EPPING BOYS' HIGH SCHOOL

Year 12 Trial Examination 2009

MATHEMATICS EXTENSION 2

Student 's Name ....   Monday 17th August, 2009

Teacher: ....

General Instructions

- Reading time – 5 minutes.
- Working time – 3 hours.
- Board-approved calculators may be used.
- All necessary working should be shown in every question.
- Full marks may not be awarded for careless or badly arranged work.
- Start each question on a new booklet.
- Attempt all questions.

Q. 1	Q. 2	Q. 3	Q. 4	Q. 5	Q. 6	Q. 7	Q.8	Total	%
/ 15	/15	/15	/15	/15	/15	/15	/15	/120	/100

Start each question on a SEPARATE **page or in a** SEPARATE **booklet.**

Question 1 (15 marks) Use a separate page/booklet **Marks**

(a) Find $\int \frac{\sin 2x}{3+\sin^2 x}dx$ **2**

(b) Find $\int \frac{1}{e^x + e^{-x}}dx$. **2**

(c) Evaluate $\int_{\sqrt{3}}^{3} \frac{1}{\sqrt{(x^2-1)}}dx$. **3**

(d) Evaluate $\int_{0}^{\pi/3} \frac{1}{1-\sin x}dx$. **4**

(e) Evaluate $\int_{0}^{4} \frac{x^2+4x+5}{(x+1)(x+3)}dx$. **4**

Question 2 (15 marks) Use a separate page/booklet **Marks**

(a) Find real x and y such that $(x+iy)^2 = 3+4i$. 2

(b) Find $|z|$ and $\arg z$ when $z = -\sqrt{3} - i$. 2

(c) (i) Express $1+i$ and $1-i$ in modulus/argument form. 2

(ii) Hence evaluate $(1+i)^{40} + (1-i)^{40}$. 2

(d) (i) Solve $z^5 = -1$. 2

(ii) Hence show that $z^5 + 1 = (z+1)\left(z^2 - 2z\cos\frac{\pi}{5} + 1\right)\left(z^2 - 2z\cos\frac{3\pi}{5} + 1\right)$. 3

(e) Indicate on an Argand diagram the region which contains the point P representing z when $|z| \le |z-4|$ and $-\dfrac{\pi}{4} \le \arg z \le \dfrac{\pi}{4}$. 2

Question 3 (15 marks) Use a separate page/booklet **Marks**

(a) Sketch (showing critical points) the graph of: $y = 3(x + \sqrt{x})$ **2**

(b) If $f(x) = x^2 - 16$, sketch the following graphs on separate axes showing all relevant points.

(i) $y = f(x)$ **1**

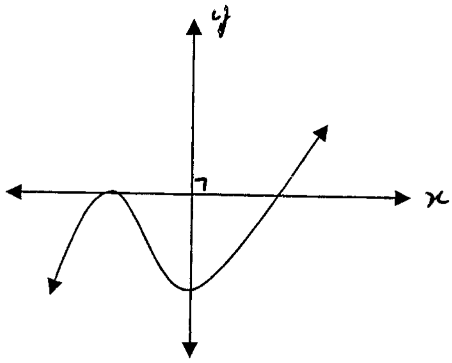
(ii) $y = \dfrac{1}{f(x)}$ **2**

(iii) $y = -\dfrac{16}{f(x)}$ **2**

(iv) $y = |f(x)| + 6$ **2**

(c) Sketch the graph $y^2 = x^2(1 - x^2)$ **2**

(d) The graph of $y = x^3 + 3x^2 - 4$ is sketched below



Sketch the curves $y = |x^3 + 3x^2 - 4|$ and $y = \ln|x^3 + 3x^2 - 4|$ on separate axes. **2**

(e) Show that $|z_1 + z_2|^2 + |z_1 - z_2|^2 = 2\left(|z_1|^2 + |z_2|^2\right)$ for any complex numbers z_1 and z_2. **2**

Question 4 (15 marks) Use a separate page/booklet **Marks**

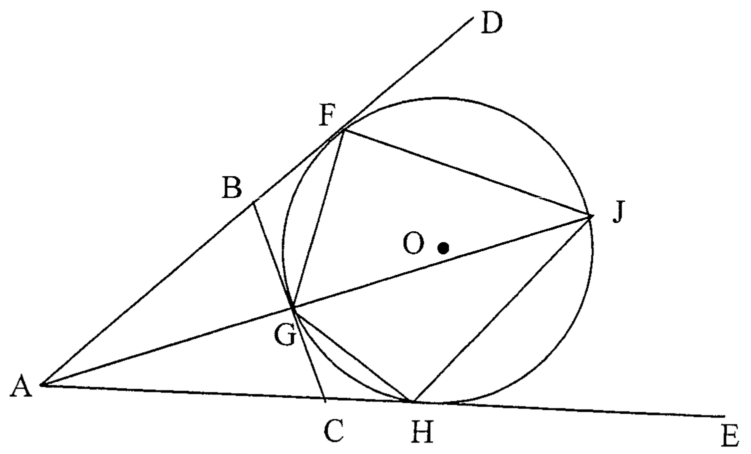
(a) A particle A_1 of mass m kg is dropped from point C and falls towards point D, which is directly underneath C. At the instant when A_1 is dropped, a second particle A_2, also of mass m kg, is projected upwards from D towards C with an initial velocity equal to twice the terminal velocity of A_1. Each particle experiences a resistance of magnitude mkv as it moves, where $v\, ms^{-1}$ is the velocity and k is a constant.

(i) Show that the terminal velocity of A_1 is $\frac{g}{k}$, where g is acceleration due to gravity. **2**

(ii) For particle A_2, show that $t = \frac{1}{k}\ln\left(\frac{3g}{g+kv}\right)$, where $v\, ms^{-1}$ is the velocity after t seconds. **3**

(iii) Suppose the particles collide at the instant when A_1 has reached 30% of its terminal velocity. Find the velocity of A_2 when they collide. Leave your answer in terms of g and k. **3**

(b) In the following diagram, the lines AH, AF and BC are all tangents to the circle.



(i) Prove that the triangles AFG and AJF are similar. **1**

(ii) Prove $(\mathrm{AF})^2 = (\mathrm{AJ})\times(\mathrm{AG})$ **1**

(iii) Prove $\mathrm{GJ} = \frac{(\mathrm{AF})\times(\mathrm{AH})}{(\mathrm{AG})} - (\mathrm{AG})$ **2**

Question 4 (continued)

(c) (i) Find the values of the constants A and B such that **1**

$4x^4 + 1 \equiv (2x^2 + Ax + 1)(2x^2 + Bx + 1)$.

