

Total marks – 120
Attempt Questions 1–8
All questions are of equal value

CSA 2009 Ext. 2 trial

Answer each question in a SEPARATE writing booklet.

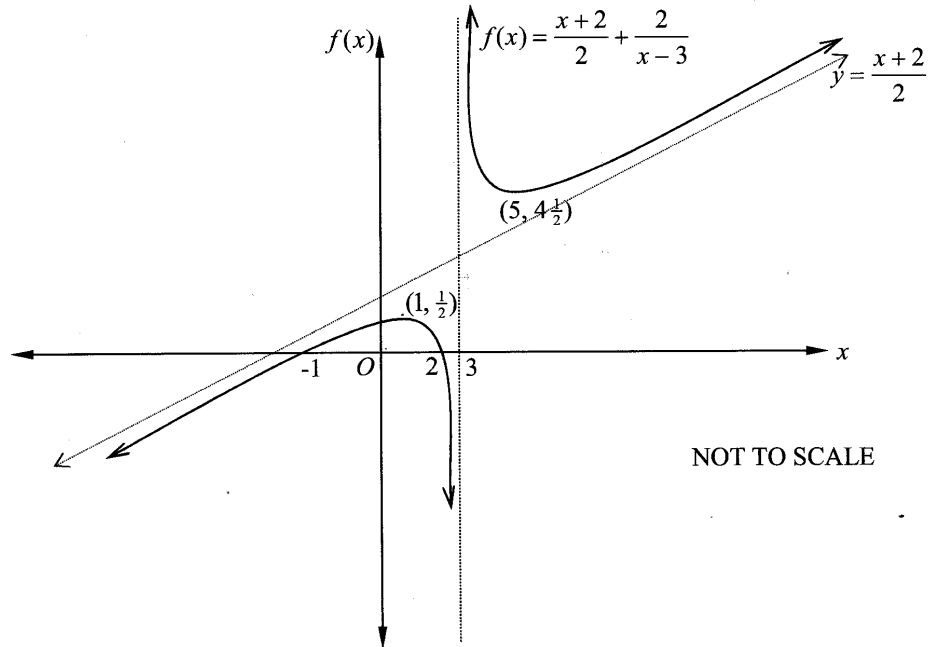
- | | | Marks |
|--|---|-------|
| Question 1 (15 marks) Use a SEPARATE writing booklet. | | |
| (a) | Evaluate $\int_3^4 x\sqrt{x-3} dx$. | 3 |
| (b) | Use the substitution $t = \tan \frac{\theta}{2}$ to find $\int \frac{\tan \theta}{1 + \cos \theta} d\theta$. | 3 |
| (c) | Use integration by parts to evaluate $\int_1^e \frac{\ln x}{\sqrt{x}} dx$. | 3 |
| (d) | (i) Find real numbers a and b such that $\frac{1}{x(x+1)} \equiv \frac{a}{x} + \frac{b}{x+1}$. | 2 |
| | (ii) Hence find $\int_1^N \frac{dx}{x(x+1)}$. | 2 |
| | (iii) Hence find the area bounded by the curve $y = \frac{1}{x(x+1)}$ and the x -axis to the right of $x = 1$. | 2 |

Marks**Question 2** (15 marks) Use a SEPARATE writing booklet.

- (a) Given that $z = 2 + i$ and $w = -2$, express $\frac{z}{z+w}$ in the form $x + iy$ where x and y are real numbers. **1**
- (b) (i) Express $w = 4 + 4i$ in modulus-argument form. **1**
- (ii) Hence, or otherwise, find all numbers z such that $z^5 = 4 + 4i$, giving your answer in modulus-argument form. **3**
- (c) (i) On a single Argand diagram, sketch the locus of points for which $|z - 3i| \leq 2$ and $\arg(z + 1) \leq \frac{\pi}{4}$. **2**
- (ii) Find the value of $\arg z$ at the point where $\arg z$ is a minimum. **2**
- (d) (i) Given that ω is one of the complex roots of $z^3 = 1$, show that $1 + \omega + \omega^2 = 0$. **2**
- (ii) Hence, or otherwise, evaluate $1 + \omega^4 + \omega^8$. **1**
- (e) The complex numbers z and w each have a modulus of 4. The arguments of z and w are $\frac{2\pi}{9}$ and $\frac{5\pi}{9}$, respectively. Find
- (i) $\arg(z + w)$. **1**
- (ii) $|z + w|$. **2**

