



EPPING BOYS' HIGH SCHOOL

Year 12 Trial Examination 2009

MATHEMATICS EXTENSION 2

Student 's Name

Monday 17th August, 2009

Teacher:

General Instructions

- Reading time – 5 minutes.
- Working time – 3 hours.
- Board-approved calculators may be used.
- All necessary working should be shown in every question.
- Full marks may not be awarded for careless or badly arranged work.
- Start each question on a new booklet.
- Attempt all questions.

| Q. 1 | Q. 2 | Q. 3 | Q. 4 | Q. 5 | Q. 6 | Q. 7 | Q.8 | Total | % |
|------|------|------|----------------|------|------|------|-----|-------|------|
| / 15 | /15 | /15 | /15 | /15 | /15 | /15 | /15 | /120 | /100 |

Start each question on a SEPARATE page or in a SEPARATE booklet.

Question 1 (15 marks) Use a separate page/booklet

Marks

(a) Find $\int \frac{\sin 2x}{3 + \sin^2 x} dx$ 2

(b) Find $\int \frac{1}{e^x + e^{-x}} dx$. 2

(c) Evaluate $\int_{\sqrt{3}}^3 \frac{1}{\sqrt{(x^2 - 1)}} dx$. 3

(d) Evaluate $\int_0^{\pi/3} \frac{1}{1 - \sin x} dx$. 4

(e) Evaluate $\int_0^4 \frac{x^2 + 4x + 5}{(x+1)(x+3)} dx$. 4

Question 2 (15 marks) Use a separate page/booklet

Marks

- (a) Find real x and y such that $(x + iy)^2 = 3 + 4i$. 2
- (b) Find $|z|$ and $\arg z$ when $z = -\sqrt{3} - i$. 2
- (c) (i) Express $1 + i$ and $1 - i$ in modulus/argument form. 2
- (ii) Hence evaluate $(1 + i)^{40} + (1 - i)^{40}$. 2
- (d) (i) Solve $z^5 = -1$. 2
- (ii) Hence show that $z^5 + 1 = (z + 1)\left(z^2 - 2z \cos \frac{\pi}{5} + 1\right)\left(z^2 - 2z \cos \frac{3\pi}{5} + 1\right)$. 3
- (e) Indicate on an Argand diagram the region which contains the point P representing z when $|z| \leq |z - 4|$ and $-\frac{\pi}{4} \leq \arg z \leq \frac{\pi}{4}$. 2

Question 3 (15 marks) Use a separate page/booklet

Marks

(a) Sketch (showing critical points) the graph of: $y = 3(x + \sqrt{x})$ 2

(b) If $f(x) = x^2 - 16$, sketch the following graphs on separate axes showing all relevant points.

(i) $y = f(x)$ 1

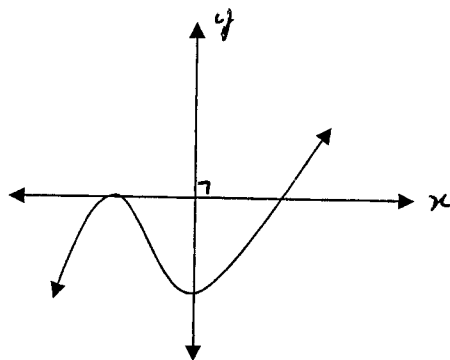
(ii) $y = \frac{1}{f(x)}$ 2

(iii) $y = -\frac{16}{f(x)}$ 2

(iv) $y = |f(x)| + 6$ 2

(c) Sketch the graph $y^2 = x^2(1 - x^2)$ 2

(d) The graph of $y = x^3 + 3x^2 - 4$ is sketched below



Sketch the curves $y = |x^3 + 3x^2 - 4|$ and $y = \ln|x^3 + 3x^2 - 4|$ on separate axes. 2

(e) Show that $|z_1 + z_2|^2 + |z_1 - z_2|^2 = 2(|z_1|^2 + |z_2|^2)$ for any complex numbers z_1 and z_2 . 2

Question 4 (15 marks) Use a separate page/booklet

Marks

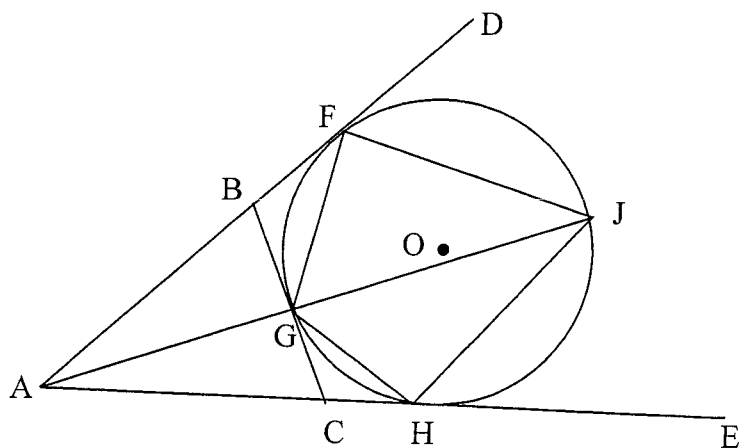
- (a) A particle A_1 of mass m kg is dropped from point C and falls towards point D, which is directly underneath C. At the instant when A_1 is dropped, a second particle A_2 , also of mass m kg, is projected upwards from D towards C with an initial velocity equal to twice the terminal velocity of A_1 . Each particle experiences a resistance of magnitude mkv as it moves, where v ms⁻¹ is the velocity and k is a constant.

(i) Show that the terminal velocity of A_1 is $\frac{g}{k}$, where g is acceleration due to gravity. 2

(ii) For particle A_2 , show that $t = \frac{1}{k} \ln\left(\frac{3g}{g + kv}\right)$, where v ms⁻¹ is the velocity after t seconds. 3

(iii) Suppose the particles collide at the instant when A_1 has reached 30% of its terminal velocity. Find the velocity of A_2 when they collide. Leave your answer in terms of g and k . 3

- (b) In the following diagram, the lines AH, AF and BC are all tangents to the circle.