(ii) Hence find the prime factors of the integer $2^{14} + 1$. **2**

Question 5 (15 marks) Use a separate page/booklet **Marks**

(a) (i) If $I_n = \int_1^e (\ln x)^n dx$ for $n \geq 0$, show that $I_n = e - nI_{n-1}$ for $n \geq 1$. **3**

(ii) Hence evaluate I_4. **2**

(b) (i) Use the substitution $u = \pi - x$ to show that

$\int_0^\pi \frac{x \sin x}{1 + \cos^2 x} dx = \int_0^\pi \frac{(\pi - x)\sin x}{1 + \cos^2 x} dx$. **1**

(ii) Deduce that $\int_0^\pi \frac{x \sin x}{1 + \cos^2 x} dx = \frac{\pi^2}{4}$ **2**

(c) The cubic equation $x^3 - x^2 + 4x - 2 = 0$ has roots α, β and γ.

(i) Find the equation whose roots are α^2, β^2 and γ^2. **2**

(ii) Find the value of $\alpha^2\beta^2 + \alpha^2\gamma^2 + \beta^2\gamma^2$. **3**

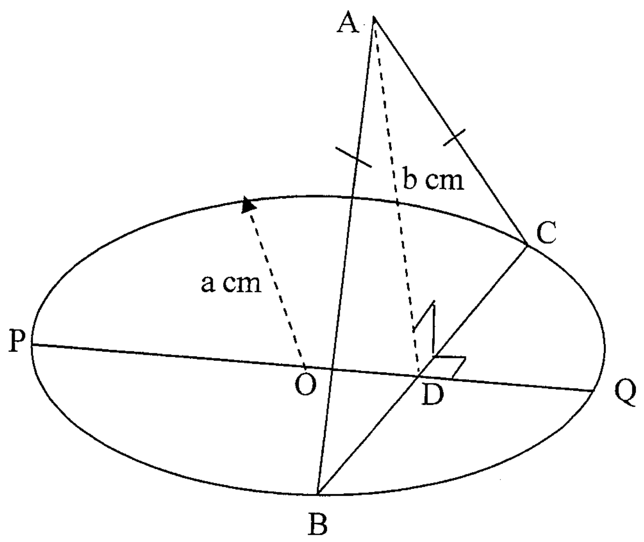
(d) Given that $\sin^{-1} x$, $\cos^{-1} x$ and $\sin^{-1}(1 - x)$ are acute, show that:

$\sin(\sin^{-1} x - \cos^{-1} x) = 2x^2 - 1$. **2**

Question 6 (15 marks) Use a separate page/booklet **Marks**

(a) The ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ is rotated about the line $y=c$ $(c>b)$. Find the volume of the solid generated by using the method of cylindrical shells. **4**

(b) The base of the solid shown in the diagram is a circle of radius a cm.



Each cross section of the solid formed by a plane perpendicular to the fixed diameter PQ is an isosceles triangle of height b cm. One such cros section is shown by ΔABC in the above diagram, where AB=AC, BC is in the base of the solid and AD= b cm. Find the volume of this solid by first drawing a diagram indicating your origin and other relevant information. **3**

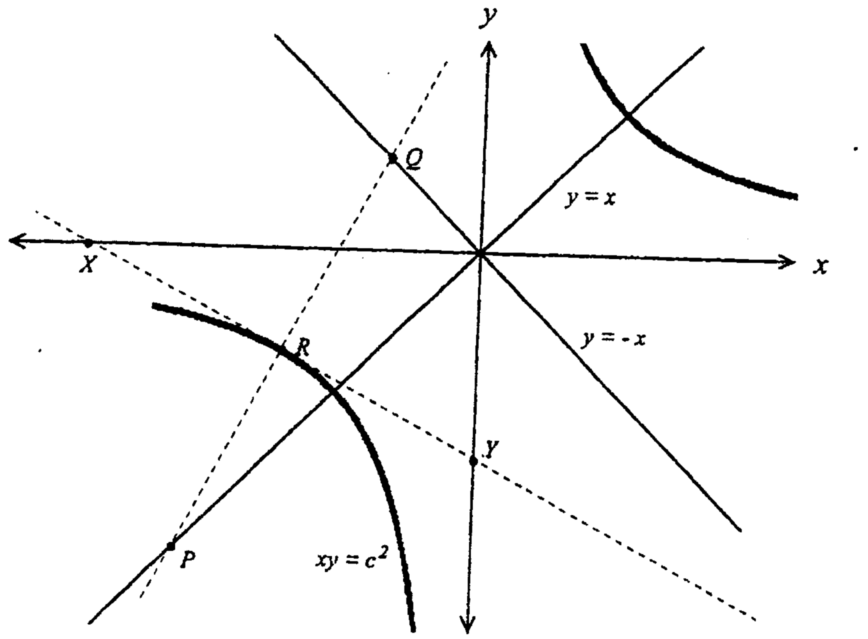
(c) (i) Write down the relations which hold between the roots α, β and χ of the equation $ax^3 + bx^2 + cx + d = 0$, $(a \neq 0)$ and the coefficients a,b,c and d. **1**

(ii) Consider the equation $36x^3 - 12x^2 - 11x + 2 = 0$. You are given the roots α, β, χ of this equation satisfy $\alpha = \beta + \chi$. Use part (i) to find α. **1**

(iii) Suppose the equation $x^3 + px^2 + qx + r = 0$ has roots λ, μ and ν which satisfy $\lambda = \mu + \nu$. Show that $p^3 - 4pq + 8r = 0$. **3**

(d) If $P(x)=2x^4+9x^3+6x^2-20x-24$ has a triple root, factorise $P(x)$ into its linear factors. **3**

Question 7 (15 marks) Use a separate page/booklet **Marks**



(a) In the diagram above $R\left(ct, \dfrac{c}{t}\right)$ is a point on the rectangular hyperbola $xy = c^2$.

The tangent to the hyperbola at R meets the x-axis at X and the y-axis at Y.

The normal to the hyperbola at R meets the line $y = x$ at P and the line $y = -x$ at Q.

You are given that the equation of the tangent at R is $x + t^2 y = 2ct$.

