Centre Number

Student Number

CATHOLIC SECONDARY SCHOOLS
ASSOCIATION OF NEW SOUTH WALES

2008
TRIAL HIGHER SCHOOL CERTIFICATE EXAMINATION

Mathematics
Extension 2

Morning Session
Monday, 11 August 2008

General Instructions

- Reading time – 5 minutes
- Working time – 3 hours
- Write using blue or black pen
- Board-approved calculators may be used
- A table of standard integrals is provided at the back of this paper
- All necessary working should be shown in every question

Total marks – 120
- Attempt Questions 1-8
- All questions are of equal value

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Total marks – 120
Attempt Questions 1–8
All questions are of equal value

Answer each question in a SEPARATE writing booklet.

Marks

Question 1 (15 marks) Use a SEPARATE writing booklet.

(a) Evaluate $\int_0^{\frac{\pi}{4}} \cos x \sin^3 x \, dx$. **2**

(b) Find $\int_2^5 \frac{2\,dx}{x^2 - 4x + 13}$. **3**

(c) (i) Find the real numbers a, b and c such that $\frac{5}{x^2(2-x)} \equiv \frac{ax+b}{x^2} + \frac{c}{2-x}$. **2**

(ii) Hence, or otherwise, find $\int \frac{20}{x^2(2-x)}\, dx$. **3**

(d) (i) Use the substitution $x = \sin\theta$ to find $\int \frac{x^2}{\sqrt{1-x^2}}\, dx$. **3**

(ii) Use integration by parts to find $\int x \sin^{-1} x \, dx$. **2**

Marks

Question 2 (15 marks) Use a SEPARATE writing booklet.

(a) Simplify $(3-4i)^3$. **1**

(b) Solve $z^2 = 5-12i$, giving your answer in the form $x + iy$, where x and y are real. **3**

(c) (i) Express $1+i$ in modulus-argument form. **1**

(ii) Hence evaluate $(1+i)^{12}$. **2**

(d) Sketch the locus of all points z such that:

(i) $\arg z = \dfrac{\pi}{3}$. **1**

(ii) $\arg \overline{z} = \dfrac{\pi}{3}$. **1**

(iii) $\arg(-z) = \dfrac{\pi}{3}$. **1**

(e) The points O, A, Z and C on the Argand diagram represent the complex numbers 0, 1, z and $z+1$ respectively, where $z = \cos\theta + i\sin\theta$ is any complex number of modulus 1, with $0 < \theta < \pi$.

(i) Explain why $OACZ$ is a rhombus. **1**

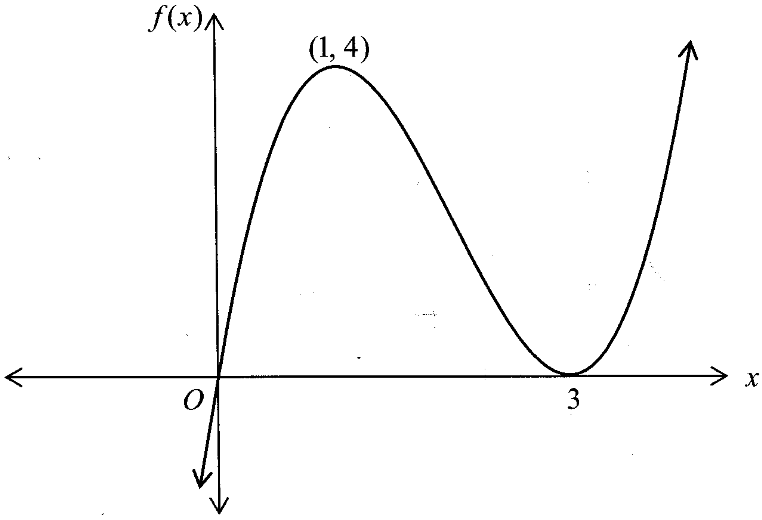
(ii) Show that $\dfrac{z-1}{z+1}$ is purely imaginary. **2**

(iii) Find the modulus and argument of $z+1$. **2**

Marks

Question 3 (15 marks) Use a SEPARATE writing booklet.