(i) Prove that the triangles AFG and AJF are similar. 1

(ii) Prove $(AF)^2 = (AJ) \times (AG)$ 1

(iii) Prove $GJ = \frac{(AF) \times (AH)}{(AG)} - (AG)$ 2

Question 4 (continued)

- (c) (i) Find the values of the constants A and B such that 1
 $4x^4 + 1 \equiv (2x^2 + Ax + 1)(2x^2 + Bx + 1).$
- (ii) Hence find the prime factors of the integer $2^{14} + 1.$ 2

Question 5 (15 marks) Use a separate page/booklet**Marks**

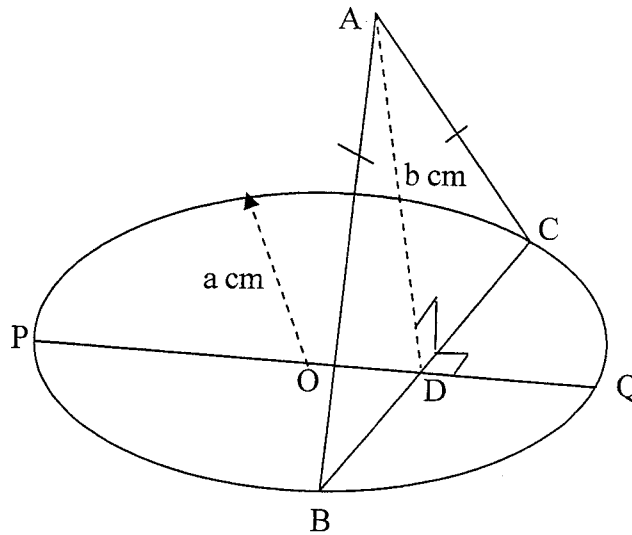
- (a) (i) If $I_n = \int_1^e (\ln x)^n dx$ for $n \geq 0$, show that $I_n = e - nI_{n-1}$ for $n \geq 1.$ 3
- (ii) Hence evaluate $I_4.$ 2
- (b) (i) Use the substitution $u = \pi - x$ to show that 1
$$\int_0^\pi \frac{x \sin x}{1 + \cos^2 x} dx = \int_0^\pi \frac{(\pi - x) \sin x}{1 + \cos^2 x} dx.$$
- (ii) Deduce that $\int_0^\pi \frac{x \sin x}{1 + \cos^2 x} dx = \frac{\pi^2}{4}$ 2
- (c) The cubic equation $x^3 - x^2 + 4x - 2 = 0$ has roots α, β and $\gamma.$
- (i) Find the equation whose roots are α^2, β^2 and $\gamma^2.$ 2
- (ii) Find the value of $\alpha^2 \beta^2 + \alpha^2 \gamma^2 + \beta^2 \gamma^2.$ 3
- (d) Given that $\sin^{-1} x, \cos^{-1} x$ and $\sin^{-1}(1 - x)$ are acute, show that:
 $\sin(\sin^{-1} x - \cos^{-1} x) = 2x^2 - 1.$ 2

Question 6 (15 marks) Use a separate page/booklet

Marks

- (a) The ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ is rotated about the line $y = c$ ($c > b$). Find the volume of the solid generated by using the method of cylindrical shells. 4

- (b) The base of the solid shown in the diagram is a circle of radius a cm.



Each cross section of the solid formed by a plane perpendicular to the fixed diameter PQ is an isosceles triangle of height b cm. One such cross section is shown by $\triangle ABC$ in the above diagram, where $AB=AC$, BC is in the base of the solid and $AD=b$ cm.

Find the volume of this solid by first drawing a diagram indicating your origin and other relevant information. 3

- (c) (i) Write down the relations which hold between the roots α , β and χ of the equation

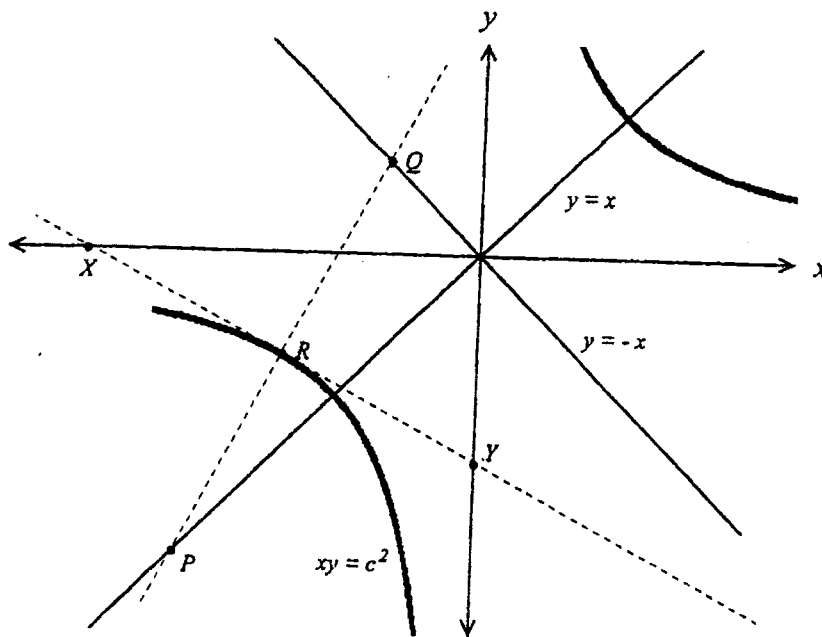
$$ax^3 + bx^2 + cx + d = 0, \quad (a \neq 0) \text{ and the coefficients } a, b, c \text{ and } d. \quad 1$$

- (ii) Consider the equation $36x^3 - 12x^2 - 11x + 2 = 0$. You are given the roots α , β , χ of this equation satisfy $\alpha = \beta + \chi$. Use part (i) to find α . 1

- (iii) Suppose the equation $x^3 + px^2 + qx + r = 0$ has roots λ , μ and ν which satisfy

$$\lambda = \mu + \nu. \text{ Show that } p^3 - 4pq + 8r = 0. \quad 3$$

- (d) If $P(x) = 2x^4 + 9x^3 + 6x^2 - 20x - 24$ has a triple root, factorise $P(x)$ into its linear factors. 3



- (a) In the diagram above $R\left(ct, \frac{c}{t}\right)$ is a point on the rectangular hyperbola $xy = c^2$.

The tangent to the hyperbola at R meets the x -axis at X and the y -axis at Y .

The normal to the hyperbola at R meets the line $y = x$ at P and the line $y = -x$ at Q .

You are given that the equation of the tangent at R is $x + t^2y = 2ct$.

- (i) Prove that the equation of the normal at R is $ty + ct^4 = t^3x + c$. 2
- (ii) It can be shown that P is the point $\left(\frac{c(t^2+1)}{t}, \frac{c(t^2+1)}{t}\right)$. Find the coordinates of X and Y .

