

EDMUND RICE COLLEGE

Extension 2 Mathematics Trial HSC Examination 2001

QUESTION 1

- a) (i) Without evaluating the integral, explain why $\int_{-1}^1 xe^{-x^2} dx = 0$.
- (ii) By using a suitable substitution or otherwise evaluate the integral $\int_0^1 xe^{-x^2} dx$.
- b) Use the method of partial fractions to evaluate $\int_0^1 \frac{6}{9-x^2} dx$.
- c) Show that $\int_2^4 \frac{dx}{x\sqrt{x-1}} = \frac{\pi}{6}$.
- d) Use the substitution $t = \tan \frac{x}{2}$ to evaluate the integral $\int_0^{\frac{\pi}{2}} \frac{dx}{1+\sin x+\cos x}$
- e) Using integration by parts show that $\int e^{ax} \cos(bx) dx = \frac{e^{ax}}{a^2+b^2} [a \cos(bx) + b \sin(bx)] + C$.
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QUESTION 2

- a) Given the complex number $\omega = 1 - \sqrt{3}i$:
- (i) Find the exact value of $|\omega|$ and $\arg \omega$.
- (ii) Find the value of ω^5 in the form $a + ib$.
- (iii) Find the two square roots of ω expressing each in the form $a + ib$ where a, b are both real.
- b) (i) Find and describe the cartesian equation of the locus represented by $|z|^2 = z + \bar{z}$.
- (ii) Sketch this locus on an argand diagram.
- c) (i) Mark clearly on an Argand diagram the region satisfied simultaneously by $|z+2| < 2$ and $0 < \arg z < \frac{3\pi}{4}$.
- (ii) Solve simultaneously $|z+2| = 2$ and $\arg z = \frac{3\pi}{4}$ writing your answer in the form $a + ib$.
- d) The point A represents the real number $z_1 = 1$. The point B represents the complex number $z_2 = (1 + \sqrt{3}) + i$. If $ABCD$ is a square in anti-clockwise rotational order, find:
- (i) The complex number z_3 represented by the point C .
- (ii) The complex number z_4 represented by the point D .
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QUESTION 3

a) If $f(x)$ is an odd function and $g(x)$ is an even function, determine whether the following are odd, even or neither.

(i) $f[g(x)]$

(ii) $g[f(x)]$

b) (i) Show that $\frac{1}{x-2} - \frac{4}{x+3} + 3 = \frac{3x^2-7}{x^2+x-6}$

(ii) Find the vertical and horizontal asymptotes of $f(x) = \frac{3x^2-7}{x^2+x-6}$

(iii) Find the turning points and determine their nature.

(iv) Sketch the curve showing all important points. You may omit the point of inflexion.

c) Find $\int \sin^2 x \cos^2 x \, dx$.

QUESTION 4

a) Given that the quartic polynomial $P(x) = x^4 - 5x^3 - 9x^2 + 81x - 108$ has a zero of multiplicity three (i.e., a treble root), then completely factorise the polynomial and find all the zeros.

b) A solid has a base in the shape of an ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$. If every cross-section perpendicular to the base is a semi-circle, with its diameter at right-angles to the major axis of the ellipse, find the volume of the solid by *slicing*.

c) A cubic polynomial is given by $P(x) = x^3 + ax + b$ where a, b are constants. It is given that the polynomial equation $P(x) = 0$ has three roots α, β and γ .

(i) Find the value of $\alpha + \beta + \delta$

(ii) Show that $\alpha^2 + \beta^2 + \delta^2 = -2a$

(iii) If the polynomial has a positive double zero, show that this double zero is $\frac{-3b}{2a}$

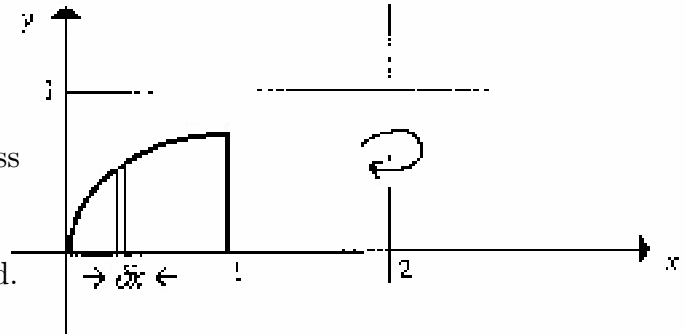
(iv) If the polynomial has three distinct zeros show that $4a^3 + 27b^2 < 0$

QUESTION 5

- a) (i) Find (as trigonometrical quantities) the five fifth roots of unity, and represent them on the Argand diagram.
- (ii) Show that the points representing the five roots on this Argand diagram represent the vertices of a pentagon. Find the area of this pentagon.
- (iii) If α is one of the complex roots, show that the other complex roots can be expressed as $\alpha^2, \alpha^{-1}, \alpha^{-2}$ and find the value of $\alpha^2 + \alpha^{-2}$.
- (iv) Factorise $x^5 - 1$ as the product of a linear and two real quadratic factors.
- b) The region shown bounded by the portion of the curve $y = \frac{x}{x+1}$ the x -axis and the line $x = 1$ is rotated about the line $x = 2$.