(a) The function defined by $f(x) = x(x-3)^2$ is drawn below.



(i) Draw separate, one-third page sketches, of the following:

(α) $y = f(|x|)$. **1**

(β) $y = \dfrac{1}{f(x)}$. **2**

(γ) $y^2 = f(x)$. **2**

(δ) $y = \tan^{-1} f(x)$. **2**

(ii) Find the values of k for which $f(x) = kx$ has exactly two distinct solutions. **2**

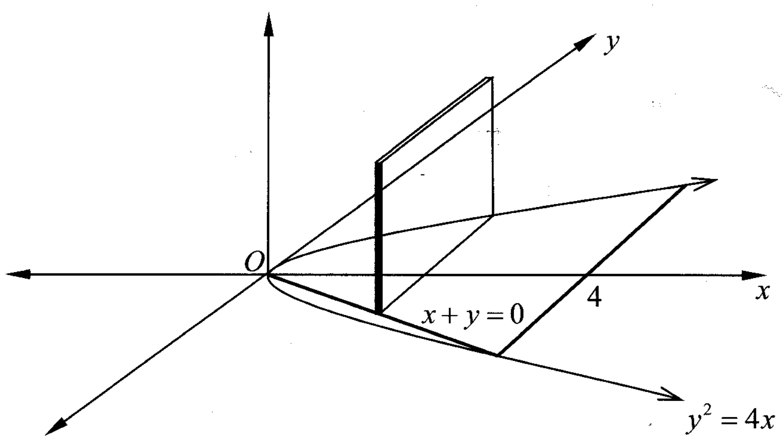
(b) When the polynomial $P(x)$ is divided by $(x+2)(x-3)$ the remainder is $4x+1$. **2**
What is the remainder when $P(x)$ is divided by $(x+2)$?

Question 3 continues on page 5

Marks

Question 3 (continued)

(c) The base of a solid is the region bounded by the curve $y^2 = 4x$ and the lines $x + y = 0$ and $x = 4$. Every cross-sectional slice perpendicular to the x axis is a square having a side with one end-point on the line $x + y = 0$ and the other on the curve $y^2 = 4x$.



(i) Show that the area of the cross-section is given by $A(x) = 4x + x^2 + 4x^{\frac{3}{2}}$. **2**

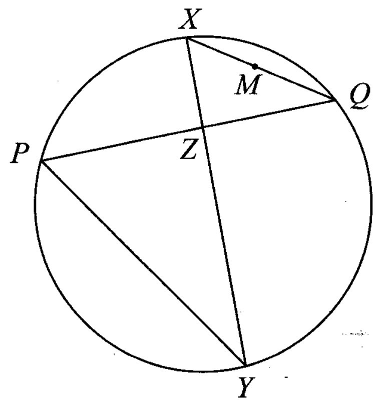
(ii) Hence find the volume of the solid formed. **2**

End of Question 3

Marks

Question 4 (15 marks) Use a SEPARATE writing booklet.

(a) Two perpendicular chords PQ and XY of a circle intersect at Z. **3**



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Copy or trace the diagram into your writing booklet.

If M is the midpoint of the chord QX, prove that MZ produced is perpendicular to the chord PY.

(b) (i) $I_n = \int x^n e^{ax} dx$, where a is a constant. **2**

Prove that $I_n = \frac{x^n e^{ax}}{a} - \frac{n}{a} I_{n-1}$.