Prove that Q is the point $\left(\frac{c(t^2-1)}{t}, \frac{c(1-t^2)}{t}\right)$. 3

- (iii) Show that PQ and XY bisect each other. 2
- (iv) Show that $PQ = XY$. 2
- (v) What type of quadrilateral is $XQYP$? Justify your answer. 1

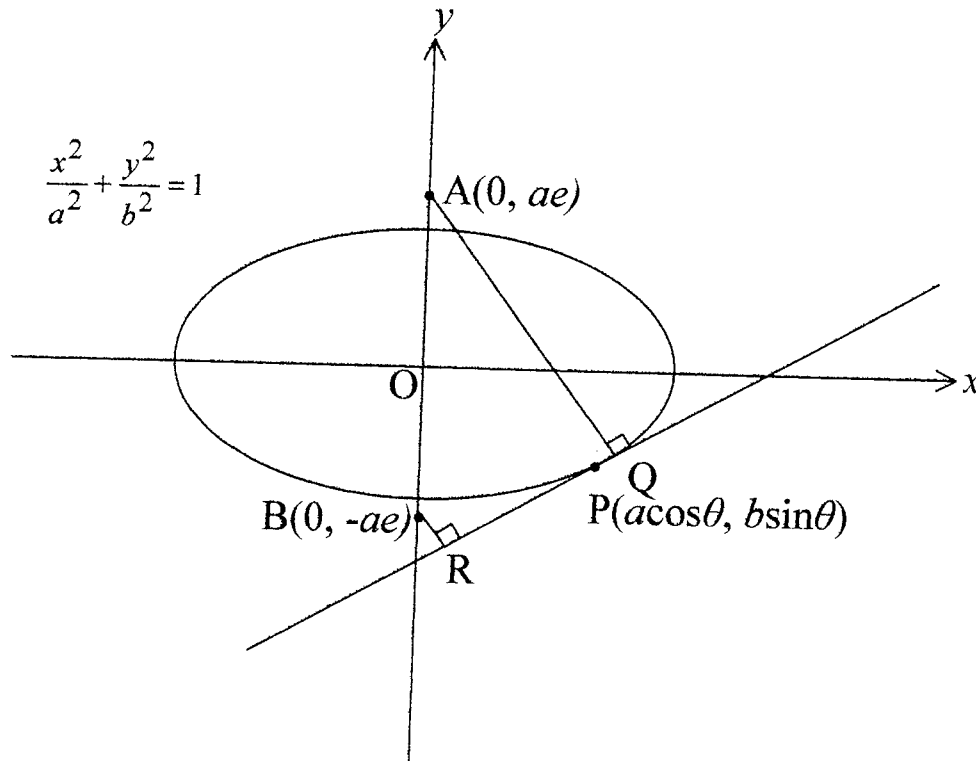
Question 7 (continued)

Marks

- (b) A sequence of numbers u_n is such that $u_1 = 3$, $u_2 = 21$ and $u_n = 7u_{n-1} - 16u_{n-2}$ for $n \geq 3$. Use the method of mathematical induction to show that $u_n = 5^n - 2^n$ for $n \geq 1$. 5

Question 8 (15 marks) Use a separate page/booklet

(a)



$P(acos\theta, bsin\theta)$ is a point on the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, $a > b > 0$, where e is the eccentricity of the ellipse.

From $A(0, ae)$ and $B(0, -ae)$ perpendiculars are drawn to meet the tangent at $P(acos\theta, bsin\theta)$ at Q and R , respectively.

- (i) Prove that the equation of the tangent at P is

$$\frac{\cos \theta}{a} x + \frac{\sin \theta}{b} y = 1$$

3

- (ii) Hence, or otherwise, show that the line $x \cos \alpha + y \sin \alpha = k$ is a tangent to the ellipse if $a^2 \cos^2 \alpha + b^2 \sin^2 \alpha = k^2$

2

- (iii) Hence, or otherwise, prove that $AQ^2 + BR^2 = 2a^2$

4

Question 8 (continued)

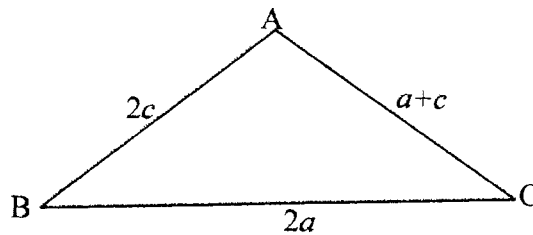
Marks

- (b) (i) Given that $\sin(X + Y) + \sin(X - Y) = 2 \sin X \cos Y$, show that

$$\sin A + \sin C = 2 \sin \frac{A+C}{2} \cos \frac{A-C}{2}$$

1

- (ii) Consider $\triangle ABC$ where



- (α) Use the sine rule to show that $\sin A + \sin C = 2 \sin B$

2

- (β) Deduce that $\sin \frac{B}{2} = \frac{1}{2} \cos \frac{A-C}{2}$

3

END OF EXAM

STANDARD INTEGRALS

$$\int x^n dx = \frac{1}{n+1} x^{n+1}, \quad n \neq -1; \quad x \neq 0, \quad \text{if } n < 0$$

$$\int \frac{1}{x} dx = \ln x, \quad x > 0$$

$$\int e^{ax} dx = \frac{1}{a} e^{ax}, \quad a \neq 0$$

$$\int \cos ax dx = \frac{1}{a} \sin ax, \quad a \neq 0$$

$$\int \sin ax dx = -\frac{1}{a} \cos ax, \quad a \neq 0$$

$$\int \sec^2 ax dx = \frac{1}{a} \tan ax, \quad a \neq 0$$

$$\int \sec ax \tan ax dx = \frac{1}{a} \sec ax, \quad a \neq 0$$

$$\int \frac{1}{a^2 + x^2} dx = \frac{1}{a} \tan^{-1} \frac{x}{a}, \quad a \neq 0$$

$$\int \frac{1}{\sqrt{a^2 - x^2}} dx = \sin^{-1} \frac{x}{a}, \quad a > 0, \quad -a < x < a$$

$$\int \frac{1}{\sqrt{x^2 - a^2}} dx = \ln \left(x + \sqrt{x^2 - a^2} \right), \quad x > a > 0$$

$$\int \frac{1}{\sqrt{x^2 + a^2}} dx = \ln \left(x + \sqrt{x^2 + a^2} \right)$$

NOTE: $\ln x = \log_e x, \quad x > 0$

Question 1

(a) Now $\int \frac{f'(x)}{f(x)} dx = \ln f(x) + c$ with $f(x) = \sin^2 x + 3$

$$\int \frac{\sin 2x}{3 + \sin^2 x} dx = \int \frac{2 \sin x \cos x}{3 + \sin^2 x} dx = \ln(\sin^2 x + 3) + c$$