(i) Prove that the equation of the normal at R is $ty + ct^4 = t^3 x + c$. **2**

(ii) It can be shown that P is the point $\left(\dfrac{c(t^2+1)}{t}, \dfrac{c(t^2+1)}{t}\right)$. Find the coordinates of X and Y.

Prove that Q is the point $\left(\dfrac{c\left(t^2-1\right)}{t}, \dfrac{c(1-t^2)}{t}\right)$. **3**

(iii) Show that PQ and XY bisect each other. **2**

(iv) Show that $PQ = XY$. **2**

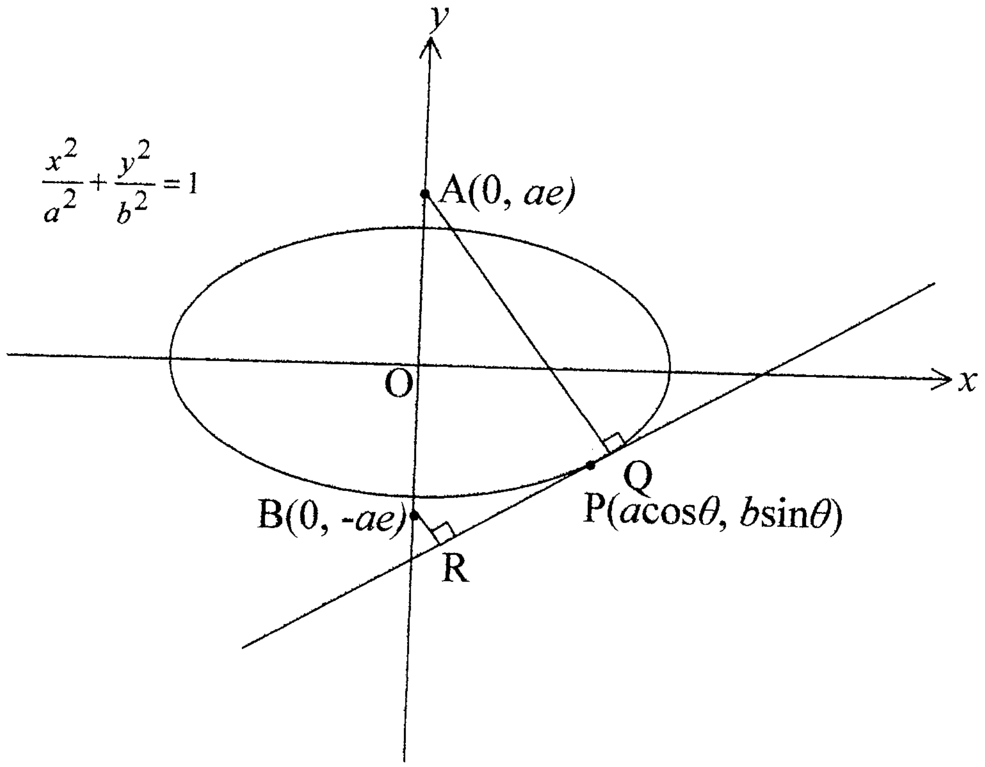
(v) What type of quadrilateral is $XQYP$? Justify your answer. **1**

Question 7 (continued) **Marks**

(b) A sequence of numbers u_n is such that $u_1 = 3$, $u_2 = 21$ and $u_n = 7u_{n-1} - 16u_{n-2}$ for $n \geq 3$. Use the method of mathematical induction to show that $u_n = 5^n - 2^n$ for $n \geq 1$. **5**

Question 8 (15 marks) Use a separate page/booklet

(a)



P($a\cos\theta$, $b\sin\theta$) is a point on the ellipse $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}=1$, $a > b > 0$, where e is the eccentricity of the ellipse.

From A(0, ae) and B(0, $-ae$) perpendiculars are drawn to meet the tangent at P($a\cos\theta$, $b\sin\theta$) at Q and R, respectively.

(i) Prove that the equation of the tangent at P is
$$\frac{\cos\theta}{a}x+\frac{\sin\theta}{b}y=1$$ **3**

(ii) Hence, or otherwise, show that the line $x\cos\alpha + y\sin\alpha = k$ is a tangent to the ellipse if $a^2\cos^2\alpha + b^2\sin^2\alpha = k^2$ **2**

(iii) Hence, or otherwise, prove that $AQ^2 + BR^2 = 2a^2$ **4**

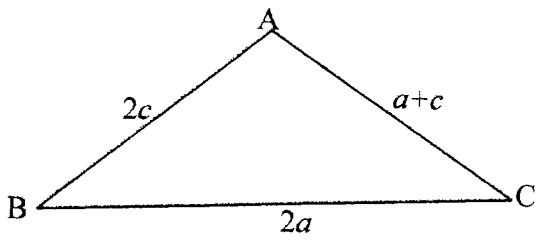
Question 8 (continued)

Marks

(b) (i) Given that $\sin(X+Y)+\sin(X-Y)=2\sin X\cos Y$, show that

$$\sin A+\sin C=2\sin\frac{A+C}{2}\cos\frac{A-C}{2}$$ **1**

(ii) Consider ΔABC where



(α) Use the sine rule to show that $\sin A+\sin C=2\sin B$ **2**

(β) Deduce that $\sin\frac{B}{2}=\frac{1}{2}\cos\frac{A-C}{2}$ **3**

END OF EXAM

STANDARD INTEGRALS

$$\int x^n dx = \frac{1}{n+1} x^{n+1}, \quad n \neq -1; \quad x \neq 0, \quad \text{if } n < 0$$

$$\int \frac{1}{x} dx = \ln x, \quad x > 0$$

$$\int e^{ax} dx = \frac{1}{a} e^{ax}, \quad a \neq 0$$

$$\int \cos ax\, dx = \frac{1}{a} \sin ax, \quad a \neq 0$$

$$\int \sin ax\, dx = -\frac{1}{a} \cos ax, \quad a \neq 0$$

$$\int \sec^2 ax\, dx = \frac{1}{a} \tan ax, \quad a \neq 0$$

$$\int \sec ax \tan ax\, dx = \frac{1}{a} \sec ax, \quad a \neq 0$$

$$\int \frac{1}{a^2 + x^2} dx = \frac{1}{a} \tan^{-1} \frac{x}{a}, \quad a \neq 0$$

$$\int \frac{1}{\sqrt{a^2 - x^2}} dx = \sin^{-1} \frac{x}{a}, \quad a > 0, \quad -a < x < a$$

$$\int \frac{1}{\sqrt{x^2 - a^2}} dx = \ln\left(x + \sqrt{x^2 - a^2} \right), \quad x > a > 0$$

$$\int \frac{1}{\sqrt{x^2 + a^2}} dx = \ln\left(x + \sqrt{x^2 + a^2} \right)$$

NOTE: $\ln x = \log_e x, \quad x > 0$

Question 1

(a) Now $\int \frac{f'(x)}{f(x)}dx = \ln f(x) + c$ with $f(x) = \sin^2 x + 3$ 1

$$\int \frac{\sin 2x}{3+\sin^2 x}dx = \int \frac{2\sin x \cos x}{3+\sin^2 x}dx = \ln(\sin^2 x + 3) + c$$ 1

