Cathholic Trial 2007, Mathematics Ext.2 HSC

Marks

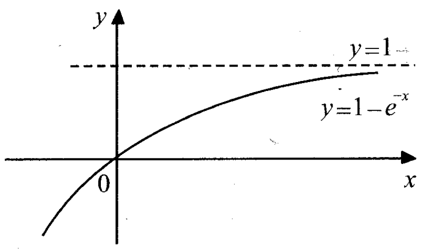
Question 1 (15 marks) Use a SEPARATE writing booklet

(a) It is given that $(2+\cos x)(2-\cos y)=3$, where $0<x<\pi$ and $0<y<\pi$.

(i) Show that $\cos y=\dfrac{1+2\cos x}{2+\cos x}$ and $\sin y=\dfrac{\sqrt{3}\sin x}{2+\cos x}$. **2**

(ii) Hence show that $\dfrac{dy}{dx}=\dfrac{\sqrt{3}}{2+\cos x}$. **3**

(b)



The diagram shows the graph of $f(x)=1-e^{-x}$. On separate diagrams sketch the graphs of the following functions, showing clearly the equations of any asymptotes:

(i) $y=[f(x)]^2$ **1**

(ii) $y=f(x^2)$ **1**

(iii) $y=\dfrac{1}{f(x)}$ **2**

(iv) $y=\ln f(x)$ **1**

(c) The function $f(x)$ is given by $f(x)=a+\dfrac{b\sin x}{x}$, $x\neq 0$ and $f(0)=0$, where a and b are non-zero real numbers.

(i) Show that $f(x)$ is an even function. **1**

(ii) Find the general solution of the equation $f(x)=a$. **2**

(iii) If $\lim\limits_{x\to\infty} f(x)=1$ and $f(x)$ is continuous at $x=0$, find the values of a and b. **2**

Marks

Question 2 (15 marks) Use a SEPARATE writing booklet

(a)(i) Find $\int \frac{1-x^{-2}}{1-x^{-1}}\,dx$. **2**

(ii) Find $\int \left(\sqrt{e^x}+1\right)^2 dx$. **2**

(b) Evaluate $\int_0^{\frac{\sqrt{3}}{2}} \frac{1-x}{\sqrt{1-x^2}}\,dx$. **3**

(c) Use the substitution $t = \tan\frac{x}{2}$ to evaluate $\int_0^{\frac{\pi}{2}} \frac{1}{4+5\sin x}\,dx$. **4**

(d)(i) If $I_n = \int_1^e (1-\ln x)^n\,dx$, $n = 0, 1, 2, \ldots$ show that $I_n = -1 + n\,I_{n-1}$, $n = 1, 2, 3, \ldots$ **2**

(ii) Hence find the value of I_3. **2**

Marks

Question 3 (15 marks) Use a SEPARATE writing booklet

(a) Find all the complex numbers $z = a + ib$, where a and b are real, such that $|z|^2 + 5\bar{z} + 10i = 0$. **3**

(b) $z_1 = 1 + i\sqrt{3}$ and $z_2 = 1 - i$ are two complex numbers.

(i) Express z_1, z_2 and $\frac{z_1}{z_2}$ in modulus / argument form. **2**

(ii) Find the smallest positive integer n such that $\frac{z_1^n}{z_2^n}$ is imaginary. For this value of n, write the value of $\frac{z_1^n}{z_2^n}$ in the form bi where b is a real number. **2**

(c)(i) On an Argand diagram shade the region where both $|z - 1| \leq 1$ and $0 \leq \arg z \leq \frac{\pi}{6}$. **2**

(ii) Find the perimeter of the shaded region. **2**

(d) On an Argand diagram the points A, B and C represent the complex numbers α, β and γ respectively. ΔABC is equilateral, named with its vertices taken anticlockwise.

(i) Show that $\gamma - \alpha = \left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right)(\beta - \alpha)$. **2**

(ii) Show that $\alpha^2 + \beta^2 + \gamma^2 = \alpha\beta + \beta\gamma + \gamma\alpha$. **2**

Marks

Question 4 (15 marks) Use a SEPARATE writing booklet

(a) $P(a\cos\theta, b\sin\theta)$ is a point in the first quadrant on the ellipse $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}=1$ and $Q(a\sec\theta, b\tan\theta)$ is a point on the hyperbola $\dfrac{x^2}{a^2}-\dfrac{y^2}{b^2}=1$, where $a>b>0$.

