(10) 3/4 U 1994 TRIAL

Question 1

(a) Differentiate with respect to x:

(i) $\sin^{-1} x$

(ii) $\ln(\tan x)$

(b) (i) Without the use of calculus, sketch

$$y = (x-1)(x^2-4)$$

(ii) Hence, solve the inequality

$$(x-1)(x^2-4) < 0$$

(c) Find the acute angle, ***to the nearest degree***, between the lines

$3x - 4y + 8 = 0$ and $x + 2y + 1 = 0$

Question 2

(a) For what values of x is $\dfrac{x+4}{x-1} < 6$

(b) Two chords AB and CD of a circle meet when produced at a point P outside the circle. Prove that triangle ADP and triangle CBP are similar.

(c) Find the indefinite integrals:

(i) $\displaystyle\int \frac{x+1}{x^2+4}\,dx$

(ii) $\displaystyle\int (1-\cos^2 x)\,dx$

Question 3

(a) Evaluate: $\displaystyle\int_0^1 \frac{x}{\sqrt{1+x}}\,dx$, ***using the substitution*** $x = u^2 - 1$

(b) A spherical balloon leaks air such that the radius decreases at a rate of 5 mm/sec. Calculate the rate of change of the volume of the balloon when the radius is 100 mm.

(c) AB is the diameter and AC a chord of a circle . The bisector of angle BAC cuts the circle at D. Prove that the tangent at D is perpendicular to AC.

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Question 4

(a) If $P(x) = x^3 - bx^2 - bx + 4$ is divisible by $(x - 2)$, find the value of "b" and hence all the zeros of $P(x)$.

(b) If α and β are roots of $x^2 + bx + q = 0$ form the equation, in general form, whose roots are $\frac{\alpha}{\beta}$ and $\frac{\beta}{\alpha}$

(c) The rate of blood flow [units/sec] through an artery was found ***experimentally*** to be:

$$r(t) = 0.4 - \sin(\pi t) \quad \text{for} \quad 0 \le t \le 2.$$

(i) What is the total blood that flows over the interval [0,2]

(ii) It is known that $r(t) = 0$ for $t \approx \frac{1}{6}$, use one step of Newton's Approximation to find an improved root to ***two decimal places***.

Question 5

(a) Differentiate with respect to x: $\tan^{-1}(\cos x)$

(b) Sketch the function $y = 2\cos^{-1}\frac{x}{3}$, stating its domain & range.

(c) One hundred grams of sugar cane in water are being converted into dextrose at a rate which is proportional to the amount at any time i.e., if M grams are converted in t minutes, then $\frac{dM}{dt} = k(100 - M)$ where k is a constant.
Show that $M = 100 + Ae^{-kt}$, where A is a constant, satisfies the differential equation. Find A, given that where $t = 0, M = 0$.
If 40 grams are converted in the first 10 minutes, find how many grams are converted in the first 30 minutes.

Question 6

(a) $P(2ap, ap^2)$ is any point on the parabola $x^2 = 4ay$.
The line **k** goes through the focus S and is parallel to the tangent at P.

(i) Find the equation of the line **k.**

(ii) The line **k** intersects the X-axis at Q. Find the equation of the locus of the midpoints of the interval QS and give a precise description of this locus.

(b) (i) Without differentiation what is the gradient of the line $y = x$ and give your reason ?

(ii) State the ***Product Rule for Differentiation***

(iii) Hence prove by Mathematical Induction:

$$\frac{d}{dx}(x^n) = nx^{n-1} \quad \text{for all } n \ge 1, \text{ \& } n \text{ is an integer.}$$

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Question 7

(a) A particle moves in such a way that its displacement x cm from the origin 0 after time t secs is given by:-

$$x = \sqrt{3}\cos 3t - \sin 3t.$$

(i) Show that the particle moves with Simple Harmonic Motion.

(ii) Evaluate the period of the motion.

(iii) Find the time at which the particle first passes through the origin.

(iv) Find the velocity when the particle is 1 cm from the origin on its first oscillation.