Question 3 (15 marks) Use a SEPARATE writing booklet.

- (a) The function defined by $f(x) = \frac{x+2}{2} + \frac{2}{x-3}$ is drawn below.



Draw separate, one-third page sketches, of the following:

- | | | |
|-------|-------------------|---|
| (i) | $y = f(x) $. | 1 |
| (ii) | $y = [f(x)]^2$. | 2 |
| (iii) | $y = f(x^2)$. | 2 |
| (iv) | $y = \ln[f(x)]$. | 2 |

Question 3 continues on page 5

Question 3 (continued)

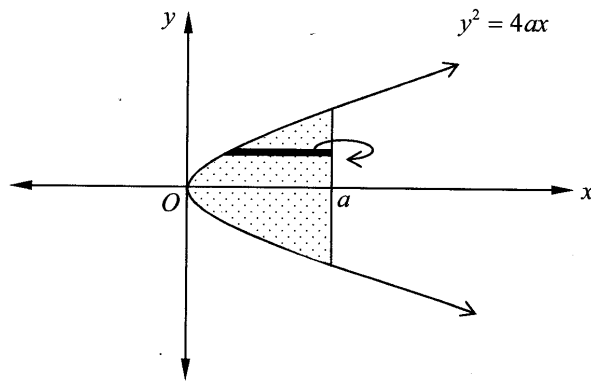
Marks

(b) The cubic equation $3x^3 - 9x^2 + 6x + 2 = 0$ has roots α, β and γ .

(i) Find the cubic equation with roots α^2, β^2 and γ^2 . 2

(ii) Hence evaluate $\alpha^3\beta\gamma + \alpha\beta^3\gamma + \alpha\beta\gamma^3$. 2

(c) The figure drawn below shows the shaded area enclosed by the parabola $y^2 = 4ax$ and the line $x = a$. 4

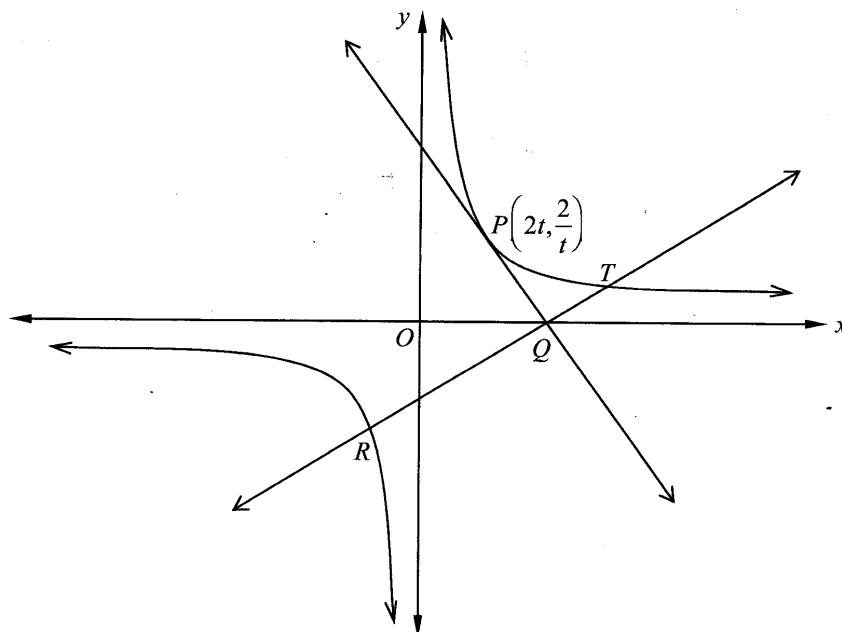


This area is rotated about the line $x = a$. By considering slices perpendicular to the axis of rotation, as shown above, find the volume of the solid of revolution.

