Name: ____________________________

Teacher: ___________________________

Class: ___________________________

FORT STREET HIGH SCHOOL

2013

HIGHER SCHOOL CERTIFICATE COURSE

ASSESSMENT TASK 3: TRIAL HSC

Mathematics Extension 1

Time allowed: 2 hours
(plus 5 minutes reading time)

Syllabus Outcomes	Assessment Area Description and Marking Guidelines	Questions
	Chooses and applies appropriate mathematical techniques in order to solve problems effectively	1-10
HE2, HE4	Manipulates algebraic expressions to solve problems from topic areas such as inverse functions, trigonometry and polynomials	11, 12
HE3, HE5 HE6	Uses a variety of methods from calculus to investigate mathematical models of real life situations, such as projectiles, kinematics and growth and decay	13
HE7	Synthesises mathematical solutions to harder problems and communicates them in appropriate form	14

Total Marks 70

Section I 10 marks
Multiple Choice, attempt all questions,
Allow about 15 minutes for this section

Section II 60 Marks
Attempt Questions 11-14,
Allow about 1 hour 45 minutes for this section

Section I	Total 10	Marks
Q1-Q10		
Section II	Total 60	Marks
Q11	/15	
Q12	/15	
Q13	/15	
Q14	/15	
	Percent	

General Instructions:

- Questions 11-14 are to be started in a new booklet.
- The marks allocated for each question are indicated.
- In Questions 11 – 14, show relevant mathematical reasoning and/or calculations.
- Marks may be deducted for careless or badly arranged work.
- Board – approved calculators may be used.

STANDARD INTEGRALS

$$\int x^n \, dx = \frac{1}{n+1} x^{n+1}, \quad n \neq -1; \quad x \neq 0, \text{ if } n < 0$$

$$\int \frac{1}{x} \, dx = \ln x, \quad x > 0$$

$$\int e^{ax} \, dx = \frac{1}{a} e^{ax}, \quad a \neq 0$$

$$\int \cos ax \, dx = \frac{1}{a} \sin ax, \quad a \neq 0$$

$$\int \sin ax \, dx = -\frac{1}{a} \cos ax, \quad a \neq 0$$

$$\int \sec^2 ax \, dx = \frac{1}{a} \tan ax, \quad a \neq 0$$

$$\int \sec ax \tan ax \, dx = \frac{1}{a} \sec ax, \quad a \neq 0$$

$$\int \frac{1}{a^2 + x^2} \, dx = \frac{1}{a} \tan^{-1} \frac{x}{a}, \quad a \neq 0$$

$$\int \frac{1}{\sqrt{a^2 - x^2}} \, dx = \sin^{-1} \frac{x}{a}, \quad a > 0, \quad -a < x < a$$

$$\int \frac{1}{\sqrt{x^2 - a^2}} \, dx = \ln\left(x + \sqrt{x^2 - a^2}\right), \quad x > a > 0$$

$$\int \frac{1}{\sqrt{x^2 + a^2}} \, dx = \ln\left(x + \sqrt{x^2 + a^2}\right)$$

NOTE : $\ln x = \log_e x, \quad x > 0$

Section I

10 marks Attempt Questions 1–10 Allow about 15 minutes for this section

Use the multiple-choice answer sheet for Questions 1–10.

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1 What is the solution to the equation $|x-2| = 2x-1$?

(A) $x = -3$
(B) $x = -1$
(C) $x = 1$
(D) $x = 3$

2 A parabola has the parametric equations $x = 12t$ and $y = -6t^2$.
What are the coordinates of the focus?

(A) $(-6, 0)$
(B) $(0, -6)$
(C) $(6, 0)$
(D) $(0, 6)$

3 What is the acute angle to the nearest degree that the line $2x-3y+5=0$ makes with the y-axis?

(A) 27°
(B) 34°
(C) 56°
(D) 63°

4 What are the coordinates of the point that divides the interval joining the points $A(1,1)$ and $B(5,3)$ externally in the ratio 2:3?