(i) Sketch the ellipse, the hyperbola and their common auxiliary circle $x^2+y^2=a^2$ on the same diagram, showing the angle θ and the related points P and Q. Show clearly how the positions of P and Q are determined by the value of θ, $0<\theta<\frac{\pi}{2}$. **2**

(ii) Prove that the tangent to the ellipse at P has equation $\dfrac{x\cos\theta}{a}+\dfrac{y\sin\theta}{b}=1$. Deduce that this tangent cuts the x-axis vertically below Q. **3**

(iii) Given that the tangent to the hyperbola at Q has equation $\dfrac{x\sec\theta}{a}-\dfrac{y\tan\theta}{b}=1$, show that this tangent and the tangent to the ellipse at P intersect at $T\left(a, b\tan\frac{\theta}{2}\right)$. Show both tangents on your sketch. **4**

(iv) Without any further working, sketch a second diagram showing both curves, the common auxiliary circle, the points P, Q and the corresponding tangents intersecting at T if $\frac{\pi}{2}<\theta<\pi$. **1**

(b) $P\left(2p, \dfrac{2}{p}\right)$ is a variable point on the hyperbola $xy=4$. The normal to the hyperbola at P meets the hyperbola again at $Q\left(2q, \dfrac{2}{q}\right)$. M is the midpoint of PQ.

(i) Show that $q=-\dfrac{1}{p^3}$. **2**

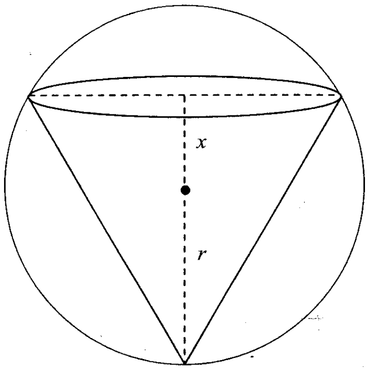
(ii) Show that M has coordinates $\left(\dfrac{1}{p}\left(p^2-\dfrac{1}{p^2}\right), p\left(\dfrac{1}{p^2}-p^2\right)\right)$. **1**

(iii) Show that as P moves on the hyperbola, the locus of M has equation $\left(x^2-y^2\right)^2=-x^3y^3$. **2**

Marks

Question 5 (15 marks) Use a SEPARATE writing booklet

(a) A right circular cone of height $(r + x)$ is inscribed in a sphere of radius r.



(i) Show that the volume V of the cone is given by $V = \frac{\pi}{3}\left(r^3 + r^2x - rx^2 - x^3\right)$. **2**

(ii) Hence show that V is a maximum when $x = \frac{1}{3}r$. **2**

(iii) Find the ratio of the maximum volume of the cone to the volume of the sphere. **2**

(b)(i) By considering $f'(x)$ where $f(x) = e^x - x$, show that $e^x > x$ for $x \geq 0$. **2**

(ii) Hence use Mathematical Induction to show that for $x \geq 0$, $e^x > \dfrac{x^n}{n!}$ for all positive integers $n \geq 1$. **3**

(c) The polynomial $P(x)$ is given by $P(x) = x^3 + ax^2 + bx + c$ where a, b and c are real. The equation $P(x) = 0$ has roots α, β and γ. S_n is defined by $S_n = \alpha^n + \beta^n + \gamma^n$ for $n = 1, 2, 3, \ldots$, and it is given that $S_1 = S_3 = 3$ and $S_2 = 7$.

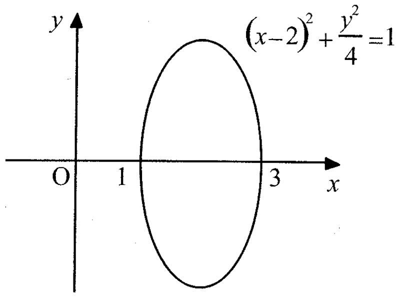
(i) Show that $a = -3$ and $b = 1$. **2**

(ii) Find the value of c. **2**

Marks

Question 6 (15 marks) Use a SEPARATE writing booklet

(a)

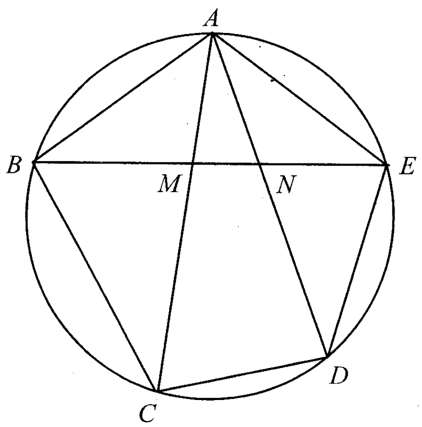


The region enclosed by the ellipse $(x-2)^2+\frac{y^2}{4}=1$ is rotated through one complete revolution about the y-axis.