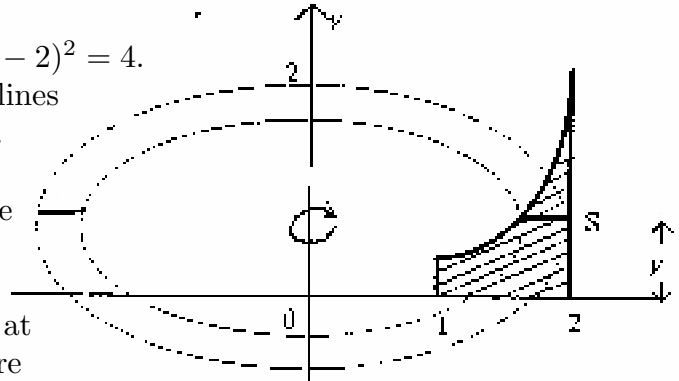
- (i) Using the method of cylindrical shells, show that the volume δV of a typical shell at a distance x from the origin and with thickness δx is given by
- $$\delta V = 2\pi(2-x)\frac{x}{1+x}\delta x$$

- (ii) Hence find the volume of the solid.

**QUESTION 6**

- a) The curve is part of the circle $x^2 + (y - 2)^2 = 4$.

The shaded region is bounded by the lines $x = 1, x = 2$, the curve and the x axis. This region is to be rotated about the y axis. When the region is rotated, the line segment at S , (height y), sweeps out an annulus.



- (i) Show that the area of the annulus at height y is given by $\pi(y - 2)^2$ where $2 - \sqrt{3} \leq y \leq 2$.
- (ii) Hence find the exact volume of the solid when the entire shaded region is rotated about the y axis (the cylindrical pipe portion of the solid has $V = \pi(6 - 3\sqrt{3})$).
- b) (i) Given the equation of the hyperbola $xy = c^2$, show that the equation of the tangent at $T(ct, \frac{c}{t})$ is given by $x + t^2y - 2ct = 0$.
- (ii) P, Q and R are three points on one branch of this hyperbola, with parameters p, q and r respectively. The tangents at P and Q intersect at U . Find the co-ordinates of U . Hence determine the equation of the line OU , where O is the origin.
- (iii) If O, U and R are collinear, show that $r^2 = pq$.
- c) The equation $x^3 + px^2 + r = 0$ has roots α, β and γ . Find the equation with roots $\frac{1}{\alpha^2}, \frac{1}{\beta^2}, \frac{1}{\gamma^2}$.

QUESTION 7

- a) (i) Show that the point $P(a \sec \theta, b \tan \theta)$ lies on the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ with asymptotes $y = \pm \frac{b}{a}x$.
- (ii) Find the equation of the tangent to the hyperbola at P .
- (iii) Show that the equation of the normal to the hyperbola at P is $\frac{ax}{\sec \theta} + \frac{by}{\tan \theta} = a^2 + b^2$.
- (iv) The tangent meets the asymptote $y = \frac{b}{a}x$ at A . The normal meets the x axis at B . The line $x = a \sec \theta$ meets the asymptote at C . Find the coordinates of B and C and prove that $\angle ACB$ is a right-angle.
- (v) Give a geometrical description of the quadrilateral $ACPB$.
- b) $P(a \cos \theta, b \sin \theta)$ is a point on the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$. A line through the origin and parallel to the tangent at P , meets the ellipse at R .
- (i) Draw a diagram and mark the given information.
- (ii) Prove that the area of triangle OPR is equal to $\frac{1}{2}ab$ for all positions of P .

QUESTION 8

- a) Consider the function $y = x^x$ for $x > 0$.
Show that the derivative is given by $(\log_e x + 1)x^x$.
- b) (i) Show that the sum of n terms of $1 + x + x^2 + x^3 + \dots = \frac{1-x^n}{1-x}$ where $x < 1$.
- (ii) Show that $1 + 2x + 3x^2 + \dots + (n-1)x^{n-2} = \frac{(n-1)x^n - nx^{n-1} + 1}{(1-x)^2}$.
- (iii) Hence find an expression for $1 + 1 + \frac{3}{4} + \frac{4}{8} + \frac{5}{16} + \dots + \frac{n-1}{2^{n-2}}$ and show that this sum is always less than 4.
- c) (i) Prove that $\int_0^a f(x) dx = \int_0^a f(a-x) dx$ (let $u = a-x$).
- (ii) Consider $f(x) = \frac{1}{1+\tan x}$ where $0 \leq x \leq \frac{\pi}{2}$ and $f(\frac{\pi}{2}) = 0$.
Show that $f(x) = f(\frac{\pi}{2} - x) = 1$ (note: $\tan(\frac{\pi}{2} - x) = \cot x$)
- (iii) Hence evaluate $\int_0^{\frac{\pi}{2}} \frac{1}{1+\tan x} dx$.