(A) $(-7, -3)$
(B) $(-7, 1)$
(C) $(-13, 1)$
(D) $(-13, -3)$

5 Which of the following is an expression for $\int \frac{e^{-2x}\,dx}{e^{-x}+1}$?

Use the substitution $u = e^{-x}+1$.

(A) $\frac{(e^{-x}+1)^2}{2} - e^{-x} + c$

(B) $\frac{(e^{-x}+1)^2}{2} + e^{-x} + c$

(C) $\log_e(e^{-x}+1) - e^{-x} + c$

(D) $\log_e(e^{-x}+1) + e^{-x} + c$

6 What is the domain and range of $y = \cos^{-1}(\frac{3x}{2})$?

(A) Domain: $-\frac{2}{3} \le x \le \frac{2}{3}$. Range: $0 \le y \le \pi$

(B) Domain: $-1 \le x \le 1$. Range: $0 \le y \le \pi$

(C) Domain: $-\frac{2}{3} \le x \le \frac{2}{3}$. Range: $-\pi \le y \le \pi$

(D) Domain: $-1 \le x \le 1$. Range: $-\pi \le y \le \pi$

7 What is the indefinite integral for $\int \left(\cos^2 x + \sec^2 x\right) dx$?

(A) $\frac{1}{2}x + \frac{1}{4}\sin 2x + \frac{1}{2}\tan x + c$

(B) $\frac{1}{2}x - \frac{1}{4}\sin 2x + \frac{1}{2}\tan x + c$

(C) $\frac{1}{2}x + \frac{1}{4}\sin 2x + \tan x + c$

(D) $\frac{1}{2}x - \frac{1}{4}\sin 2x + \tan x + c$

8 A football is kicked at an angle of α to the horizontal. The position of the ball at time t seconds is given by $x = Vt\cos\alpha$ and $y = Vt\sin\alpha - \frac{1}{2}gt^2$ where g m/s^2 is the acceleration due to gravity and v m/s is the initial velocity of projection. What is the maximum height reached by the ball?

(A) $\dfrac{V\sin\alpha}{g}$

(B) $\dfrac{g\sin\alpha}{V}$

(C) $\dfrac{V^2\sin^2\alpha}{2g}$

(D) $\dfrac{g\sin^2\alpha}{2V^2}$

9 At a dinner party, the host, hostess and their six guests sit at a round table. In how many ways can they be arranged if the host and hostess are separated?

(A) 720 (B) 1440

(C) 3600 (D) 5040

10 The velocity of a particle moving in a straight line is given by $v = 2x + 3$ where x metres is the distance from fixed point O and v is the velocity in metres per second. What is the acceleration of the particle when it is 4 metres from O?

(A) $a = 11$ ms^{-2}

(B) $a = 19.5$ ms^{-2}

(C) $a = 22$ ms^{-2}

(D) $a = 72$ ms^{-2}

Section II

60 marks Attempt Questions 11–14 Allow about 1 hour and 45 minutes for this section

Answer each question in a SEPARATE writing booklet. Extra writing booklets are available.

In Questions 11–14, your responses should include relevant mathematical reasoning and/or calculations.

Question 11 (15 marks) Use a SEPARATE writing booklet.

(a) Evaluate $\int_{-1}^{1} \frac{dx}{\sqrt{2-x^2}}$. [3]

(b) For what values of x is $\frac{x}{x-2} < 2$? [3]

(c) Differentiate $\left(1-x^2\right)\ln\left(x^2-1\right)$ with respect to x. [2]

(d) In how many ways can a committee of 2 men and 3 women be chosen from a group of 7 men and 9 women? [1]

(e) Let $f\left(x\right) = x^3 + 5x^2 + 17x - 10$. The equation $f\left(x\right) = 0$ has only one real root.

i. Show that the root lies between 0 and 2. [1]

ii. Use one application of Newtons Method with an initial estimate of $x_0 = 1$ to find a better approximation of the root (to 2 decimal places). [2]

(f) Evaluate $\int_{0}^{1} \frac{2x}{2x+1}\,dx$ using the substitution $u = 2x+1$. [3]

End of Question 11

Question 12 (15 marks) Use a SEPARATE writing booklet.

