CSSA HSC Trial, Mathematics Extension 2, 2004

Question 1

(a) Consider the function $f(x) = x(x-3)^2$.

(i) Sketch the graph of the curve $y = f(x)$ showing clearly the coordinates and nature of any turning points and the intercepts on the x and y axes. Find the set of possible values of the real numbers k such that the equation $f(x) = k$ has three real, distinct solutions.

(ii) On separate axes, sketch the graphs of the following curves, showing clearly the coordinates and nature of any turning points and the equations of any asymptotes:

$$y = \{f(x)\}^2 \qquad y = \frac{1}{f(x)} \qquad y^2 = f(x)$$

(b) A curve is given parametrically in terms of the real number t by the equations $x = \frac{3t}{1+t^3}$ and $y = \frac{3t^2}{1+t^3}$.

(i) Express t in terms of x and y. Hence show that the curve has Cartesian equation $x^3 + y^3 = 3xy$. Deduce that the curve is symmetrical about the line $y = x$.

(ii) Show that $\frac{dy}{dx} = \frac{y-x^2}{y^2-x}$. Hence show that the curve has a horizontal tangent when $x = \sqrt[3]{2}$. Write down the coordinates of a point on the curve where the tangent is vertical.

Question 2

(a) **(i)** Find $\int (\cos x + \sin x)^2\, dx$.

(ii) Find $\int \frac{1}{1-x^2}\, dx$.

(b) Use the substitution $u = e^x - 1$ to find $\int \frac{e^{2x}}{(e^x-1)^2}\, dx$.

(c) **(i)** Use the substitution $t = \tan\frac{x}{2}$ to evaluate $\int_0^{\frac{\pi}{2}} \frac{1}{1+\sin x}\, dx$.

(ii) Hence find the value of $\int_0^{\frac{\pi}{2}} \frac{\sin x}{1+\sin x}\, dx$.

(d) **(i)** If $I_n = \int_1^e x^3(\ln x)^n\, dx$ for $n = 0, 1, 2, ...$, show that $I_n = \frac{e^4}{4} - \frac{n}{4} I_{n-1}$ for $n = 1, 2, 3, ...$

(ii) Hence find the value of $\int_1^e x^3(\ln x)^2\, dx$.

Question 3

(a) Solve the equation $|z|^2 + 2i\overline{z} = 4i + 7$, expressing any answers in the form $z = a + ib$ where a and b are real.

(b) A, B and C are the angles of a triangle.
Show that $(\cos A + i\sin A)(\cos B + i\sin B) + (\cos C - i\sin C) = 0$.

(c) $z_1 = 2(\cos\frac{\pi}{6} + i\sin\frac{\pi}{6})$ and $z_2 = 2(\cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3})$ are two complex numbers.

(i) On an Argand diagram draw the vectors $\overrightarrow{OA}, \overrightarrow{OB}$ and $\overrightarrow{OS}$ to represent z_1, z_2 and $z_1 + z_2$ respectively.

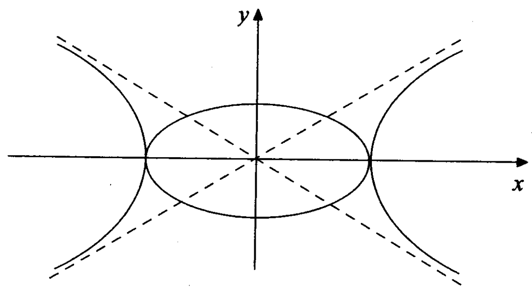
(ii) Hence express $z_1 + z_2$ in modulus/argument form.

(d) **(i)** On an Argand diagram shade the reion where both $|z| \leq 4$ and $|z - 4| \leq 4$.

(ii) Find the exact area of the shaded region.

Question 4

(a)



$P(a\cos\theta, b\sin\theta)$ lies on the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, where $a > b > 0$. The tangent to the ellipse at P passes through a focus of the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ with eccentricity e.

(i) Show that the tangent to the ellipse at P has equation $bx\cos\theta + ay\sin\theta = ab$.

(ii) Show that P lies on a directrix of the hyperbola.

(iii) Show that the tangent to the ellipse at P has gradient ± 1.

(b) **(i)** Sketch the graph of the rectangular hyperbola $xy = 1$, showing clearly the coordinates of the foci and the equations of the directrices.

(ii) $P(p, \frac{1}{p})$ and $Q(q, \frac{1}{q})$ are two points on the rectangular hyperbola $xy = 1$. Show that the chord PQ has equation $x + pqy - (p + q) = 0$.

(iii) If O is the origin, show that $\triangle OPQ$ has area $\frac{|p^2 - q^2|}{2|pq|}$ square units.

Question 5

(a) **(i)** Use the substitution $x = 10\sqrt{2}\sin\theta$ to show that $\int_{-10}^{10} \sqrt{200 - x^2}\, dx = 100 + 50\pi$, then use a geometrical argument to verify this result.

