ respectively.

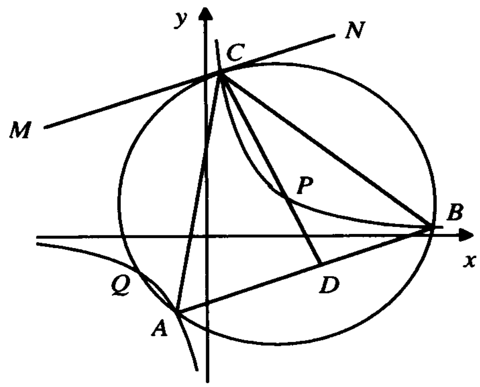
(i) Show that PQR is a right-angled triangle. **2**

(ii) Find in terms of z_1 and z_2 the complex number represented by the point S such that $PQRS$ is a rectangle. **2**

Marks

Question 4 **Begin a new page**

(a)



$P\left(c\theta, \frac{c}{\theta}\right)$ and $Q\left(-c\theta, -\frac{c}{\theta}\right)$, where $\theta > 0$ and $c > 0$, are two points on the rectangular hyperbola $xy = c^2$. The circle with centre P and radius PQ cuts the hyperbola again at points $A\left(c\alpha, \frac{c}{\alpha}\right)$, $B\left(c\beta, \frac{c}{\beta}\right)$ and $C\left(c\gamma, \frac{c}{\gamma}\right)$. CP produced meets AB at D. MCN is tangent to the circle at C.

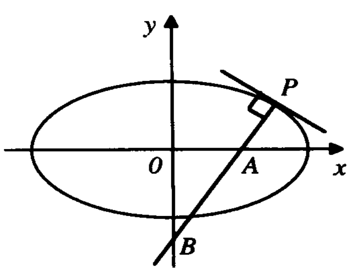
(i) Show that the circle cuts the hyperbola at points $\left(ct, \frac{c}{t}\right)$ where t satisfies the equation $t^4 - 2t^3\theta - 3t^2\left(\theta^2 + \frac{1}{\theta^2}\right) - \frac{2}{\theta}t + 1 = 0$. Hence deduce that $\alpha\beta\gamma\theta = -1$. **3**

(ii) Show that $CPD \perp AB$. Hence show that $MCN \parallel AB$. **2**

(iii) Show that $CA = CB$. **3**

(iv) What word best classifies triangle ABC? Justify your answer. **1**

(b)



$P(a\cos\theta, b\sin\theta)$, where $0 < \theta < \frac{\pi}{2}$, is a point on the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, where $a > b > 0$. The normal to the ellipse at P has equation $ax\sin\theta - by\cos\theta = \left(a^2 - b^2\right)\sin\theta\cos\theta$. This normal cuts the x axis at A and the y axis at B.

(i) Show that ΔOAB has area $\frac{\left(a^2 - b^2\right)^2}{2ab}\sin\theta\cos\theta$. **3**

(ii) Find the maximum area of ΔOAB and the coordinates of P when this maximum occurs. **3**

Marks

Question 5 **Begin a new page**

(a) The equation $x^4 - x^3 + 2x^2 - 2x + 1 = 0$ has roots $\alpha, \beta, \gamma, \delta$.

(i) Show that none of $\alpha, \beta, \gamma, \delta$ is an integer. **2**

(ii) Find the monic equation of degree four with roots $\alpha - 1, \beta - 1, \gamma - 1, \delta - 1$, and hence find the value of $(\alpha+\beta+\gamma)(\beta+\gamma+\delta)(\gamma+\delta+\alpha)(\delta+\alpha+\beta)$. **4**

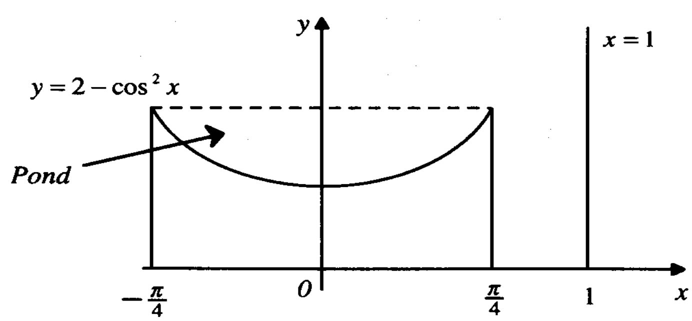
(b) (i) Express the roots of the equation $z^5 + 32 = 0$ in modulus / argument form. **3**