(a) Two points $P\left(2ap, ap^2\right)$ and $Q\left(2aq, aq^2\right)$ lie on the parabola $x^2 = 4ay$.
The general tangent at any point on the parabola with parameter t is given by $y = tx - at^2$ (do NOT prove this).

i. Find the co-ordinates of the point of intersection T of the tangents to the parabola at P and Q. [2]

ii. You are given that the tangents at P and Q intersect at an angle of 45°. Show that [1]

$$p - q = 1 + pq$$

iii. By evaluating the expression $x^2 - 4ay$, or otherwise, find the locus of the point T when the tangents at P and Q meet as described in part ii above. [2]

(b) For the function given by $f\left(x\right) = -1 + \sqrt{x+4}$

i. State the domain for the function $f\left(x\right)$. [1]

ii. Find the inverse function $f^{-1}\left(x\right)$ for the given function $f\left(x\right)$. [2]

iii. Find the restrictions on the domain and range for $f^{-1}\left(x\right)$ to be the inverse function of $f\left(x\right)$. [1]

(c) Prove by Mathematical Induction that $n^3 + 2n$ is divisible by 3, for all positive integer n. [3]

(d) For $0 \le \theta \le 2\pi$, find all the solutions of $\sin 2\theta = -\cos\theta$. [3]

End of Question 12

Question 13 (15 marks) Use a SEPARATE writing booklet.

(a) The rate at which a body warms in air is proportional to the difference in temperature *T* of the body and the constant temperature *A* of the surrounding air. This rate is given by the differential equation

$$\frac{dT}{dt} = k\left(T - A\right)$$

where t is the time in minutes and k is a constant.

i. Show that $T = A + A_0 e^{kt}$, where A_0 is a constant, is a solution of this equation. [1]

ii. A cold body, initially at 5° C, warms to 10° C in 20 minutes. The air temperature around the body is 25° C. Find the temperature of the body after a further 40 minutes have elapsed. Give your answer to the nearest degree. [3]

(b) The acceleration of a particle moving in a straight line is given by

$$\frac{d^2x}{dt^2} = 2x - 3$$

where *x* is the displacement, in metres, from the origin *O* and *t* is the time in seconds. Initially the particle is at rest at $x = 4$.

i. If the velocity of the particle is *v* m/s, show that [1]

$$v^2 = 2\left(x^2 - 3x - 4\right)$$

ii. Show that the particle does not pass through the origin. [1]

iii. Determine the position of the particle when $v = 10$. Justify your answer. [2]

(c) For the graph of $y = \dfrac{2x+1}{x-1}$

i. Find the horizontal asymptote of the graph. [1]

ii. Without the use of calculus, sketch the graph of $y = \dfrac{2x+1}{x-1}$, showing the asymptote found in part (i). [2]

Question 13 continues on page 9

(d) The velocity v m/s of a particle moving in simple harmonic motion along the x-axis is given by

$$v^2 = 8 + 2x - x^2$$

i. Between which two points is the particle oscillating? [1]

ii. What is the amplitude of the motion? [1]