(b) Using the substitution $e^x = u, du = e^x dx$

$$\int \frac{1}{e^x + e^{-x}}dx = \int \frac{e^x}{e^{2x}+1}dx$$ 1

$$= \int \frac{1}{u^2+1}du = \tan^{-1} u + c = \tan^{-1} e^x + c$$ 1

(c) Using $\int \frac{1}{\sqrt{(x^2-a^2)}}dx = \ln\left|x + \sqrt{(x^2-a^2)}\right| + c$ with $a = 1$, ✓ 3

$$\int_{\sqrt{3}}^{3} \frac{1}{\sqrt{(x^2-1)}}dx = \ln\left|x+\sqrt{x^2-1}\right|\Bigg\|_{\sqrt{3}}^{3} = \ln(3+\sqrt{8}) - \ln(\sqrt{3}+\sqrt{2}) = \ln\left(\frac{3+2\sqrt{2}}{\sqrt{3}+\sqrt{2}}\right).$$ ✓ ✓

(d) Let $t = \tan\frac{x}{2}, 0 < x < \frac{\pi}{3}, 0 < t < \frac{1}{\sqrt{3}}, x = 2\tan^{-1} t, dx = \frac{2}{1+t^2}dt$. 1

Now $\sin x = \frac{2\tan(x/2)}{1+\tan^2(x/2)}$, 1

$$\int_0^{\pi/3} \frac{1}{1-\sin x}dx = \int_0^{\pi/3} \frac{1}{1-\frac{2\tan(x/2)}{1+\tan^2(x/2)}}dx = \int_0^{1/\sqrt{3}} \frac{1}{1-\frac{2t}{1+t^2}} \times \frac{2}{1+t^2}dt = \int_0^{1/\sqrt{3}} \frac{1+t^2}{1+t^2-2t} \times \frac{2}{1+t^2}dt$$ 1

$$= 2\int_0^{1/\sqrt{3}} \frac{1}{(1-t)^2}dt = \left[\frac{2}{1-t}\right]_0^{1/\sqrt{3}} = \frac{2}{1-1/\sqrt{3}} - 2 = \frac{2\sqrt{3}}{\sqrt{3}-1} - 2 = \frac{2}{\sqrt{3}-1}$$ 1

$$= \sqrt{3} + 1$$

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Trial Exam Solution

Question 1 (cont.)

(e)

$$\frac{x^2+4x+5}{(x+1)(x+3)} = 1 + \frac{2}{(x+1)(x+3)}.$$

$$\frac{2}{(x+1)(x+3)} = \frac{a}{(x+1)} + \frac{b}{(x+3)}$$

$2 = a(x+3) + b(x+1)$. ✓

Let $x=-1 :\to 2 = 2a \Rightarrow a = 1$.

Let $x=-3 :\to 2 = -2b \Rightarrow b = -1$. ✓

$$\int_0^4 \frac{x^2+4x+5}{(x+1)(x+3)}dx$$

$$= \int_0^4 1dx + \int_0^4 \frac{2}{(x+1)(x+3)}dx$$

$$= [x]_0^4 + \int_0^4 \frac{1}{x+1}dx - \int_0^4 \frac{1}{x+3}dx$$ ✓

$$= 4 + \left[\ln|x+1|\right]_0^4 - \left[\ln|x+3|\right]_0^4$$

$$= 4 + \ln 5 - \ln 1 - (\ln 7 - \ln 3).$$

$$= 4 + \ln\frac{15}{7}$$ ✓

4

Question 2

(a)

$(x+iy)^2 = 3+4i$

$\left(x^2-y^2\right)+(2xy)i = 3+4i$

Equating real parts:

$x^2-y^2=3$

Equating imaginary parts

$2xy=4$

$\therefore x^4-x^2y^2=3x^2$ and $x^2y^2=4$

$\therefore x^4-3x^2-4=0$

$\left(x^2-4\right)\left(x^2+1\right)=0$ ✓ 1

$\therefore x=\pm 2,\ y=\pm 1$ ✓ 1

(b)

$z=-\sqrt{3}-i=2\left(-\frac{\sqrt{3}}{2}-\frac{1}{2}i\right)$ 2

$=2\left(\cos\left(-\frac{5\pi}{6}\right)+i\sin\left(-\frac{5\pi}{6}\right)\right).$

$|z|=2, \arg z=-\frac{5\pi}{6}$ ✓
✓

(c) i) 2

$z=1+i.\ z=\sqrt{2}\left(\frac{1}{\sqrt{2}}+i\frac{1}{\sqrt{2}}\right)=\sqrt{2}\left(\cos\frac{\pi}{4}+i\sin\frac{\pi}{4}\right)$ ✓

$1-i=\bar{z}=\sqrt{2}\left(\cos\frac{\pi}{4}-i\sin\frac{\pi}{4}\right).$ ✓

ii) Using *de* Moivre's theorem $z^{40}=2^{20}\operatorname{cis}(10\pi)$,

$(\bar{z})^{40}=2^{20}\operatorname{cis}(-10\pi)$.

$z^{40}+(\bar{z})^{40}=z^{40}+\overline{\left(z^{40}\right)}=2\operatorname{Re}\left(z^{40}\right)=2^{21}\cos(10\pi).$ ✓ 1

$\therefore (1+i)^{40}+(1-i)^{40}=2^{21}\cos(10\pi).$

$=2^{21}$ ✓ 1

Question 2 (cont.)

(d) i) (i) $|-1|=1$ and $\arg(-1)=\pi$.