(b) (i) Prove $\dfrac{d^2x}{dt^2} = \dfrac{d}{dx}\left(\dfrac{1}{2}v^2\right)$

(ii) Prove $\dfrac{d}{dx}(x\ln x) = 1 + \ln x$

(iii) The acceleration of a particle moving in a straight line and starting at 1 cm on the positive side of the origin, at rest is given by:

$$\frac{d^2x}{dt^2} = 1 + \ln x$$

derive the equation relating **v** and **x**.
Hence evaluate **v** when $\mathbf{x = e^2}$

End of Examination

Q1(a) (i) $y = \sin^{-1}x$

1 $\dfrac{dy}{dx} = \dfrac{1}{\sqrt{1-x^2}}$

(ii) $y = \ln(\tan x)$

$\dfrac{dy}{dx} = \dfrac{\sec^2 x}{\tan x}$

2 $= \dfrac{1}{\sin x \cos x}$.

(b) (i) $y = (x-1)(x^2-4)$

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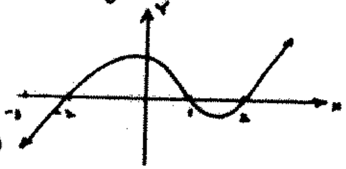


$(x-1)(x^2-4) < 0$

2 $\{x: x < -2 \cup 1 < x < 2\}$

(c) $l_1: 3x - 4y + 8 = 0$ $m_1 = \frac{3}{4}$

$l_2: x + 2y + 1 = 0$ $m_2 = -\frac{1}{2}$

Let ψ = acute angle betw l_1 & l_2

$\therefore \tan\psi = \left|\dfrac{m_1 - m_2}{1 + m_1 m_2}\right|$

$= \left|\dfrac{\frac{3}{4} + \frac{1}{2}}{1 - \frac{3}{4}\cdot\frac{1}{2}}\right|$

$= \left|\dfrac{\frac{5}{4}}{\frac{5}{8}}\right|$

$\tan\psi = 2$

$\therefore \psi = 63°26'$

3 $\psi = 63°$ (to nearest degree)

Q2 (a) $\dfrac{x+4}{x-1} < 6$.

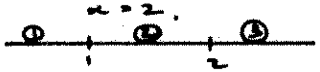
$x - 1 \neq 0 \therefore x \neq 1$

Solve $\dfrac{x+4}{x-1} = 6$

$x + 4 = 6x - 6$

$10 = 5x$

$x = 2$.



Test points from regions:—

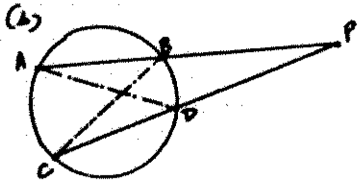
$x = 0$: $-4 < 6$ ✓

$x = 1\frac{1}{2}$: $11 \nless 6$ ✗

$x = 3$: $\frac{7}{2} < 6$ ✓

4 $\therefore \{x: x < 1 \text{ \& } x > 2\}$

(b)



Aim: To prove $\triangle ADP \,|||\, \triangle CBP$.

Data: AB & DC are chords meeting externally at P.

Proof: In $\triangle ADP$, $\triangle CBP$:—

1 $A\hat{P}B = C\hat{P}B$ (Common $\angle$).

2 $D\hat{A}P = B\hat{C}P$ (Angles standing on same arc).

[3 $A\hat{D}P = C\hat{B}P$ (3rd angle)] optional.

3 $\therefore \triangle DAP \,|||\, \triangle BCP$ (3 angles).

(c) (i) $\displaystyle\int \frac{x+1}{x^2+4}\,dx = \frac{1}{2}\int \frac{2x}{x^2+4}\,dx + \int \frac{1}{x^2+4}\,dx$

2 $= \frac{1}{2}\ln(x^2+4) + \frac{1}{2}\tan^{-1}\frac{x}{2} + c$.

(ii) $\displaystyle\int (1 - \cos^2 x)\,dx$

$= \displaystyle\int \sin^2 x\,dx$

$= \frac{1}{2}\displaystyle\int 1 - \cos 2x\,dx$

3 $= \frac{1}{2}\left[x - \frac{1}{2}\sin 2x\right] + c$.