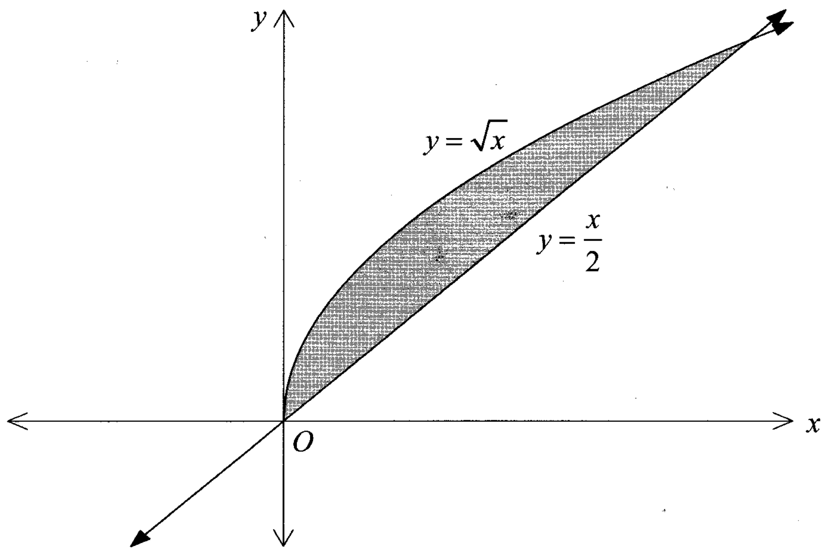
(ii) Hence find the value of $\int_0^1 x^3 e^{2x}\, dx$. **3**

Question 4 continues on page 7

Marks

Question 4 (continued)

(c) By the method of cylindrical shells, find the volume of the solid generated by rotating the region bounded by $y = \frac{x}{2}$ and $y = \sqrt{x}$ about the x axis. **4**



(d) The polynomial equation $x^5 - ax^2 + b = 0$ has a multiple root. **3**

Show that $108a^5 = 3125b^3$.

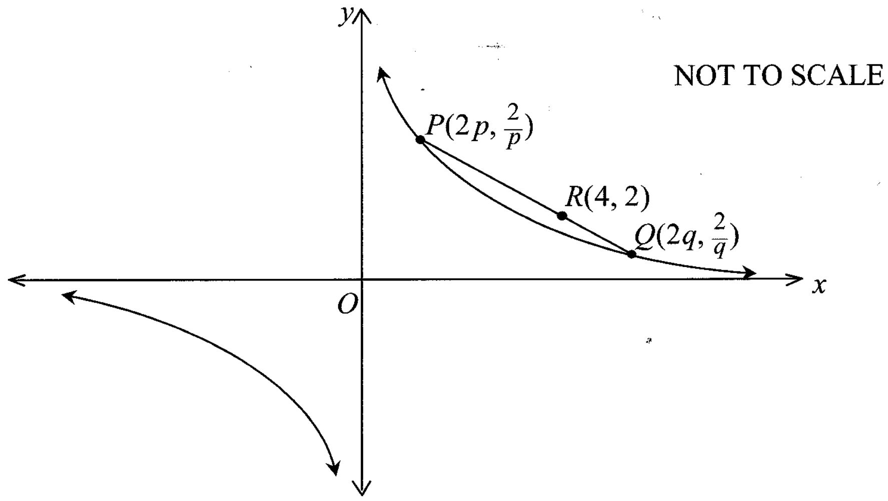
End of Question 4

Marks

Question 5 (15 marks) Use a SEPARATE writing booklet.

(a) If a, b and c are positive real numbers, prove that $(a+b)(b+c)(c+a) \geq 8abc$. **3**

(b) $P(2p, \frac{2}{p})$ and $Q(2q, \frac{2}{q})$ are points on the rectangular hyperbola $xy = 4$. M is the midpoint of the chord PQ. P and Q move on the hyperbola so that the chord PQ always passes through the point $R(4, 2)$.



(i) Show that the equation of the chord PQ is $x + pqy = 2(p+q)$. **2**

(ii) Show that $pq = p + q - 2$. **1**

(iii) Hence sketch the locus of M, as P and Q move on the curve $xy = 4$. **3**

(c) (i) By considering $z^9 - 1$ as the difference of two cubes, or otherwise, write $1 + z + z^2 + z^3 + z^4 + z^5 + z^6 + z^7 + z^8$ as a product of two polynomials with real coefficients, one of which is a quadratic. **2**