End of Question 3

Question 4 (15 marks) Use a SEPARATE writing booklet.

- (a) $P\left(2t, \frac{2}{t}\right)$ is a point on the rectangular hyperbola $xy = 4$. The tangent at P cuts the x -axis at Q and the line through Q , perpendicular to the tangent at P , cuts the hyperbola at the points R and T as shown.



- (i) Show that the equation of the tangent at P is $x + t^2y = 4t$. 2
- (ii) Show that the line through Q , perpendicular to the tangent at P , has equation $t^2x - y = 4t^3$. 3
- (iii) If M is the midpoint of RT , show that M has coordinates $(2t, -2t^3)$. 3
- (iv) Find the equation of the locus of M , as P moves on the curve $xy = 4$. 1

Question 4 continues on page 7

Question 4 (continued)

Marks

- (b) (i) State why the sum of n terms of $1+x+x^2+x^3+\dots+x^{n-1}$ is equal to $\frac{1-x^n}{1-x}$.

1

- (ii) Show that, for all $n > 1$,

2

$$1+2x+3x^2+\dots+(n-1)x^{n-2} = \frac{(n-1)x^n - nx^{n-1} + 1}{(1-x)^2}.$$

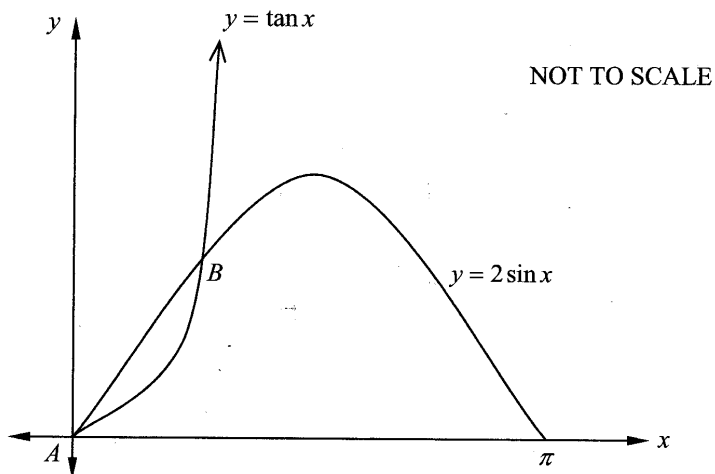
- (iii) Hence find an expression for $1+1+\frac{3}{4}+\frac{4}{8}+\frac{5}{16}+\dots+\frac{n-1}{2^{n-2}}$ and show that this sum is always less than 4.

3

End of Question 4

Question 5 (15 marks) Use a SEPARATE writing booklet.

- (a) The graphs of $y = 2 \sin x$ and $y = \tan x$ are shown below.



The base of a solid is the region bounded by the curves $y = 2 \sin x$ and $y = \tan x$ in the first quadrant between $A(0, 0)$ and $B\left(\frac{\pi}{3}, \sqrt{3}\right)$.

The solid is formed such that every cross-sectional slice perpendicular to the x -axis is a semicircle having a diameter with one end-point on the curve $y = 2 \sin x$ and the other on $y = \tan x$.