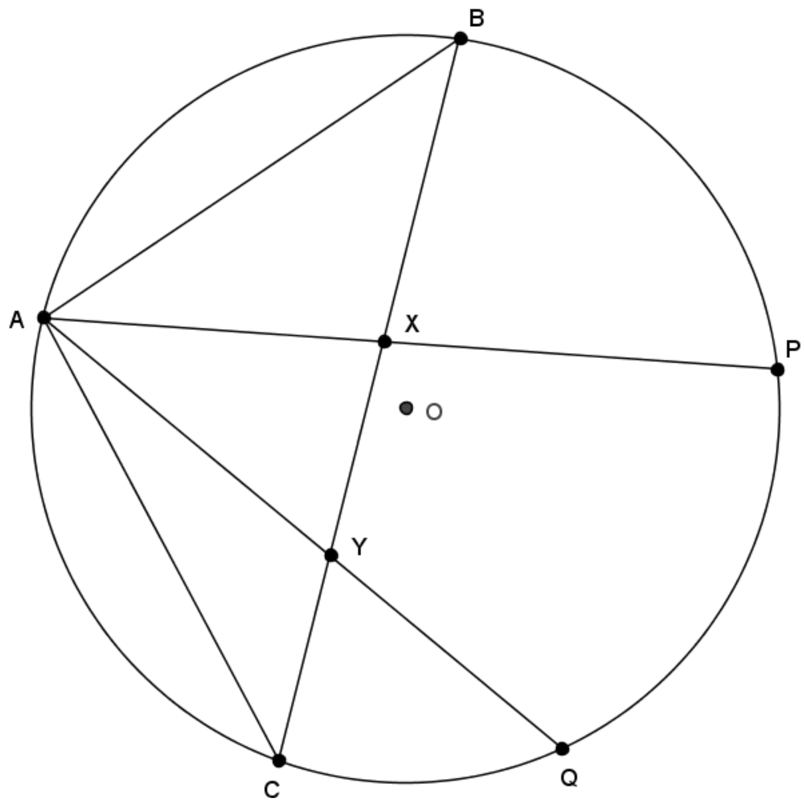
iii. Find the acceleration of the particle is terms of x. [1]

iv. Find the period of oscillation. [1]

End of Question 13

Question 14 (15 marks) Use a SEPARATE writing booklet.

(a) Let ABPQC be a circle such that AB=AC, AP meets BC at X and AQ meets BC at Y, as shown below. Let $\angle BAP = \alpha$ and $\angle ABC = \beta$.



i. Copy the diagram into your writing booklet, marking the information given above, and state why $\angle AXC = \alpha + \beta$. [1]

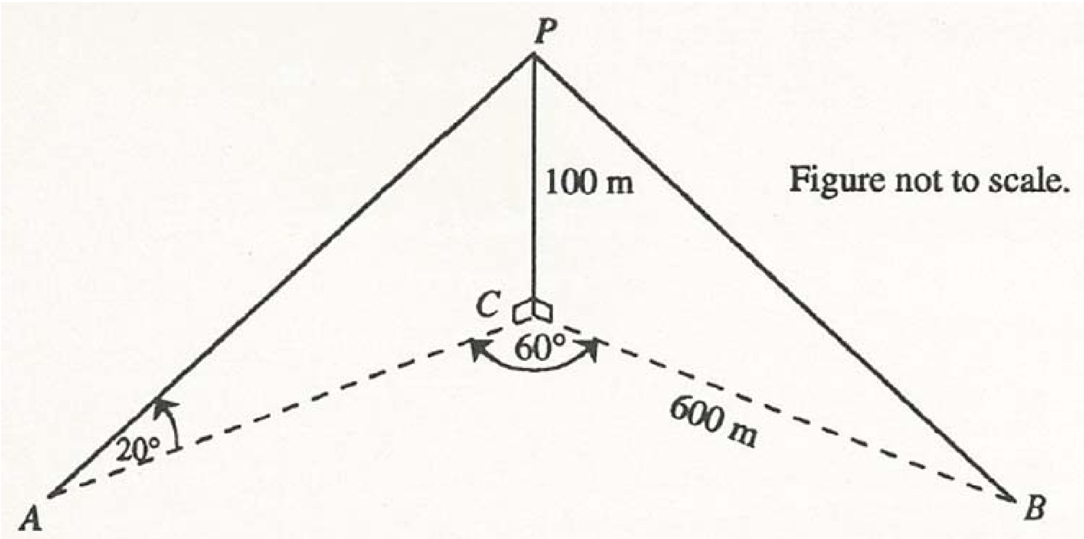
ii. Prove $\angle BQP = \alpha$. [1]

iii. Prove $\angle BQA = \beta$. [1]

iv. Prove the quadrilateral $PQYX$ is cyclic. [2]

Question 14 continues on page 11

(b) Two yachts, *A* and *B*, subtend an angle of 60° at the base *C* of a cliff C.