Therefore the complex 5th roots of -1 are $\cos\frac{\pi}{5}\pm i\sin\frac{\pi}{5}$, $\cos\frac{3\pi}{5}\pm i\sin\frac{3\pi}{5}$, and -1. Note the roots are equally spaced around a unit circle in an Argand diagram by an angle of $\frac{2\pi}{5}$ and modulus of 1. ✓ 2

(ii) $z^5+1=(z+1)\left(z-\text{cis}\frac{\pi}{5}\right)\left(z-\text{cis}\left(-\frac{\pi}{5}\right)\right)\left(z-\text{cis}\frac{3\pi}{5}\right)\left(z-\text{cis}\left(-\frac{3\pi}{5}\right)\right)$. ✓ 3

Note

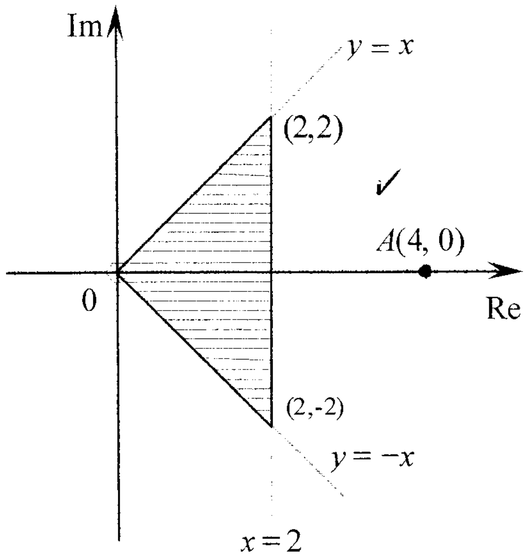
$\left(z-\text{cis}\frac{\pi}{5}\right)\left(z-\text{cis}\left(-\frac{\pi}{5}\right)\right)=\left(\left(z-\cos\frac{\pi}{5}\right)-i\sin\frac{\pi}{5}\right)\left(\left(z-\cos\frac{\pi}{5}\right)+i\sin\frac{\pi}{5}\right)=\left(z-\cos\frac{\pi}{5}\right)^2+\left(\sin\frac{\pi}{5}\right)^2=$

$=z^2-2z\cos\frac{\pi}{5}+1$ ✓

and $\left(z-\text{cis}\frac{3\pi}{5}\right)\left(z-\text{cis}\left(-\frac{3\pi}{5}\right)\right)=z^2-2z\cos\frac{3\pi}{5}+1$.

$\therefore z^5+1=(z+1)\left(z^2-2z\cos\frac{\pi}{5}+1\right)\left(z^2-2z\cos\frac{3\pi}{5}+1\right)$. ✓

(e)



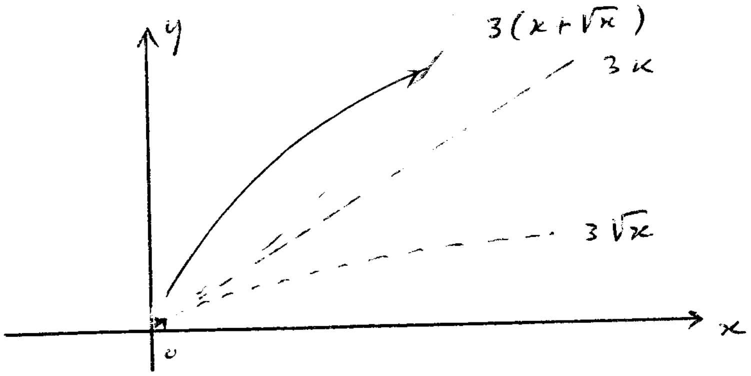
$|z|=|z-4|$ is the perpendicular bisector of OA. 2

$\arg z=\frac{\pi}{4}$ is the ray $y=x,\ x>0$.

$\arg z=-\frac{\pi}{4}$ is the ray $y=-x,\ x>0$. ✓

Question 3

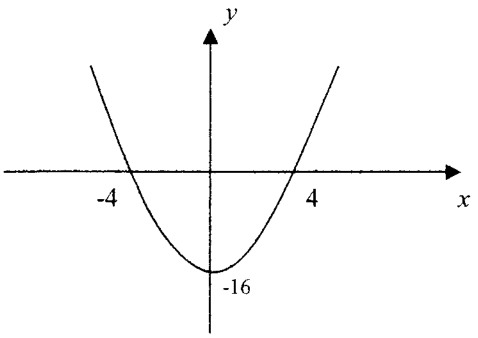
a



$y = 3(x + \sqrt{x})$ 2

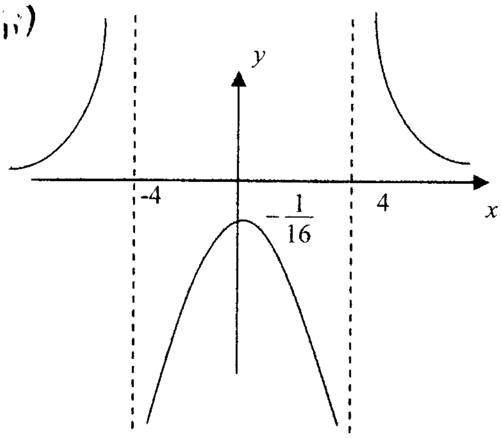
(b)

i)



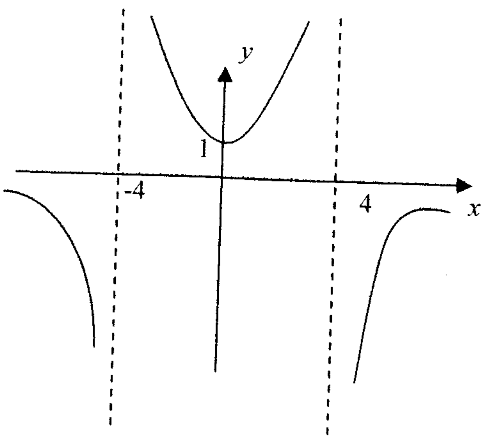
$f(x) = x^2 - 16$ 1

ii)



$y = \frac{1}{f(x)}$ 2

iii)

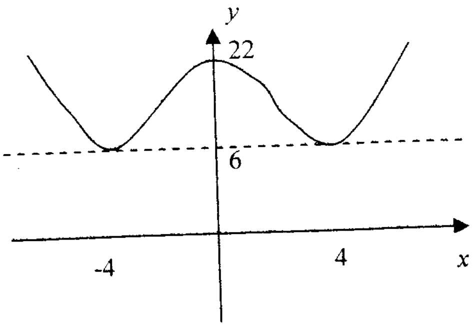


$y = -\frac{16}{f(x)}$ 2

Question 3 (cont.)