Q3 (a) 5 $\displaystyle\int_0^1 \frac{x}{\sqrt{1+x}}\,dx$

$x = u^2 - 1$

$\dfrac{dx}{du} = 2u$

$dx = 2u\,du$

At $x = 0$: $0 = u^2 - 1$

$u = \pm 1$

At $x = 1$:

$1 = u^2 - 1$

$2 = u^2$

$u = \pm\sqrt{2}$

$= \displaystyle\int_1^{\sqrt{2}} \frac{u^2-1}{\sqrt{u^2}}\,2u\,du$

$= \displaystyle\int_1^{\sqrt{2}} \frac{u^2-1}{u}\cdot 2u\,du$

$= 2\left(\frac{u^3}{3} - u\right]_1^{\sqrt{2}}$

$= 2\left[\frac{2\sqrt{2}}{3} - \sqrt{2} - \frac{1}{3} + 1\right]$

$= 2\left(\frac{2-\sqrt{2}}{3}\right)$

(b) 3 $\dfrac{dr}{dt} = 5$ mm/s

& $\dfrac{dV}{dt} = \dfrac{dV}{dr}\cdot\dfrac{dr}{dt}$

But $V = \frac{4}{3}\pi r^3$

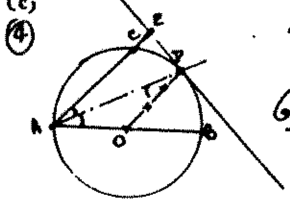
$\dfrac{dV}{dr} = 4\pi r^2$

At $r = 100$ mm

$\dfrac{dV}{dt} = 4\pi(100)^2 \cdot 5$

$= 200000\pi$ mm³/s.

(c) 4



Aim: To prove $AE \perp ED$.

Given AD bisects $B\hat{A}C$.

Construction: OD and extend AC to tangent at E.

Proof: $\triangle OAD$ is isosceles - radii of circle

$\therefore O\hat{D}A = D\hat{A}O = D\hat{A}C$ (Data).

But $C\hat{A}D$ & $O\hat{D}A$ are alternate equal angles in $\parallel$ lines AC & OD

OD is $\perp$ tangent at D.

$\therefore$ AE is $\perp$ tangent at E.

Q4 (a) $P(x) = x^3 - bx^2 - bx + 4$ 2

4 $P(2) = 8 - 4b - 2b + 4 = 0$.

$-6b = -12$

$b = 2$.

$\therefore P(x) = x^3 - 2x^2 - 2x + 4$.

$$
\begin{array}{r}
x^2 - 2 \\
x-2\,\overline{)\,x^3 - 2x^2 - 2x + 4} \\
\underline{x^3 - 2x^2 } \\
-2x + 4 \\
\underline{-2x + 4}
\end{array}
$$

$\therefore P(x) = (x^2 - 2)(x - 2)$

$= (x - \sqrt{2})(x + \sqrt{2})(x - 2)$ optional

$\therefore P(x) = 0$

$(x - \sqrt{2})(x + \sqrt{2})(x - 2) = 0$

$x = \pm\sqrt{2}, 2$.

3

(b) $x^2 + bx + q = 0$.

$\alpha + \beta = -b$

$\alpha\beta = q$

Eq: $x^2 - \left(\dfrac{\alpha}{\beta} + \dfrac{\beta}{\alpha}\right)x + \dfrac{\alpha}{\beta}\times\dfrac{\beta}{\alpha} = 0$

$\Rightarrow x^2 - \left(\dfrac{\alpha^2 + \beta^2}{\alpha\beta}\right)x + 1 = 0$.

$x^2 - \left(\dfrac{(\alpha+\beta)^2 - 2\alpha\beta}{\alpha\beta}\right)x + 1 = 0$.

$x^2 - \left(\dfrac{b^2 - 2q}{q}\right)x + 1 = 0$

$qx^2 - (b^2 - 2q)x + q = 0$.

(c) 2 $r(t) = 0.4 - \sin(\pi t)$

Total Blood:

(i) $\displaystyle\int_0^2 r(t)\,dt$

$= \displaystyle\int_0^2 0.4 - \sin\pi t\,dt$

$= 0.4t + \frac{1}{\pi}\cos\pi t\Big]_0^2$

$= 0.8 + \dfrac{\cos 2\pi}{\pi} - \dfrac{\cos 0}{\pi}$

$= 0.8 + \frac{1}{\pi} - \frac{1}{\pi}$

$= 0.8$ unit of blood.