(b) Using the substitution $e^x = u, du = e^x dx$

$$\begin{aligned} \int \frac{1}{e^x + e^{-x}} dx &= \int \frac{e^x}{e^{2x} + 1} dx \\ &= \int \frac{1}{u^2 + 1} du = \tan^{-1} u + c = \tan^{-1} e^x + c \end{aligned}$$

(c) Using $\int \frac{1}{\sqrt{x^2 - a^2}} dx = \ln \left| x + \sqrt{x^2 - a^2} \right| + c$ with $a = 1$, ✓

$$\int_{\sqrt{3}}^3 \frac{1}{\sqrt{x^2 - 1}} dx = \ln \left| x + \sqrt{x^2 - 1} \right| \Big|_{\sqrt{3}}^3 = \ln(3 + \sqrt{8}) - \ln(\sqrt{3} + \sqrt{2}) = \ln\left(\frac{3 + 2\sqrt{2}}{\sqrt{3} + \sqrt{2}}\right). \checkmark$$

(d) Let $t = \tan \frac{x}{2}, 0 < x < \frac{\pi}{3}, 0 < t < \frac{1}{\sqrt{3}}, x = 2 \tan^{-1} t, dx = \frac{2}{1+t^2} dt$.

Now $\sin x = \frac{2 \tan(x/2)}{1 + \tan^2(x/2)}$,

$$\int_0^{\pi/3} \frac{1}{1 - \sin x} dx = \int_0^{\pi/3} \frac{1}{1 - \frac{2 \tan(x/2)}{1 + \tan^2(x/2)}} dx = \int_0^{1/\sqrt{3}} \frac{1}{1 - \frac{2t}{1+t^2}} \times \frac{2}{1+t^2} dt = \int_0^{1/\sqrt{3}} \frac{1+t^2}{1+t^2-2t} \times \frac{2}{1+t^2} dt$$

$$= 2 \int_0^{1/\sqrt{3}} \frac{1}{(1-t)^2} dt = \left[\frac{2}{1-t} \right]_0^{1/\sqrt{3}} = \frac{2}{1-1/\sqrt{3}} - 2 = \frac{2\sqrt{3}}{\sqrt{3}-1} - 2 = \frac{2}{\sqrt{3}-1}$$

$= \sqrt{3} + 1$

2009. Extension 2 Mathematics
Trial Exam Solutions

Question 1 (cont.)

(2) $\frac{x^2 + 4x + 5}{(x+1)(x+3)} = 1 + \frac{2}{(x+1)(x+3)}$

$$\frac{2}{(x+1)(x+3)} = \frac{a}{x+1} + \frac{b}{x+3}$$

$$2 = a(x+3) + b(x+1) \quad \checkmark$$

Let $x = -1 \Rightarrow 2 = 2a \Rightarrow a = 1$.

Let $x = -3 \Rightarrow 2 = -2b \Rightarrow b = -1$. \checkmark

$$\int_0^4 \frac{x^2 + 4x + 5}{(x+1)(x+3)} dx$$

$$= \int_0^4 1 dx + \int_0^4 \frac{2}{(x+1)(x+3)} dx$$

$$= [x]_0^4 + \int_0^4 \frac{1}{x+1} dx - \int_0^4 \frac{1}{x+3} dx \quad \checkmark$$

$$= 4 + [\ln|x+1|]_0^4 - [\ln|x+3|]_0^4$$

$$= 4 + \ln 5 - \ln 1 - (\ln 7 - \ln 3)$$

$$= 4 + \ln \frac{15}{7} \quad \checkmark$$

4

Question 2

(a)

$$(x + iy)^2 = 3 + 4i$$

$$(x^2 - y^2) + (2xy)i = 3 + 4i$$

Equating real parts:

$$x^2 - y^2 = 3$$

Equating imaginary parts

$$2xy = 4$$

$$\therefore x^4 - x^2 y^2 = 3x^2 \text{ and } x^2 y^2 = 4$$

$$\therefore x^4 - 3x^2 - 4 = 0$$

$$(x^2 - 4)(x^2 + 1) = 0 \quad \checkmark$$

$$\therefore x = \pm 2, y = \pm 1 \quad \checkmark$$

(b)

$$z = -\sqrt{3} - i = 2\left(-\frac{\sqrt{3}}{2} - \frac{1}{2}i\right)$$

$$= 2\left(\cos\left(-\frac{5\pi}{6}\right) + i\sin\left(-\frac{5\pi}{6}\right)\right)$$

$$|z| = 2, \arg z = -\frac{5\pi}{6} \quad \checkmark$$

(c)

i) $z = 1 + i. z = \sqrt{2}\left(\frac{1}{\sqrt{2}} + i\frac{1}{\sqrt{2}}\right) = \sqrt{2}\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right) \quad \checkmark$

$$1 - i = \bar{z} = \sqrt{2}\left(\cos\frac{\pi}{4} - i\sin\frac{\pi}{4}\right). \quad \checkmark$$

ii) Using de Moivre's theorem $z^{40} = 2^{20} \text{cis}(10\pi)$,

$$(\bar{z})^{40} = 2^{20} \text{cis}(-10\pi).$$

$$z^{40} + (\bar{z})^{40} = z^{40} + \overline{(z^{40})} = 2 \operatorname{Re}(z^{40}) = 2^{21} \cos(10\pi). \quad \checkmark$$

$$\therefore (1+i)^{40} + (1-i)^{40} = 2^{21} \cos(10\pi).$$

$$= 2^{21} \quad \checkmark$$

Question 2 (cont.)

(d) i) (i) $|-1|=1$ and $\arg(-1)=\pi$.