(ii) A mould for a model railway tunnel is made by rotating the region bounded by the curve $y = \sqrt{200 - x^2}$ and the x axis between the lines $x = -10$ and $x = 10$ through $180°$ about the line $x = 100$ (where all measurements are in cm). Use the method of cylindrical shells to show that the volume $V\,\text{cm}^3$ of the tunnel is given by $V = \pi \int_{-10}^{10} (100 - x)\sqrt{200 - x^2}\, dx$. Hence find the volume of the tunnel in m^3 correct to 2 significant figures.

(b) **(i)** Show that the roots of the equation $z^{10} = 1$ are given by $z = \cos\frac{r\pi}{5} + i\sin\frac{r\pi}{5}$, $r = 0, 1, 2, ..., 9$.

(ii) Explain why the equation $\left(\frac{z-1}{z}\right)^{10} = 1$ has only nine roots. Show that the roots of $\left(\frac{z-1}{z}\right)^{10} = 1$ are given by $z = \frac{1}{2}\left(1 + i\cot\frac{r\pi}{10}\right),\ r = 1, 2, 3, ..., 9$.

Question 6

(a) A particle of mass m is moving vertically in a resisting medium in which the resistance to motion has magnitude $\frac{1}{10}mv^2$ where the particle has speed $v\,\text{m.s}^{-1}$. The acceleration due to gravity is $g\,\text{m.s}^{-2}$.

(i) If the particle falls vertically downwards from rest, show that its acceleration $a\,\text{m.s}^{-2}$ is given by $a = g - \frac{1}{10}v^2$. Hence show that its terminal speed $V\,\text{m.s}^{-1}$ is given by $V = \sqrt{10g}$.

(ii) If the particle is projected vertically upwards with speed $V\tan\alpha\ \text{m.s}^{-1}$ (for some $0 < \alpha < \frac{\pi}{2}$), show that its acceleration $a\,\text{m.s}^{-2}$ is given by $a = -(g + \frac{1}{10}v^2)$. Hence show that it reaches a maximum height H metres given by $H = 5\ln\sec^2\alpha$, and that it returns to its point of projection with speed $V\sin\alpha\ \text{m.s}^{-1}$.

(b) The equation $x^3 + 2x + 1 = 0$ has roots α, β and γ.

(i) Find the monic cubic equation with roots $\frac{1}{\alpha}, \frac{1}{\beta}$ and $\frac{1}{\gamma}$.

(ii) Find the monic cubic equation with roots $\frac{\beta+\gamma}{\alpha^2}, \frac{\gamma+\alpha}{\beta^2}$ and $\frac{\alpha+\beta}{\gamma^2}$.

Question 7

(a) **(i)** Newton's method is being used to find an approximation to the positive root $x = \sqrt{c}$ of the equation $x^2 - c = 0$. The initial approximation is $x = a$ (for some $a > 0$). The error in this initial approximation is $\varepsilon_0 = |a - \sqrt{c}|$. Show that the error ε_1 in the next approximation (obtained by one application of Newton's method) is given by $\varepsilon_1 = \frac{\varepsilon_0^2}{2a}$.

(ii) Find the values of a (in terms of c) such that $\varepsilon_1 = \varepsilon_0$.

(b) **(i)** If $4 - \tan\theta = 5\sin\theta\cos\theta$, show that $x = \tan\theta$ is a root of the equation $x^3 - 4x^2 + 6x - 4 = 0$.

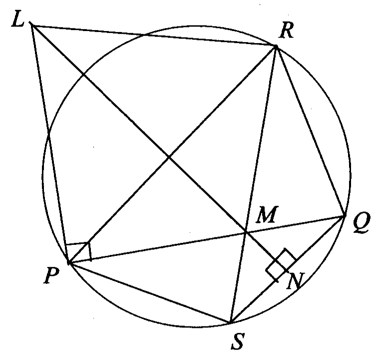
(ii) Solve the equation $4 - \tan\theta = 5\sin\theta\cos\theta$ for $0° \leq \theta \leq 360°$ giving answers correct to the nearest degree.

(c) **(i)** By writing $n!$ as a product, show that $n! < (n+1)^n$ for all positive integers n.

(ii) Hence show that $\sqrt[n]{n!} < \sqrt[n+1]{(n+1)!}$ for all positive integers n.

Question 8

(a)



PQ and RS are two chords of a circle which intersect at M inside the circle. MN is the perpendicular from M to SQ. L is the point on NM produced such that LP is perpendicular to PQ.

(i) Copy the diagram.

(ii) Show that $\triangle PML ||| \triangle NMQ$

(iii) Hence show that $LR \perp RS$.

(b) The number x and the real number θ are such that $x + \frac{1}{x} = 2\cos\theta$. Use mathematical induction to show that $x^n + \frac{1}{x^n} = 2\cos\ n\theta$ for all positive integers $n \geq 2$.