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(ii) Improved root z_2:
(3)

$$z_2 = z_1 - \frac{r(z_1)}{r'(z_1)}$$

$r(\tfrac{1}{6}) = 0.4 - \sin\frac{\pi}{6}$.
$= 0.4 - \tfrac{1}{2}$.
$= -0.1$.

$r'(\tfrac{1}{6}) = -\pi\cos\frac{\pi}{6}$
$= -\frac{\pi\sqrt{3}}{2}$.

$\therefore z_2 = \frac{1}{6} - \left[\frac{-0.1}{-\frac{\sqrt{3}}{2}\pi}\right]$

$= 0.13$ (2 dp).

Q5(a) (3) $y = \tan^{-1}(\cos x)$.

Let $u = \cos x$, $y = \tan^{-1} u$

$\frac{du}{dx} = -\sin x$, $\frac{dy}{du} = \frac{1}{1+u^2}$

$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$.

$= \frac{1}{1+u^2} \times -\sin x$

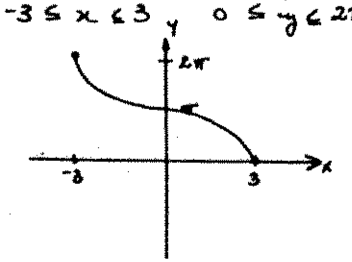
$= \frac{-\sin x}{1+\cos^2 x}$.

(b) (2) $y = 2\cos^{-1}\frac{x}{3}$

$-1 \le X \le 1$ $\quad 0 \le Y \le \pi$

$-1 \le \frac{x}{3} \le 1$ $\quad 0 \le \frac{y}{2} \le \pi$

$-3 \le x \le 3$ $\quad 0 \le y \le 2\pi$.



(c) (5) $M = 100 + Ae^{-kt}$ (5)

$\frac{dM}{dt} = -Ake^{-kt}$

$= k(100 - (100 + Ae^{-kt}))$

$\frac{dM}{dt} = k(100 - M)$.

When $t = 0$, $M = 0$:

$0 = 100 + Ae^{0}$

$A = -100$

$\therefore M = 100 - 100e^{-kt}$

& $40 = 100 - 100e^{-10k}$

$-60 = -100e^{-10k}$

$e^{-10k} = 0.6$.

$-10k \ln e = \ln 0.6$.

$k = \frac{\ln 0.6}{-10}$

$k = 0.05108$

When $t = 30$;

$M = 100 - 100e^{-30 \times 0.05108}$

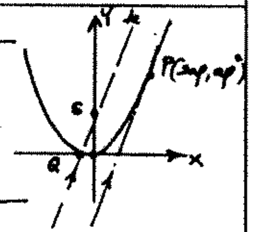
$\therefore M = 78.4$ grams converted!

Q6 (2)
(a) Gradient of k:—
(i) $m_k = p$

Eqn of k:—

$y - a = p(x - 0)$

$y = px + a$



(ii) Coordinates of Q:—
(3) $y = 0 \rightarrow x = -\frac{a}{p}$

$\therefore Q = (\frac{-a}{p}, 0)$

Midpoint of QS:—

$X = \frac{-\frac{a}{p} + 0}{2}$ $\quad Y = \frac{a + a}{2}$

$X = \frac{-a}{2p}$ $\quad Y = \frac{a}{2}$

Locus is straight line $y = \frac{a}{2}$
(as it is independent of p).

(b)(i) $y = x$
(1) has a gradient of 1
Because:
(1) line makes an angle of 45°
with X axis & $\tan 45° = 1$.
or (1) in form $y = mx + b \rightarrow m = 1$
or (1) 1st PRINCIPLE.

(ii) Product Rule for $y = uv$
(1) where u & v are functions of x.

$\therefore \frac{d}{dx}(uv) = v\frac{du}{dx} + u\frac{dv}{dx}$.

(iii)
Step (1) To prove true for $n = 1$.
(4) $\frac{d}{dx}(x^1) = 1x^0 = 1$.

& gradient of line $y = x$ is 1.

$\therefore$ True for $n = 1$.

(2) Assume true for $n = k$.

$\frac{d}{dx}(x^k) = kx^{k-1}$.