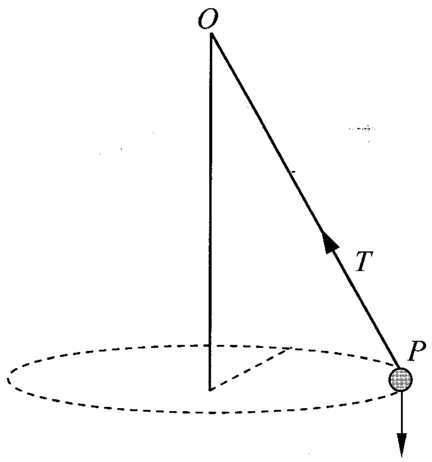
(ii) Solve $z^9 - 1 = 0$ and determine the six solutions of $z^6 + z^3 + 1 = 0$. **2**

(iii) Hence show that $\cos\frac{2\pi}{9} + \cos\frac{4\pi}{9} = \cos\frac{\pi}{9}$. **2**

Marks

Question 6 (15 marks) Use a SEPARATE writing booklet.

(a) A body P of mass 0.5 kg is suspended from a fixed point O by means of a light rod of length 1 m. The mass is rotated in a horizontal circle at a constant speed v m s^{-1} and the rod makes an angle of θ with the downward direction of the vertical. Assume $g = 9.8$ ms^{-2} and $\theta = 30^\circ$.



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(i) Resolve the horizontal and vertical forces at P and show that $\tan\theta = \dfrac{v^2}{rg}$ where r is the radius of the circle. **3**

(ii) Find the tension T in the rod. **1**

(iii) Find the speed v of P. **1**

(iv) Find the period of the motion. **1**

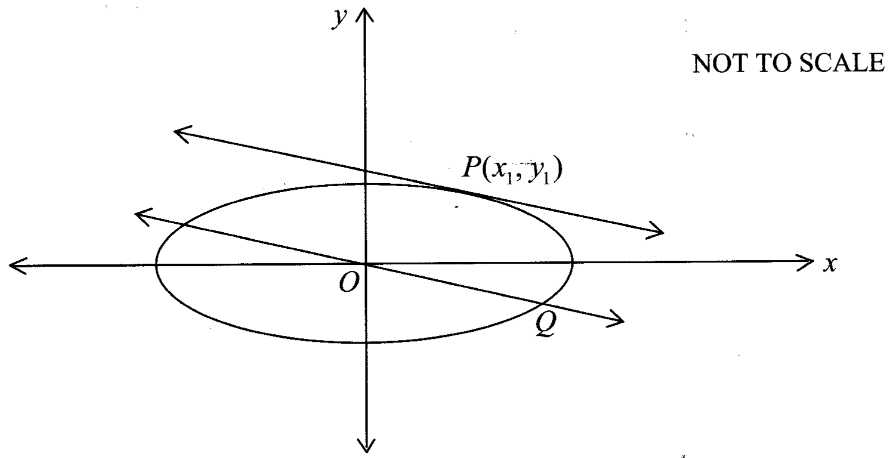
Question 6 continues on page 10

Marks

Question 6 (Continued)

(b) $P(x_1, y_1)$ is a point on the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ with centre O. A line drawn from O, parallel to the tangent to the ellipse at P, meets the ellipse at Q.

The equation of the tangent to the ellipse at P is $\frac{xx_1}{a^2}+\frac{yy_1}{b^2}=1$.



(i) Show that the equation of the line OQ is $\frac{xx_1}{a^2}+\frac{yy_1}{b^2}=0$. **2**

(ii) Show that the coordinates of Q, in terms of x_1 and y_1, are $\left(\frac{ay_1}{b}, \frac{-bx_1}{a}\right)$. **3**

(iii) Show that the distance between the tangent at P and the line OQ is $\frac{ab}{OQ}$. **3**

(iv) Hence prove that the area of the triangle OPQ is independent of the position P. **1**

End of Question 6

Marks

Question 7 (15 marks) Use a SEPARATE writing booklet.