- (i) Show that the area of the cross-section of the solid is given by 2

$$A(x) = \frac{\pi}{8} (4 \sin^2 x - 4 \sin x \tan x + \tan^2 x).$$

- (ii) Hence find the volume of the solid formed. 4

Question 5 continues on page 9

Question 5 (continued)

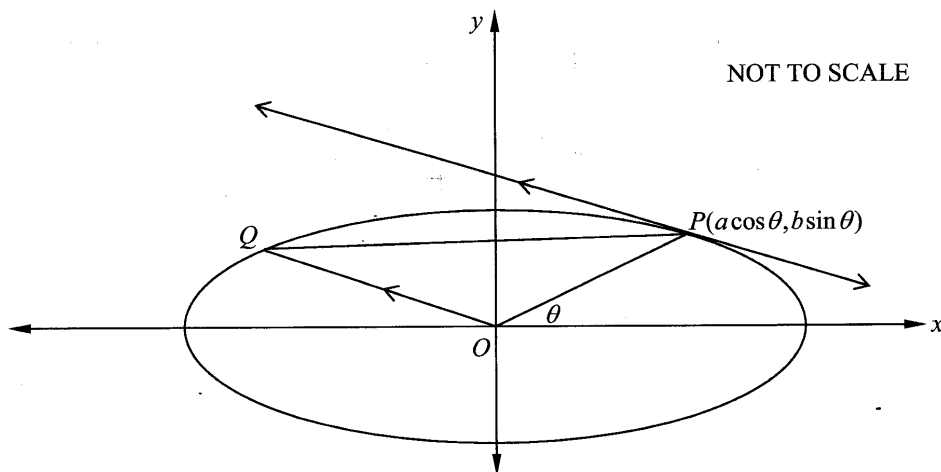
Marks

- (b) (i) Use De Moivre's theorem to show that $\cos 3\theta = \cos^3 \theta - 3\cos \theta \sin^2 \theta$ and $\sin 3\theta = 3\cos^2 \theta \sin \theta - \sin^3 \theta$. **2**
- (ii) Deduce that $\tan 3\theta = \frac{\tan^3 \theta - 3 \tan \theta}{3 \tan^2 \theta - 1}$. **2**
- (iii) Hence, or otherwise, show that $\tan \frac{\pi}{12}$ is a root of the equation $x^3 - 3x^2 - 3x + 1 = 0$. **3**
- (iv) Show that $\tan \frac{\pi}{12} + \tan \frac{5\pi}{12} = 4$. **2**

End of Question 5

Question 6 (15 marks) Use a SEPARATE writing booklet.

- (a) $P(a\cos\theta, b\sin\theta)$ is a point on the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, $a \geq b$ where a and b are positive. A line through the origin and parallel to the tangent at P meets the ellipse at Q .



- (i) Show that the gradient of the tangent at P is $\frac{-b}{a} \cot\theta$. 1
- (ii) Show that the coordinates of Q are $(-a\sin\theta, b\cos\theta)$. 2
- (iii) Find the length of PQ and state the maximum and minimum values of this length for $-\pi \leq \theta \leq \pi$. 3

Question 6 continues on page 11

Question 6 (continued)

Marks

- (b) (i) Given that $x > 0$ and $y > 0$ show that $x + y \geq 2\sqrt{xy}$. 1
- (ii) Hence show that for $x > 0$, $y > 0$, $z > 0$ and $w > 0$. 2
 $x + y + z + w \geq 4\sqrt[4]{xyzw}$.
- (iii) Consider x, y, z and $w = \frac{x+y+z}{3}$. Apply the result in (ii) to show 2
that $\frac{x+y+z}{3} \geq \sqrt[3]{xyz}$.
- (c) Consider $I = \int_{-1}^1 \frac{x^2 e^x}{e^x + 1} dx$ and $J = \int_{-1}^1 \frac{x^2}{e^x + 1} dx$.
- (i) Use the substitution $u = -x$ in I to show that $I = J$. 3
- (ii) Hence evaluate I . 1

End of Question 6

Question 7 (15 marks) Use a SEPARATE writing booklet.

(a) (i) Show that $2 \cos A \sin B = \sin(A + B) - \sin(A - B)$.