(3) Prove true for $n = k+1$

i.e. $\frac{d}{dx}(x^{k+1}) = (k+1)x^k$.

But $\frac{d}{dx}(x^{k+1}) = \frac{d}{dx}(x^k \cdot x^1)$

$= \left[\frac{d}{dx}(x^k)\right] \cdot x + \left[\frac{d}{dx}(x)\right] x^k$.

$= kx^{k-1} \cdot x + x^k$

$= kx^k + x^k$

$= (k+1)x^k$ QED.

(4) If true for $n = 1$, then true for $n = 1+1 = 2$ and so on for all positive integral values of n.

Q7 (a)(i) $x = \sqrt{3}\cos 3t - \sin 3t$
(2) $\frac{dx}{dt} = -3\sqrt{3}\sin 3t - 3\cos 3t$

$\frac{d^2x}{dt^2} = -9\sqrt{3}\cos 3t + 9\sin 3t$

$= -9(\sqrt{3}\cos 3t - \sin 3t)$

$\ddot{x} = -9x$ $\quad (n^2 = 9)$

(ii) Period of motion $T = \frac{2\pi}{n}$
(1) $\therefore T = \frac{2\pi}{3}$

(iii) $\sqrt{3}\cos 3t - \sin 3t = 0$.
(2)

$\tan\alpha = \frac{1}{\sqrt{3}}$

$\therefore \alpha = \frac{\pi}{6}$

$\therefore 2\left(\frac{\sqrt{3}}{2}\cos 3t - \frac{1}{2}\sin 3t\right) = 0$.

$2\cos(3t + \frac{\pi}{6}) = 0$

$\cos(3t + \frac{\pi}{6}) = 0$

$3t + \frac{\pi}{6} = \frac{\pi}{2}$

$t = \frac{\pi}{9}$ s

or using (4)
$\tan 3t = \sqrt{3}$

$3t = \frac{\pi}{3}$

$t = \frac{\pi}{9}$ s.

(iv) (2)
When $2\cos(3t + \frac{\pi}{6}) = 1$

$\cos(3t + \frac{\pi}{6}) = \frac{1}{2}$.

$3t + \frac{\pi}{6} = \frac{\pi}{3}$

$3t = \frac{\pi}{6}$

$t = \frac{\pi}{18}$.

At $t = \frac{\pi}{18}$, $v = \frac{dx}{dt}$

$= -3\sqrt{3}\sin\frac{3\pi}{18} - 3\cos\frac{3\pi}{18}$

$= -3\sqrt{3}\sin\frac{\pi}{6} - 3\cos\frac{\pi}{6}$

$= -\frac{3\sqrt{3}}{2} - \frac{3\sqrt{3}}{2}$.

$= -\frac{6\sqrt{3}}{2}$

$v = -3\sqrt{3}$ cm/sec.

(b) (i) $\frac{d^2x}{dt^2} = \frac{d}{dx}(\tfrac{1}{2}v^2)$

$\frac{d}{dx}(\tfrac{1}{2}v^2) = \frac{d}{dv}(\tfrac{1}{2}v^2)\frac{dv}{dx}$

(1) $= v\frac{dv}{dx}$

$= \frac{dx}{dt} \cdot \frac{dv}{dx}$

$= \frac{dv}{dt} = \frac{d^2x}{dt^2}$

(ii) LHS $= \frac{d}{dx}(x\ln x) = 1 + \ln x$

$\left[\frac{d}{dx}(x)\right]\ln x + \left[\frac{d}{dx}(\ln x)\right][x]$.

$= 1\ln x + \frac{1}{x} \cdot x$

$= \ln x + 1$. (1)

(iii) $\int \frac{d}{dx}(\tfrac{1}{2}v^2)\,dx = \int 1 + \ln x\,dx$
(2)

$\tfrac{1}{2}v^2 = \int \frac{d}{dx}(x\ln x)\,dx$

$\tfrac{1}{2}v^2 = x\ln x + c$

$v = 0$, $x = 1 \Rightarrow c = 0$.

$\therefore v = \pm\sqrt{2x\ln x}$ (1)

At $x = e^2$, $v = \pm\sqrt{2e^2\ln e^2} = 2e$ cm/s

(5)