(a) A rock of mass 5 kg is propelled vertically upward into the air from the ground with initial speed 12 ms^{-1}. The rock is subject to air resistance of $\frac{v^2}{2}$ Newtons in the opposite direction to its velocity, v ms^{-1}. The rock is also subject to a downward gravitational force of 50 Newtons.

The equation of motion of the rock until it reaches its highest point is $\ddot{x} = -\frac{v^2}{10} - 10$, where x metres is the height of the rock above the ground when its velocity is $v\,\text{ms}^{-1}$.

(i) Find the time taken by the rock to reach its maximum height. **3**

(ii) Show that $v^2 = 244e^{-\frac{x}{5}} - 100$ while the rock is ascending. **3**

(iii) Find the maximum height reached by the rock. **2**

(b) The Fibonacci sequence 1, 1, 2, 3, 5, 8, . . . can be defined as $u_1 = 1$, $u_2 = 1$ and $u_{n+2} = u_n + u_{n+1}$ for integers $n > 0$.

(i) Use induction to prove that $u_n < a^n$ for any $a > \frac{1+\sqrt{5}}{2}$. **3**

(ii) Assuming that $\frac{u_{n+1}}{u_n}$ approaches a limit as $n \to \infty$, **4**

show that $\frac{u_{n+1}}{u_n} \to \frac{1+\sqrt{5}}{2}$.

Question 8 (15 marks) Use a SEPARATE writing booklet. **Marks**

(a) Consider the function $f(x) = (ax-b)^2 + (cx-d)^2$ where a, b, c and d are real.

(i) Explain why $f(x) \geq 0$ for all x. **1**

(ii) Hence, or otherwise, prove that $|ab+cd| \leq \sqrt{a^2+c^2}\sqrt{b^2+d^2}$. **3**

(b) (i) Write out the binomial expansion of $\left(1+\frac{1}{n}\right)^n$, where n is a positive integer. **1**

(ii) Show that the $(k+1)^{th}$ term, T_{k+1} in your expansion, is given by **2**

$$T_{k+1} = \frac{1}{k!}\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)\cdots\left(1-\frac{k-1}{n}\right).$$

(iii) Let the $(k+1)^{th}$ term in the expansion of $\left(1+\frac{1}{n}\right)^{n+1}$ be U_{k+1}. **3**

Show that $U_{k+1} > T_{k+1}$.

Question 8 continues on page 13

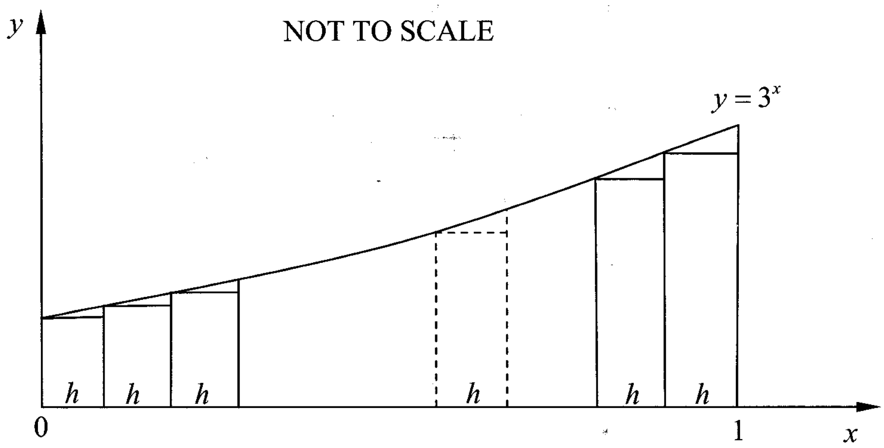
Marks

Question 8 (continued)

(c) The diagram shows the curve with equation $y = 3^x$ for $0 \le x \le 1$.

The area A under the curve between these limits is divided into n strips,

each of width h where $nh = 1$.



(i) Show that $A > \dfrac{2h}{3^h - 1}$. **2**

(ii) Hence show that $\dfrac{h}{3^h - 1} < \dfrac{1}{\ln 3} < \dfrac{h\,3^h}{3^h - 1}$. **3**

End of paper