1

(ii) Show that $\frac{\sin\left(\frac{3x}{2}\right) - \sin\left(\frac{x}{2}\right)}{2 \sin\left(\frac{x}{2}\right)} = \cos x$.

2

(iii) Prove by induction that for all positive integers n ,

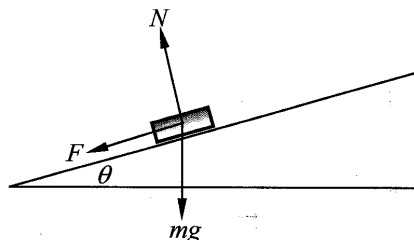
4

$$\cos x + \cos 2x + \cos 3x + \dots + \cos nx = \frac{\sin\left(\frac{2n+1}{2}x\right)}{2 \sin\left(\frac{x}{2}\right)} - \frac{1}{2}$$

Question 7 continues on page 13

Question 7 (continued)

- (b) A car of mass m is travelling around a circular banked track and is subject to a reaction force, N , perpendicular to the track and a frictional force, F , parallel to the track as indicated in the diagram.



The track has a radius of 300 metres and is banked at an angle of $\theta = 5^\circ$ to the horizontal.

- (i) Use a diagram to show that 2

$$mg = N \cos \theta - F \sin \theta$$
 and
$$\frac{mv^2}{r} = F \cos \theta + N \sin \theta.$$
- (ii) By squaring the equations above find an expression for $N^2 + F^2$. 2
- (iii) If the car is travelling at a speed of 72 km/h and the ratio of N to F is 6:1, show that in this situation
$$N = \frac{6m}{\sqrt{37}} \sqrt{g^2 + \frac{160\,000}{r^2}}.$$
 2
- (iv) A driver wants to reduce the frictional force acting on the car to zero. 2
 Find the speed at which the car must travel around the track in this situation. (You may assume that $g = 10 \text{ m s}^{-2}$.)

End of Question 7

Question 8 (15 marks) Use a SEPARATE writing booklet.

- (a) The only force acting on a particle moving horizontally in a straight line is a resistance of $mk(c + v)$ acting in that line, where m is the mass of the particle, v is the velocity and k and c are positive constants. Initially the particle moves with positive velocity U and comes to rest at time T .

At time $\frac{1}{2}T$ its velocity is $\frac{1}{4}U$.

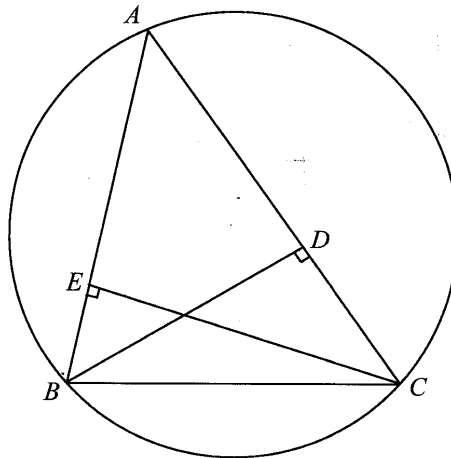
- (i) Show that the acceleration is given by $a = -k(c + v)$. 1
- (ii) Show that $c = \frac{1}{8}U$. 4
- (iii) Show that at time t , $\frac{8v}{U} = 9e^{-kt} - 1$. 2

Question 8 continues on page 15

Question 8 (continued)

Marks

- (b) In the diagram, BC is a fixed chord of a circle and A is a variable point on the major arc of the chord BC . The point D lies on AC , between A and C , such that BD is perpendicular to AC . The point E lies on AB , between A and B , such that CE is perpendicular to AB .



NOT TO SCALE

Copy or trace this diagram into your writing booklet.

- (i) Show that $BCDE$ is a cyclic quadrilateral on a circle with BC as diameter. 2
- (ii) Show that as A varies ED has a constant length. 3
- (iii) Show that the locus of the midpoint of ED is part of a circle whose centre is the midpoint of BC . 3

End of paper