(i) Use the method of cylindrical shells to show that the volume V of the solid of revolution is given by . $V=8\pi\int_1^3 x\sqrt{1-(x-2)^2}\,dx$ **2**

(ii) Hence find the volume of the solid of revolution in simplest exact form. **4**

(b)



$ABCDE$, where $AB=AE$, is a pentagon inscribed in a circle. BE meets AC and AD at M and N respectively.

(i) Show that $\angle BEA=\angle ACE$. **2**

(ii) Hence show that $CDNM$ is a cyclic quadrilateral. **3**

Marks

(c) The polynomial $P(x)$ is given by $P(x) = x^4 + bx^2 + 1$ where b is a real number. The equation $P(x) = 0$ has a real root α, where $\alpha \neq 0$.

(i) Express the other three roots of the equation $P(x) = 0$ in terms of α and deduce that all four roots are real. **2**

(ii) Find the set of possible values of b. **2**

Question 7 (15 marks) Use a SEPARATE writing booklet

(a)(i) Show that the equation $z^7 - 1 = 0$ has roots $1, \omega, \omega^2, \omega^3, \omega^4, \omega^5$ and ω^6, where $\omega = \cos\frac{2\pi}{7} + i\sin\frac{2\pi}{7}$. **2**

(ii) Hence show that the equation $z^6 + z^5 + z^4 + z^3 + z^2 + z + 1 = 0$ has roots $\omega, \omega^2, \omega^3, \omega^4, \omega^5$ and ω^6. **1**

(iii) Find the value of $\cos\frac{2\pi}{7} + \cos\frac{4\pi}{7} + \cos\frac{6\pi}{7}$. **2**

(iv) Find the monic quadratic equation with numerical coefficients whose roots are $\omega + \omega^2 + \omega^4$ and $\omega^3 + \omega^5 + \omega^6$. **2**

(b) A particle of mass m is moving vertically in a resisting medium in which the resistance to motion has magnitude $\frac{1}{10}mv^2$ when the particle has velocity $v\,\text{ms}^{-1}$. The acceleration due to gravity is $10\,\text{ms}^{-2}$.

(i) The particle is projected vertically upwards with speed $U\,\text{ms}^{-1}$. Show that during its upward motion, its acceleration $a\,\text{ms}^{-2}$ is given by $a = -\frac{1}{10}(100 + v^2)$. **1**

(ii) Hence show that its maximum height, H metres, is given by $H = 5\ln\left(\dfrac{U^2 + 100}{100}\right)$. **3**

(iii) The particle falls vertically from rest. Show that during its downward motion its acceleration $a\,\text{ms}^{-2}$ is given by $a = \frac{1}{10}(100 - v^2)$. **1**

(iv) Hence show that it returns to its point of projection with speed $V\,\text{ms}^{-1}$ given by $V = \dfrac{10U}{\sqrt{U^2 + 100}}$. **3**

Marks

Question 8 (15 marks) Use a SEPARATE writing booklet

(a) Consider the function $f(x)=\sum_{k=1}^{n}(a_k x-1)^2$ where $a_1>0,\ a_2>0,\dots,\ a_n>0$ are real.

(i) Express $f(x)$ in the form $f(x)=Ax^2+Bx+C$ for real numbers A, B and C. **1**

(ii) Show that $\sum_{k=1}^{n}a_k^{\ 2}\geq\frac{1}{n}\left(\sum_{k=1}^{n}a_k\right)^2$. **2**

(iii) Hence show that $1^2+3^2+\dots+(2n-1)^2\geq n^3$ and $1^4+3^4+\dots+(2n-1)^4\geq n^5$. **2**

(b) Two players A, B play a game of chance comprising several turns in which each player throws a fair 6-sided die. The possible outcomes are a draw (A, B throw the same score), A wins (A's score is higher than B's) or B wins. The game is over when either player first records two wins. Let p_n be the probability the game ends on the n^{th} turn. Let q_n be the probability the game does not end in n or fewer turns.

(i) Explain why, on each turn, the probability that A wins is $\frac{5}{12}$. **1**

(ii) Explain why $p_2+q_2=1$. **1**

(iii) Explain why $p_n+q_n=q_{n-1}$, $n=3,4,5,\dots$ and deduce that $\sum_{k=2}^{n}p_k=1-q_n$, $n\geq 2$. **3**

(iv) Show that $q_n=\frac{25n^2-5n+4}{4\times 6^n}$ and $p_n=\frac{25(n-1)(5n-8)}{4\times 6^n}$, $n\geq 2$. **3**

(v) What is the probability the game will never end? Justify your answer. **2**