Therefore the complex 5th roots of -1 are $\cos \frac{\pi}{5} \pm i \sin \frac{\pi}{5}$, $\cos \frac{3\pi}{5} \pm i \sin \frac{3\pi}{5}$, and -1 . Note the roots are equally spaced around a unit circle in an Argand diagram by an angle of $\frac{2\pi}{5}$ and modulus of 1. ✓ 2

(ii) $z^5 + 1 = (z+1)\left(z - \operatorname{cis} \frac{\pi}{5}\right)\left(z - \operatorname{cis} \left(-\frac{\pi}{5}\right)\right)\left(z - \operatorname{cis} \frac{3\pi}{5}\right)\left(z - \operatorname{cis} \left(-\frac{3\pi}{5}\right)\right)$. ✓ 3

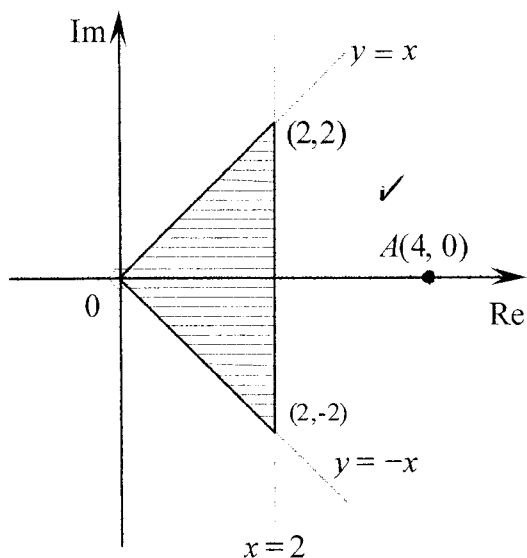
Note

$$\left(z - \operatorname{cis} \frac{\pi}{5}\right)\left(z - \operatorname{cis} \left(-\frac{\pi}{5}\right)\right) = \left(\left(z - \cos \frac{\pi}{5}\right) - i \sin \frac{\pi}{5}\right)\left(\left(z - \cos \frac{\pi}{5}\right) + i \sin \frac{\pi}{5}\right) = \left(z - \cos \frac{\pi}{5}\right)^2 + \left(\sin \frac{\pi}{5}\right)^2 = z^2 - 2z \cos \frac{\pi}{5} + 1$$
 ✓

and $\left(z - \operatorname{cis} \frac{3\pi}{5}\right)\left(z - \operatorname{cis} \left(-\frac{3\pi}{5}\right)\right) = z^2 - 2z \cos \frac{3\pi}{5} + 1$.

$\therefore z^5 + 1 = (z+1)\left(z^2 - 2z \cos \frac{\pi}{5} + 1\right)\left(z^2 - 2z \cos \frac{3\pi}{5} + 1\right)$. ✓

(e)



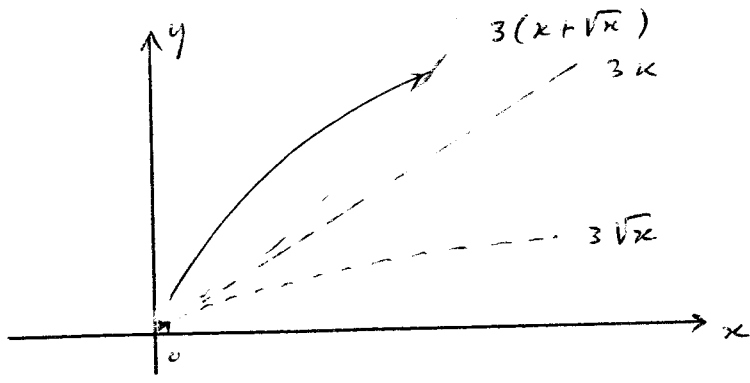
$|z|=|z-4|$ is the perpendicular bisector of OA .

$\arg z = \frac{\pi}{4}$ is the ray $y = x, x > 0$.

$\arg z = -\frac{\pi}{4}$ is the ray $y = -x, x > 0$. ✓ 2

Question 3

a

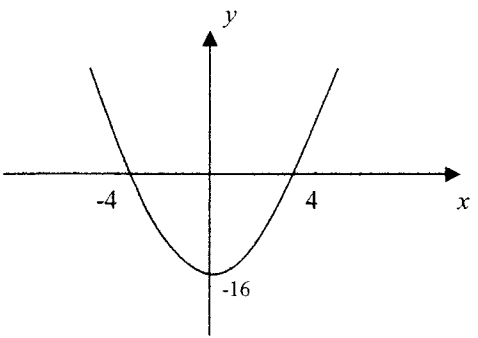


$$y = 3(x + \sqrt{x})$$

2

(b)

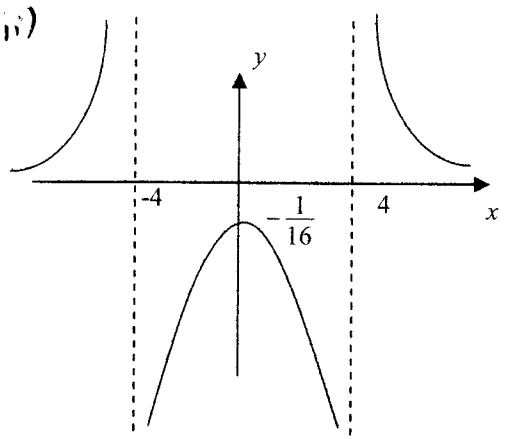
i)



$$f(x) = x^2 - 16$$

1

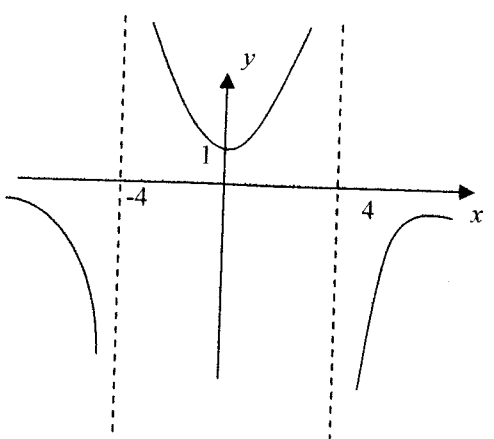
ii)



$$y = \frac{1}{f(x)}$$

2

iii)

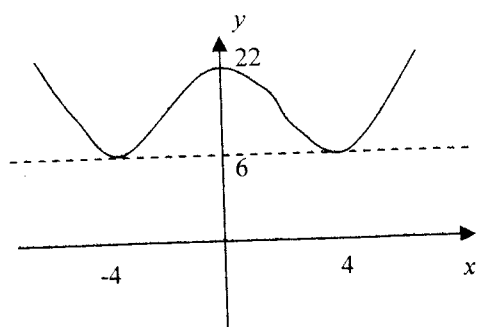


$$y = -\frac{16}{f(x)}$$

2

Question 3 (cont.)

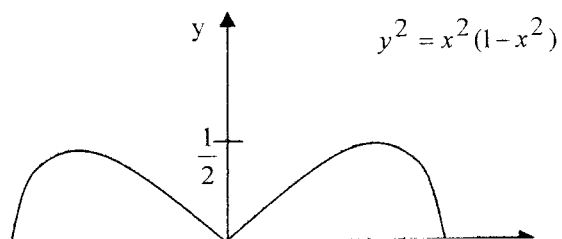
(b)



$$y = |f(x)| + 6$$

2

(c)



$$y^2 = x^2(1-x^2)$$

2

Question 4

Question 4 (a)(i)

Criteria

- One for $\ddot{x} = 0$ and, one for simplification.