(b)

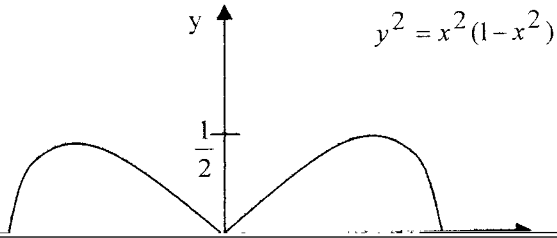
(iv)



$y = |f(x)| + 6$

2

(c)



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Question 4

Question 4 (a)(i)

Criteria
• One for $\ddot{x}=0$ and, one for simplification.

2

Answer: For A_1 : $\ddot{x}=g-kv$. Terminal velocity is achieved when $\ddot{x}=0$ i.e. when $0=g-kv$.

$\therefore$ terminal velocity $=\dfrac{g}{k}$

Question 4 (a) (ii)

Criteria
• One for integration, one for constant, one for simplification.

3

Answer:

For A_2 :

$$\frac{dv}{dt}=-g-kv$$

$$t=\int\frac{-1}{g+kv}dv$$

$$=\frac{-1}{k}\ln(g+kv)+c_1$$

$$When\,t=0, v=\frac{2g}{k}$$

$$\therefore c_1=\frac{1}{k}\ln(3g)$$

$$\therefore t=\frac{1}{k}\ln(\frac{3g}{g+kv})$$

Question 4 (a) (iii)

Criteria
• One for $t=\dfrac{1}{k}\ln\left(\dfrac{g}{g-kv}\right)$, one for $\dfrac{1}{k}\ln\dfrac{10}{7}=\dfrac{1}{k}\ln\left(\dfrac{3g}{g+kv}\right)$, one for simplification.

3

Answer:

For A_1

$$\frac{dv}{dt}=g-kv$$

$$t=\int\frac{1}{g-kv}dv$$

$$=-\frac{1}{k}\ln(g-kv)+c_2$$

$$when\,t=0, v=0\Rightarrow c_2=\frac{1}{k}\ln g$$

$$t=\frac{1}{k}\ln\left(\frac{g}{g-\dfrac{3g}{10}}\right)=\frac{1}{k}\ln\frac{10}{7}$$

$$\frac{1}{k}\ln\frac{10}{7}=\frac{1}{k}\ln\left(\frac{3g}{g+kv}\right)$$

$$t=\frac{1}{k}\ln\left(\frac{g}{g-kv}\right)$$

When the particles collide, A_1 has a velocity

$$\frac{3g}{10k}$$

$$\frac{10}{7}=\frac{3g}{g+kv}$$

$$10g+10kv=21g\Rightarrow 10kv=11g\Rightarrow v=\frac{11g}{10k}$$

At the instant the particle collide, A_2 has a velocity

$$\frac{11g}{10k}ms^{-1}$$

Question 4

(b)

Answer

In Δ's AFG, AJF

$\angle AFG = \angle AJF$ (angle between tangent and chord= angle in alternate segment) 1

$\angle FAG = \angle FAJ$ (common angle)

Third angles are equal

$\therefore \Delta$ AFG is similar to ΔAJF (AAA)

(ii)

$\frac{AF}{AG} = \frac{AJ}{AF}$ (corresponding sides of similar triangles in the same ratio) 1

$\therefore AF^2 = AG \times AJ$

(iii)

Now GJ=AJ-AG also $AF^2 = AG \times AJ \Rightarrow AJ = \frac{AF^2}{AG}$ 2

$\therefore GJ = \frac{AF^2}{AG} - AG$

Now AF=AH (Tangent from a common point are equal)

$\therefore GJ = \frac{AF \times AH}{AG} - AG$

i) $4x^4 + 1 \equiv (2x^2 + Ax + 1)(2x^2 + Bx + 1)$

$\equiv 4x^4 + 2(A+B)x^3 + (4 + AB)x^2 + (A+B)x + 1$

$\therefore \quad A + B = 0$

$\left.\begin{array}{l} \\ AB = -4 \end{array}\right\} \Rightarrow A = 2, \quad B = -2$ ✓ 1

$\therefore 4x^4 + 1 \equiv (2x^2 + 2x + 1)(2x^2 - 2x + 1)$

ii) Let $x = 8 \Rightarrow 4x^4 + 1 = 4 \times 8^4 + 1$ 1

$= 2^2 \times 2^{12} + 1$

$= 2^{14} + 1$

And RHS $= (2 \times 8^2 + 2 \times 8 + 1)(2 \times 8^2 - 8 \times 2 + 1)$

$= 145 \times 113$

$= 5 \times 29 \times 113$ 1

i.e., $2^{14} + 1 = 5 \times 29 \times 113$

Question 5

(a) i) $I_n = \int_1^e (\ln x)^n \, dx$.

Let $u = (\ln x)^n \Rightarrow du = n(\ln x)^{n-1}\frac{dx}{x}$ ✓

$dv = dx \Rightarrow v = x$

∴ $\int u\,dv = uv - \int v\,du$

∴ $I_n = \left[x(\ln x)^n\right]_1^e - n\int_1^e (\ln x)^{n-1}\,dx$ ✓

$I_n = e - nI_{n-1}$ ✓

ii) $I_0 = \int_1^e 1.dx = e - 1$ ✓

∴ $I_4 = e - 4\times I_3 = e - 4(e - 3I_2)$

$= -3e + 12(e - 2I_1) = 9e - 24(e - I_0)$

$= 9e - 24e + 24(e - 1)$

$I_4 = 9e - 24$ ✓

(b) i) $\int_0^\pi \frac{x\sin x\,dx}{1+\cos^2 x} = I$

Let $u = \pi - x \Rightarrow x = \pi - u$

$dx = -du$

∴ $I = \int_\pi^0 \frac{(\pi-u)\sin(\pi-u)\cdot -du}{1+\cos^2(\pi-u)}$

$I = \int_0^\pi \frac{(\pi-u)\sin u\,du}{1+\cos^2 u}$

∴ $\int_0^\pi \frac{x\sin x}{1+\cos^2 x}dx = \int_0^\pi \frac{(\pi-x)\sin x}{1+\cos^2 x}dx$ ✓

ii) Now

$\int_0^\pi \frac{x\sin x}{1+\cos^2 x}dx = \int_0^\pi \frac{(\pi-x)\sin x}{1+\cos^2 x}dx$