(ii) Hence show that $z^4 - 2z^3 + 4z^2 - 8z + 16 = \left\{z^2 - \left(4\cos\frac{\pi}{5}\right)z + 4\right\}\left\{z^2 - \left(4\cos\frac{3\pi}{5}\right)z + 4\right\}$. **2**

(iii) Hence find the exact values of $\cos\frac{\pi}{5}$ and $\cos\frac{3\pi}{5}$ in simplest surd form. **4**

Question 6 **Begin a new page**

(a)



A mould for a circular fish pond is made by rotating the region bounded by the curve $y = 2 - \cos^2 x$ and the x axis between $x = -\frac{\pi}{4}$ and $x = \frac{\pi}{4}$ through one complete revolution about the line $x = 1$. All measurements are in metres.

(i) Use the method of cylindrical shells to show that the volume of the fish pond is given by
$$V = \pi \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} (1-x)\cos 2x \, dx .$$ **3**

(ii) Hence find the capacity of the fish pond correct to the nearest litre. **3**

(b) A particle of mass m kilograms is dropped from rest in a medium where the resistance to motion has magnitude $\frac{1}{10}mv^2$ Newtons when the speed of the particle is $v\,\text{ms}^{-1}$. After t seconds, the particle has fallen x metres, and has velocity $v\,\text{ms}^{-1}$ and acceleration $a\,\text{ms}^{-2}$. The particle hits the ground $\ln\left(1+\sqrt{2}\right)$ seconds after it is dropped. Take $g = 10\,\text{ms}^{-2}$.

(i) Draw a diagram showing the forces acting on the particle. Deduce that $a = \frac{1}{10}\left(100 - v^2\right)$. **2**

(ii) Express v as a function of t. Hence find the speed with which the particle hits the ground, giving the answer in simplest exact form. **4**

(iii) Find in simplest exact form the distance fallen by the particle before it hits the ground. **3**

Marks

Question 7 **Begin a new page**

(a) a, b, c denote the lengths of the sides of a triangle.

(i) Express $4b^2c^2 - (b^2 + c^2 - a^2)^2$ as the product of four factors. **3**

(ii) Hence show that $(b^2 + c^2 - a^2)^2 < 4b^2c^2$. **1**

(b) Consider the function $f(x) = \cos^{-1} x$.

(i) Show that the function $E(x) = f(x) + f(-x)$ is even, and $O(x) = f(x) - f(-x)$ is odd. **2**

(ii) Hence express $\cos^{-1} x$ as the sum of an even function and an odd function. On the same diagram, sketch the graphs of these two functions. **3**

(c) A sequence $u_1, u_2, u_3, u_4 \ldots$ satisfies the relationship $u_n = u_{n-1} + u_{n-2}$ for $n \geq 3$.

(i) Show that $u_1 u_2 + u_2 u_3 = u_3^2 - u_1^2$. **2**

(ii) Use Mathematical Induction to show that **4**

$u_1 u_2 + u_2 u_3 + u_3 u_4 + u_4 u_5 + \ldots + u_{2n-1} u_{2n} + u_{2n} u_{2n+1} = u_{2n+1}^2 - u_1^2$ for $n \geq 1$.

Question 8 **Begin a new page**

(a) Code numbers of three digits are made from the digits 1, 2, 3, 4, 5, 6, 7, 8, 9 where no digit is repeated.

(i) Find the number of different code numbers that can be formed. **1**

(ii) How many of these code numbers are such that the three digits do not occur in increasing order of magnitude, reading from left to right? **3**

(b) Consider the function $f(x) = x - \dfrac{3\sin x}{2 + \cos x}$.

(i) Show that $f'(x) = \left(\dfrac{1 - \cos x}{2 + \cos x}\right)^2$. **2**

(ii) Hence show that $x > \dfrac{3\sin x}{2 + \cos x}$ for $x > 0$. **3**

(c)(i) Show that $\sin(2r+1)\theta - \sin(2r-1)\theta = 2\sin\theta\cos 2r\theta$. Hence show that **3**

$$\sin\theta \sum_{r=1}^{n} \cos 2r\theta = \tfrac{1}{2}\{\sin(2n+1)\theta - \sin\theta\}.$$

(ii) Hence evaluate $\displaystyle\sum_{r=1}^{100} \cos^2\left(\frac{r\pi}{100}\right)$. **3**