2

Answer: For A_1 : $\ddot{x} = g - kv$. Terminal velocity is achieved when $\ddot{x} = 0$ i.e. when $0 = g - kv$.

$$\therefore \text{terminal velocity} = \frac{g}{k}$$

Question 4 (a) (ii)

Criteria

- One for integration, one for constant, one for simplification.

3

Answer:

For A_2 :

$$\frac{dv}{dt} = -g - kv$$

$$t = \int \frac{-1}{g + kv} dv$$

$$= -\frac{1}{k} \ln(g + kv) + c_1$$

$$\text{When } t = 0, v = \frac{2g}{k}$$

$$\therefore c_1 = \frac{1}{k} \ln(3g)$$

$$\therefore t = \frac{1}{k} \ln\left(\frac{3g}{g + kv}\right)$$

Question 4 (a) (iii)

Criteria

- One for $t = \frac{1}{k} \ln\left(\frac{g}{g - kv}\right)$, one for $\frac{1}{k} \ln \frac{10}{7} = \frac{1}{k} \ln\left(\frac{3g}{g + kv}\right)$, one for simplification.

3

Answer:

For A_1

$$\frac{dv}{dt} = g - kv$$

$$t = \int \frac{1}{g - kv} dv$$

$$= -\frac{1}{k} \ln(g - kv) + c_2$$

$$\text{when } t = 0, v = 0 \Rightarrow c_2 = \frac{1}{k} \ln g$$

$$t = \frac{1}{k} \ln\left(\frac{g}{g - \frac{3g}{10}}\right) = \frac{1}{k} \ln \frac{10}{7}$$

$$\frac{1}{k} \ln \frac{10}{7} = \frac{1}{k} \ln\left(\frac{3g}{g + kv}\right)$$

$$t = \frac{1}{k} \ln\left(\frac{g}{g - kv}\right)$$

When the particles collide, A_1 has a velocity

$$\frac{3g}{10k}$$

$$\frac{10}{7} = \frac{3g}{g + kv}$$

$$10g + 10kv = 21g \Rightarrow 10kv = 11g \Rightarrow v = \frac{11g}{10k}$$

At the instant the particle collide, A_2 has a velocity

$$\frac{11g}{10k} \text{ ms}^{-1}$$

Question 4

(b)

Answer

In Δ 's AFG, AJF

$\angle AFG = \angle AJF$ (angle between tangent and chord = angle in alternate segment)

$\angle FAG = \angle FAJ$ (common angle)

Third angles are equal

$\therefore \Delta AFG$ is similar to ΔAJF (AAA)

(ii)

$\frac{AF}{AG} = \frac{AJ}{AF}$ (corresponding sides of similar triangles in the same ratio)

$\therefore AF^2 = AG \times AJ$

(iii)

Now $GJ = AJ - AG$ also $AF^2 = AG \times AJ \Rightarrow AJ = \frac{AF^2}{AG}$

$\therefore GJ = \frac{AF^2}{AG} - AG$

Now $AF = AH$ (Tangent from a common point are equal)

$\therefore GJ = \frac{AF \times AH}{AG} - AG$

i) $4x^4 + 1 \equiv (2x^2 + Ax + 1)(2x^2 + Bx + 1)$
 $\equiv 4x^4 + 2(A+B)x^2 + (A+AB)x^2 + (A+B)x + 1$

$\therefore \left. \begin{matrix} A+B = 0 \\ AB = -4 \end{matrix} \right\} \Rightarrow A = 2, B = -2 \quad \checkmark$

$\therefore 4x^4 + 1 \equiv (2x^2 + 2x + 1)(2x^2 - 2x + 1)$

ii) Let $x = 8 \Rightarrow 4x^4 + 1 = 4 \times 8^4 + 1$
 $= 2^2 \times 2^{12} + 1$
 $= 2^{14} + 1$

And RHS = $(2 \times 8^2 + 2 \times 8 + 1)(2 \times 8^2 - 8 \times 2 + 1)$
 $= 145 \times 113$
 $= 5 \times 29 \times 113$

i.e., $2^{14} + 1 = 5 \times 29 \times 113$

Question 5

(a) i) $L_n = \int_1^e (\ln x)^n dx$

Let $u = (\ln x)^n \Rightarrow du = n(\ln x)^{n-1} \frac{dx}{x}$ ✓

$dv = dx \Rightarrow v = x$

$\therefore \int u dv = uv - \int v du$

$\therefore L_n = \left[x (\ln x)^n \right]_1^e - n \int_1^e (\ln x)^{n-1} dx$ ✓

$L_n = e - n L_{n-1}$ ✓

ii) $L_0 = \int_1^e 1 dx = e - 1$ ✓

$\therefore L_1 = e - 4 \times L_0 = e - 4(e - 1)$
 $= -3e + 4 = 4 - 3e$

$L_1 = 4 - 3e$ ✓

(b) i) $\int_0^\pi \frac{x \sin x}{1 + \cos^2 x} dx = I$

Let $u = \pi - x \Rightarrow x = \pi - u$
 $du = -dx$

$\therefore I = \int_\pi^0 \frac{(\pi - u) \sin(\pi - u)}{1 + \cos^2(\pi - u)} (-du)$

$I = \int_0^\pi \frac{(\pi - u) \sin u}{1 + \cos^2 u} du$

$\therefore \int_0^\pi \frac{x \sin x}{1 + \cos^2 x} dx = \int_0^\pi \frac{(\pi - x) \sin x}{1 + \cos^2 x} dx$ ✓

ii) Now

$\int_0^\pi \frac{x \sin x}{1 + \cos^2 x} dx = \int_0^\pi \frac{(\pi - x) \sin x}{1 + \cos^2 x} dx$

$\therefore I = \int_0^\pi \frac{\pi \sin x}{1 + \cos^2 x} dx = I$

(b) ii) $\Rightarrow 2I = \int_0^\pi \frac{\pi \sin x}{1 + \cos^2 x} dx$

$2I = \pi \left[\tan^{-1} t \right]_{-1}^1 = \pi \left[\frac{\pi}{4} - \left(-\frac{\pi}{4} \right) \right]$