∴ $I = \int_0^\pi \frac{\pi\sin x}{1+\cos^2 x}dx - I$

(b) ii) $\Rightarrow 2I = \int_0^\pi \frac{\pi\sin x}{1+\cos^2 x}dx$,

$2I = \pi\left[\tan^{-1} t\right]_{-1}^{1}$ ✓ $= \pi\left[\frac{\pi}{4} - \left(-\frac{\pi}{4}\right)\right]$

∴ $I = \int_0^\pi \frac{x\sin x}{1+\cos^2 x}dx = \frac{\pi^2}{4}$ ✓

(c) i) $x^3 - x^2 + 4x - 2 = 0$ has roots α, β, γ. if $t = \alpha^2$

∴ $\alpha = \sqrt{t}$

∴ $\alpha^3 - \alpha^2 + 4\alpha - 2 = 0$

∴ $t\sqrt{t} + 4\sqrt{t} - t - 2 = 0$ ✓

∴ $\sqrt{t}(t+4) = t+2$

$t(t^2 + 8t + 16) = t^2 + 4t + 4$

∴ $t^3 + 7t^2 + 12t - 4 = 0$

∴ $x^3 + 7x^2 + 12x - 4 = 0$ ✓

ii)

$\alpha^2+\beta^2+\gamma^2 = (\alpha+\beta+\gamma)^2 - 2(\alpha\beta+\beta\gamma+\alpha\gamma)$

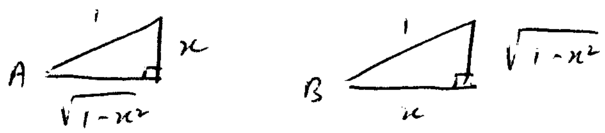
∴ $\alpha^2\beta^2 + \beta^2\gamma^2 + \alpha^2\gamma^2 =$ ✓

$(\alpha\beta + \beta\gamma + \alpha\gamma)^2 - 2\alpha\beta\gamma(\alpha+\beta+\gamma)$

$= 16 - 2\times2\times1 = 12$ ✓

(d) $A = \sin^{-1} x \Rightarrow x = \sin A$

$B = \cos^{-1} x \Rightarrow x = \cos B$



∴ $\cos A = \sqrt{1-x^2}$, $\sin B = \sqrt{1-x^2}$ ✓

$\sin(\sin^{-1}x - \cos^{-1}x) = \sin(A - B)$

$= \sin A\cos B - \cos A\sin B$

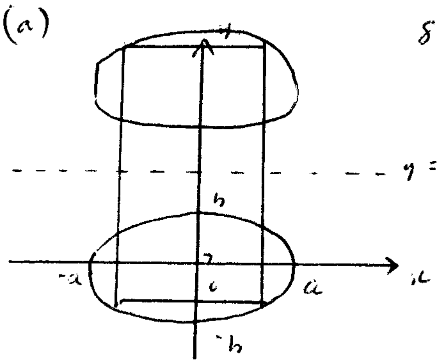
$= x^2 - (1 - x^2)$

$= 2x^2 - 1$ ✓

(9)

Question 6

(a)



$\delta v = 2\pi (c-y)\, 2x\, dy$

$V = 4\pi \int_{-b}^{b} x(c-y)\,dy \; ; \; x = \frac{a}{b}\sqrt{b^2-y^2}$

$\therefore \; V = \frac{4\pi a}{b}\int_{-b}^{b}(c-y)\sqrt{b^2-y^2}\,dy$

$V = \frac{4\pi a}{b} \times c \int_{-b}^{b}\sqrt{b^2-y^2}\,dy - \frac{4\pi a}{b}\int_{-b}^{b} y\sqrt{b^2-y^2}\,dy$

$= \frac{4\pi ac}{b} \times \frac{\pi b^2}{2} - \frac{4\pi a}{b} \times 0$

$\therefore \; V = 2\pi^2 abc$

(b) $BC = 2\sqrt{a^2-x^2}$

$A_{\Delta ABC} = \frac{1}{2} \times 2\sqrt{a^2-x^2} \times b$

$= b\sqrt{a^2-x^2}$

$\therefore \; V = b\int_{-a}^{a}\sqrt{a^2-x^2}\,dx = b \times \pi \times \frac{a^2}{2} =$

$\therefore \; V = \frac{\pi}{2}a^2 b$

(c) $ax^3 + bx^2 + cx + d = 0$

i) $\alpha + \beta + \lambda = -b/a$

$\alpha\beta + \beta\lambda + \alpha\lambda = c/a$

$\alpha\beta\lambda = -\frac{d}{a}$

(c) ii)

$36x^3 - 12x^2 - 11x + 2 = 0$ has zeroes α, β, γ and $\alpha = \beta + \gamma$

$\therefore \; \alpha + \beta + \lambda = 2\alpha = \frac{12}{36} \Rightarrow \alpha = \frac{1}{6}$

iii) $x^3 + px^2 + qx + r = 0$

has roots λ, μ and ν

$\lambda = \mu + \nu$.

Then $\lambda + \mu + \nu = 2\lambda = -p$

$\Rightarrow \lambda = -\frac{p}{2}$

$\therefore \; \lambda^3 + p\lambda^2 + q\lambda + r = 0$

$\therefore \; -\frac{p^3}{8} + \frac{p^3}{4} + -\frac{pq}{2} + r = 0$

$\therefore \; p^3 - 4pq + 8r = 0$

(d)

$p(x) = 2x^4 + 9x^3 + 6x^2 - 20x - 24$

has a triple root.

$\therefore \; p'(x) = 8x^3 + 27x^2 + 12x - 20$

has a double root.

$\therefore \; p''(x) = 24x^2 + 54x + 12$ has a 1 fold root.

$\therefore \; p''(x) = 0 \Rightarrow 24x^2 + 54x + 12 = 0$

$\therefore \; 4x^2 + 9x + 2 = 0$

$\therefore \; (4x+1)(x+2) = 0$

$x = -\frac{1}{4}, \quad x = -2$

Subs. $x = -\frac{1}{4}$ or $x = -2$

$p(-2) = 0 \Rightarrow x = -2$ is a triple root.