$\therefore I = \int_0^\pi \frac{x \sin x}{1 + \cos^2 x} dx = \frac{\pi^2}{4}$ ✓

(c) i) $x^3 - x^2 + 4x - 2 = 0$ has roots α, β, γ . If $t = \alpha^2$

$\therefore \alpha = \sqrt{t}$

$\therefore \alpha^3 - \alpha^2 + 4\alpha - 2 = 0$

$\therefore t\sqrt{t} + 4\sqrt{t} - t - 2 = 0$ ✓

$\therefore \sqrt{t}(t+4) = t+2$

(c) $t(t^2 + 8t + 16) = t^2 + 4t + 4$

$\therefore t^3 + 7t^2 + 12t - 4 = 0$

$\therefore x^3 + 7x^2 + 12x - 4 = 0$ ✓

ii)

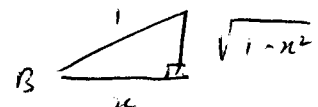
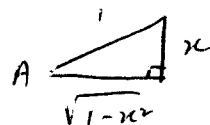
$\alpha^2 + \beta^2 + \gamma^2 = (\alpha + \beta + \gamma)^2 - 2(\alpha\beta + \beta\gamma + \alpha\gamma)$

$\therefore \alpha^2\beta^2 + \beta^2\gamma^2 + \alpha^2\gamma^2 =$

$(\alpha\beta + \beta\gamma + \alpha\gamma)^2 - 2\alpha\beta\gamma(\alpha + \beta + \gamma)$ ✓
 $= 16 - 2 \times 2 \times 1 = 12$ ✓

(d) $A = \sin^{-1} x \Rightarrow x = \sin A$

$B = \cos^{-1} x \Rightarrow x = \cos B$



$\therefore \cos A = \sqrt{1-x^2}, \sin B = \sqrt{1-x^2}$ ✓

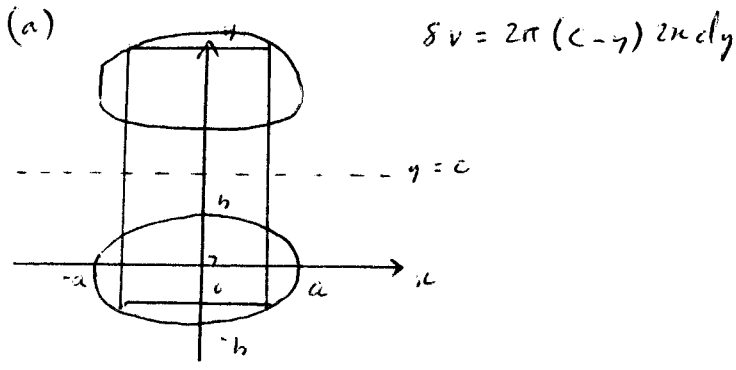
$\sin(\sin^{-1} x - \cos^{-1} x) = \sin(A - B)$

$= \sin A \cos B - \cos A \sin B$

$= x^2 - (1 - x^2)$

$= 2x^2 - 1$ ✓

Question 6



$$v = 4\pi \int_{-b}^b x(c-y) dy \quad ; \quad x = \frac{a}{b} \sqrt{b^2 - y^2}$$

$$\therefore v = \frac{4\pi a}{b} \int_{-b}^b (c-y) \sqrt{b^2 - y^2} dy \quad \checkmark$$

$$v = \frac{4\pi a}{b} \times c \int_{-b}^b \sqrt{b^2 - y^2} dy - \frac{4\pi a}{b} \int_{-b}^b y \sqrt{b^2 - y^2} dy \quad \checkmark$$

$$= \frac{4\pi a c}{b} \times \frac{\pi b^2}{2} - \frac{4\pi a}{b} \times 0$$

$$\therefore v = 2\pi^2 abc \quad \checkmark$$

(b) $BC = 2\sqrt{a^2 - x^2}$

$$A_{\Delta ABC} = \frac{1}{2} \times 2\sqrt{a^2 - x^2} \times b$$

$$= b\sqrt{a^2 - x^2} \quad \checkmark$$

$$\therefore v = b \int_{-a}^a \sqrt{a^2 - x^2} dx \quad \checkmark = b \times \pi \times \frac{a^2}{2}$$

$$\therefore v = \frac{\pi}{2} a^2 b \quad \checkmark$$

(c) $ax^3 + bx^2 + cx + d = 0$

i) $\alpha + \beta + \lambda = -b/a$

$$\alpha\beta + \beta\lambda + \alpha\lambda = c/a$$

$$\alpha\beta\lambda = -\frac{d}{a} \quad \checkmark$$

(c) ii)

$36x^3 - 12x^2 - 11x + 2 = 0$ has roots α, β, λ and $\alpha = \beta + \lambda$

$$\therefore \alpha + \beta + \lambda = 2\alpha = \frac{12}{36} \Rightarrow \alpha = \frac{1}{6} \quad \checkmark$$

iii) $x^3 + px^2 + qx + r = 0$

has roots λ, μ and ν

$$\lambda = \mu + \nu$$

Then $\lambda + \mu + \nu = 2\lambda = -p$

$$\Rightarrow \lambda = -\frac{p}{2} \quad \checkmark$$

$$\therefore \lambda^3 + p\lambda^2 + q\lambda + r = 0$$

$$\therefore -\frac{p^3}{8} + \frac{p^3}{4} + -\frac{pq}{2} + r = 0 \quad \checkmark$$

$$\therefore p^3 - 4pq + 8r = 0 \quad \checkmark$$

(d)

$p(x) = 2x^4 + 9x^3 + 6x^2 - 20x - 24$ has a triple root.

$\therefore p'(x) = 8x^3 + 27x^2 + 12x - 20$ has a double root.