$\therefore \; p(x) = (x+2)^3(2x-3)$

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Question 7

(a) equ. of tangent is

i) $x + t^2 y = 2ct \Rightarrow m_1 = -\frac{1}{t^2}$

$\therefore$ Gradient m_2 of normal is $m_2 = t^2$. ✓

$\therefore$ Equ. of normal is

$$\frac{y - c/t}{x - ct} = t^2$$

$\therefore \quad yt - c = t^3 x - ct^4$

$\therefore \quad ty + ct^4 = t^3 x + c$ ✓

ii) Tangent $x + t^2 y = 2ct$ meets x-axis and y-axis at X and Y respectively

$\therefore X(2ct, 0), \quad Y\left(0, \frac{2c}{t}\right)$ ✓

Solve $\begin{cases} ty + ct^4 = t^3 x + c \\ y = -x \end{cases}$ simultaneously ✓

$\therefore \quad x_Q = \frac{c(t^2-1)}{t}, \quad y_Q = \frac{c(1-t^2)}{t}$

$\therefore \quad Q\left(\frac{c}{t}(t^2-1), \frac{c}{t}(1-t^2)\right)$ ✓

iii) Mid-point of $X(2ct, 0)$ and $Y\left(0, \frac{2c}{t}\right)$ is $\left(ct, \frac{c}{t}\right)$ which is ✓ R. Also, applying mid-point formula to find mid-point of interval joining P and Q we also obtain $R\left(ct, \frac{c}{t}\right)$. ✓

$\therefore$ PQ and XY bisect each other at R.

iv) $XY^2 = 4c^2t^2 + \frac{4c^2}{t^2} =$

$= 4c^2\left(t^2 + \frac{1}{t^2}\right)$ ✓

$PQ^2 = 4c^2\left(t^2 + \frac{1}{t^2}\right) \Rightarrow XY^2 = PQ^2$

$\therefore \quad XY = PQ$ ✓

v) Since $PQ \perp XY$ and PQ, XY bisecting each other. Also $PQ = XY$.

$\therefore$ $PXQY$ is a square. ✓

(b) $u_1 = 3, \quad u_2 = 29,$

$u_n = 7u_{n-1} - 10u_{n-2}$.

$n = 1 \Rightarrow u_1 = 5^1 + 2^1 = 7 \Rightarrow$ true for $n = 1$. ✓ Assume the statement is true for $n = k$, i.e.,

$u_k = 5^k + 2^k$. Now for $n = k+1$,

$u_{k+1} = 7u_k - 10u_{k-1}$ ✓

$= 7(5^k + 2^k) - 10 \times (5^{k-1} + 2^{k-1})$ ✓

$= (5+2) \times 5^k + (5+2) \times 2^k$ ✓

$- 2 \times 5^k - 5 \times 2^k$

$\Rightarrow \quad u_{k+1} = 5^{k+1} + 2^{k+1}$ ✓

Hence statement also true for $n = k+1$.

$\therefore$ Proved by mathematical induction.

Question 8

(a)

i) $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

$\therefore \frac{2x}{a^2} + \frac{2y}{b^2} \times \frac{dy}{dx} = 0$

$\therefore \frac{dy}{dx} = -\frac{b^2}{a^2} \times \frac{x}{y} \Rightarrow m = -\frac{b\cos\theta}{a\sin\theta}$ ✓

Tangent is

$y - b\sin\theta = -\frac{b\cos\theta}{a\sin\theta}(x - a\cos\theta)$ ✓

$\therefore a\sin\theta\, y - ab\sin^2\theta =$
$-b\cos\theta\, x + ab\cos^2\theta$.

$\therefore b\cos\theta\, x + a\sin\theta\, y = ab$ ✓

$\therefore \frac{\cos\theta}{a} x + \frac{\sin\theta}{b} y = 1$

ii) $x\cos\alpha + y\sin\alpha = k$ is a tangent if

$\cos\theta = \frac{\cos\alpha}{b}$, $\sin\theta = \frac{\sin\alpha}{a}$

and $k = ab$. ✓

$\Rightarrow \sin^2\theta + \cos^2\theta = \frac{\cos^2\alpha}{b^2} + \frac{\sin^2\alpha}{a^2} = 1$

$\therefore a^2\cos^2\alpha + b^2\sin^2\alpha = a^2b^2 = k^2$ ✓

iii)

$AQ^2 = a^2e^2\sin^2\alpha + k^2 - 2aek\sin\alpha$ ✓

$BR^2 = a^2e^2\sin^2\alpha + k^2 + 2aek\sin\alpha$ ✓

$\therefore AQ^2 + BR^2 = 2a^2e^2\sin^2\alpha + 2k^2$

$= 2a^2\left(1 - \frac{b^2}{a^2}\right)\sin^2\alpha + 2(a^2\cos^2\alpha + b^2\sin^2\alpha)$ ✓

$\therefore AQ^2 + BR^2 = 2a^2(\sin^2\alpha + \cos^2\alpha)$

$= 2a^2$ ✓

(b)

Now

$\sin(x+y) + \sin(x-y) = 2\sin x\cos y$

Let $\left.\begin{array}{l} x+y = A \\ x-y = C \end{array}\right\} \Rightarrow \begin{array}{l} x = \frac{A+C}{2} \\ y = \frac{A-C}{2} \end{array}$

$\therefore \sin A + \sin C = 2\sin\frac{A+C}{2}\cos\frac{A-C}{2}$ ✓

ii)

(α) $\frac{\sin A}{2a} = \frac{\sin B}{a+c} = \frac{\sin C}{2c}$ ✓

$\therefore \frac{\sin A + \sin C}{2a + 2c} = \frac{\sin B}{a+c}$

$\therefore \sin A + \sin C = \frac{2(a+c)}{a+c} \times \sin B$

$\therefore \sin A + \sin C = 2\sin B$ $(*)$ ✓

(β) Now

$\sin A + \sin C = 2\sin\frac{A+C}{2}\cos\frac{A-C}{2}$

Hence $(*)$ can be written as

$2\sin\frac{A+C}{2}\cos\frac{A-C}{2} = 4\sin\frac{B}{2}\cos\frac{B}{2}$ $(**)$ ✓

Since $A + B + C = 180°$

$\therefore \frac{A+C}{2} = 90° - \frac{B}{2} \Rightarrow \sin\frac{A+C}{2} = \cos\frac{B}{2}$ ✓

$\therefore (**)$ can be written as

$4\sin\frac{B}{2}\cancel{\cos\frac{B}{2}} = 2\cancel{\cos\frac{B}{2}}\cos\frac{A-C}{2}$

$\therefore \sin\frac{B}{2} = \frac{1}{2}\cos\left(\frac{A-C}{2}\right)$ ✓