$\therefore p''(x) = 24x^2 + 54x + 12$ has a 1 fold root. \checkmark

$$\therefore p''(x) = 0 \Rightarrow 24x^2 + 54x + 12 = 0$$

$$\therefore 4x^2 + 9x + 2 = 0$$

$$\therefore (4x+1)(x+2) = 0$$

$$x = -\frac{1}{4}, \quad x = -2 \quad \checkmark$$

Subst. $x = -\frac{1}{4}$ or $x = -2$

$p(-2) = 0 \Rightarrow x = -2$ is a triple root.

$$\therefore p(x) = (x+2)^3(2x-3) \quad \checkmark$$

Question 7

(a) equ. of tangent is

$$i) x + t^2 y = 2ct \Rightarrow m_1 = -\frac{1}{t^2}$$

ii. Gradient m_2 of normal is $m_2 = t^2$ ✓

iii. Equ. of normal is

$$\frac{y - c/t}{x - ct} = t^2$$

$$i) yt - c = t^3 x - ct^4$$

$$ii) ty + ct^4 = t^3 x + c \quad \checkmark$$

ii) Tangent $x + t^2 y = 2ct$ meets x -axis and y -axis at X and Y respectively

$$i) X(2ct, 0), Y(0, \frac{2c}{t}) \quad \checkmark$$

$$\text{Solve } \begin{cases} ty + ct^4 = t^3 x + c \\ y = -x \end{cases} \text{ simultaneously} \quad \checkmark$$

$$i) X_Q = \frac{c(t^2-1)}{t}, Y_Q = \frac{c(1-t^2)}{t}$$

$$ii) Q \left(\frac{c}{t}(t^2-1), \frac{c}{t}(1-t^2) \right) \quad \checkmark$$

iii) Mid-point of $X(2ct, 0)$ and $Y(0, \frac{2c}{t})$ is $(ct, \frac{c}{t})$ which is ✓

R . Also, applying mid-point formula to find mid-point of interval joining P and Q we also obtain $R(ct, \frac{c}{t})$. ✓

iv. PQ and XY bisect each other at R .

$$iv) x y^2 = 4c^2 t^2 + \frac{4c^2}{t^2} = 4c^2 \left(t^2 + \frac{1}{t^2} \right) \quad \checkmark$$

$$PQ^2 = 4c^2 \left(t^2 + \frac{1}{t^2} \right) \Rightarrow x y^2 = PQ^2$$

$$i) XY = PQ \quad \checkmark$$

vi) Since $PQ \perp XY$ and PQ, XY bisecting each other. Also $PQ = XY$.

ii. $PXQY$ is a square. ✓

$$(b) u_1 = 3, u_2 = 29,$$

$$u_n = 7u_{n-1} - 10u_{n-2}$$

$$n=1 \Rightarrow u_1 = 5^1 + 2^1 = 7 \Rightarrow \text{true for } n=1. \quad \checkmark$$

Assume the statement is

true for $n=k$, i.e.,

$$u_k = 5^k + 2^k. \text{ Now for } n=k+1,$$

$$u_{k+1} = 7u_k - 10u_{k-1} \quad \checkmark$$

$$= 7(5^k + 2^k) - 10(5^{k-1} + 2^{k-1}) \quad \checkmark$$

$$= (5+2) \times 5^k + (5+2) \times 2^k \quad \checkmark$$

$$= 2 \times 5^k + 5 \times 2^k$$

$$\Rightarrow u_{k+1} = 5^{k+1} + 2^{k+1} \quad \checkmark$$

Hence statement also true for

$$n = k+1.$$

ii. Proved by mathematical induction.

Question 8

(a)

$$i) \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

$$ii) \frac{2x}{a^2} + \frac{2y}{b^2} \times \frac{dy}{dx} = 0$$

$$iii) \frac{dy}{dx} = -\frac{b^2}{a^2} \times \frac{x}{y} \Rightarrow m = -\frac{bcos\theta}{asin\theta} \checkmark$$

Tangent is

$$y - bsin\theta = -\frac{bcos\theta}{asin\theta} (x - acos\theta) \checkmark$$

$$ii) asin\theta y - absin^2\theta = -bcos\theta x + abcos^2\theta$$

$$iii) bcos\theta x + asin\theta y = ab \checkmark$$

$$iv) \frac{cos\theta}{a} x + \frac{sin\theta}{b} y = 1$$

ii) $x cos\alpha + y sin\alpha = k$ is a tangent if

$$cos\theta = \frac{cos\alpha}{b}, \quad sin\theta = \frac{sin\alpha}{a}$$

$$\text{and } k = ab \checkmark$$

$$\Rightarrow sin^2\theta + cos^2\theta = \frac{cos^2\alpha}{b^2} + \frac{sin^2\alpha}{a^2} = 1$$

$$ii) a^2 cos^2\alpha + b^2 sin^2\alpha = a^2 b^2 = k^2 \checkmark$$

iii)

$$AQ^2 = a^2 e^2 sin^2\alpha + k^2 - 2aek sin\alpha \checkmark$$

$$BR^2 = a^2 e^2 sin^2\alpha + k^2 + 2aek sin\alpha \checkmark$$

$$ii) AQ^2 + BR^2 = 2a^2 e^2 sin^2\alpha + 2k^2$$

$$= 2a^2 \left(1 - \frac{b^2}{a^2}\right) sin^2\alpha + 2(a^2 cos^2\alpha + b^2 sin^2\alpha) \checkmark$$

$$ii) AQ^2 + BR^2 = 2a^2 (sin^2\alpha + cos^2\alpha) = 2a^2 \checkmark$$

(b)

Now

$$sin(x+y) + sin(x-y) = 2sinx cosy$$

$$\text{Let } \left. \begin{aligned} x+y &= A \\ x-y &= C \end{aligned} \right\} \Rightarrow \begin{aligned} x &= \frac{A+C}{2} \\ y &= \frac{A-C}{2} \end{aligned}$$

$$ii) sinA + sinC = 2sin \frac{A+C}{2} cos \frac{A-C}{2} \checkmark$$

$$ii) \frac{sinA}{2a} = \frac{sinB}{a+c} = \frac{sinC}{2c} \checkmark$$

$$(a) \frac{sinA}{2a} = \frac{sinB}{a+c} = \frac{sinC}{2c} \checkmark$$

$$ii) \frac{sinA + sinC}{2a + 2c} = \frac{sinB}{a+c}$$

$$ii) sinA + sinC = \frac{2(a+c)}{a+c} \times sinB$$

$$ii) sinA + sinC = 2sinB (*) \checkmark$$

(b) Now

$$sinA + sinC = 2sin \frac{A+C}{2} cos \frac{A-C}{2}$$

Hence (*) can be written as

$$2sin \frac{A+C}{2} cos \frac{A-C}{2} = 4sin \frac{B}{2} cos \frac{B}{2} (***) \checkmark$$

Since $A+B+C = 180^\circ$

$$ii) \frac{A+C}{2} = 90^\circ - \frac{B}{2} \Rightarrow sin \frac{A+C}{2} = cos \frac{B}{2} \checkmark$$

ii) (***) can be written as

$$4sin \frac{B}{2} cos \frac{B}{2} = 2cos \frac{B}{2} cos \frac{A-C}{2}$$

$$ii) sin \frac{B}{2} = \frac{1}{2} cos \left(\frac{A-C}{2}\right) \checkmark$$