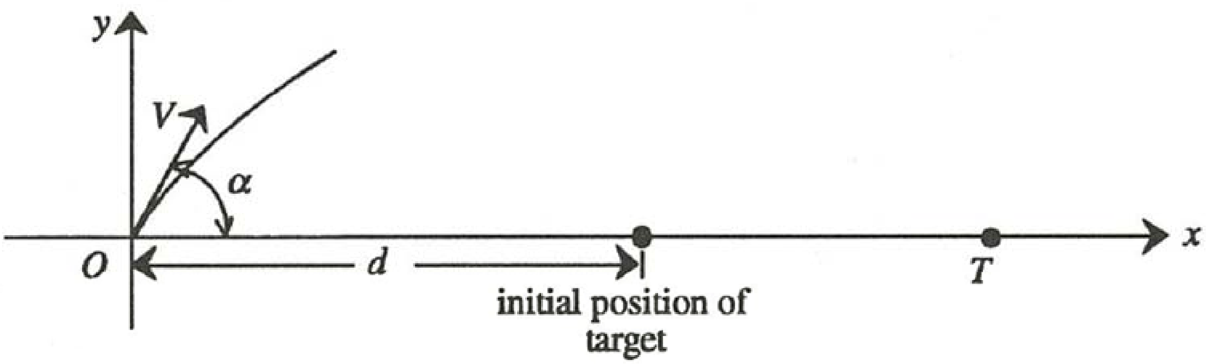
From yacht *A*, the angle of elevation to point *P*, 100 m vertically above *C*, is 20°.
Yacht *B* is 600m from *C*.

i. Calculate length *AC*. [1]

ii. Calculate the distance between the two yachts. [2]

Question 14 continues on page 12

(c) A projectile, with initial speed V_0 m/s, is fired at an angle of elevation α from the origin at *O* towards a target *T*, which is moving away from *O* along the *x*-axis.



You may assume that the projectiles trajectory is defined by the equations

$$x = Vt\cos\alpha \quad y = Vt\sin\alpha - \frac{1}{2}gt^2$$

where *x* and *y* are the horizontal and vertical displacements of the projectile in metres at time *t* seconds after firing, and where *g* is the acceleration due to gravity.

i. Show that the projectile is above the x-axis for a total of $\frac{2V\sin\alpha}{g}$ seconds. [1]

ii. Show that the horizontal range of the projectile is $\frac{2V^2\sin\alpha\cos\alpha}{g}$ metres. [1]

iii. At the instant the projectile is fired, the target T is d metres from O and it is moving away at a constant speed of u m/s.

Suppose that the projectile hits the target when fired at an angle of elevation α. Show that [3]

$$u = V\cos\alpha - \frac{gd}{2V\sin\alpha}$$

iv. Deduce that the projectile will not hit the target if [2]

$$u > \frac{\sqrt{2}\left(V^2 - gd\right)}{2V}.$$

End of Question 14

1	$\lvert x-2\rvert = 2x-1$ $x-2=2x-1$ or $x-2=-(2x-1)$ $x=-3$ $x-2=-2x+1$ $x=1$ Test solutions $x=-3$ $\lvert x-2\rvert=2x-1$ $x=1$ $\lvert x-2\rvert=2x-1$ $\lvert -1-2\rvert = 2\times -1-1$ $\lvert 1-2\rvert = 2\times 1-1$ $3=-3$ (incorrect) $1=1$ (correct) Solution is $x=1$	1 Mark: C
2	$x=12t$ and $y=-6t^2$ $a=6$ and the parabola is concave downwards Focus is $(0,-6)$	1 Mark: B
3	For $2x-3y+5=0$ then $m=\frac{2}{3}$ Angle the line makes with the *x*-axis $\tan\theta=\frac{2}{3}$ $\theta=33.69006753...\approx 34^\circ$ Angle the line makes with the *y*-axis $90^\circ-34^\circ=56^\circ$	1 Mark: C
4	$x=\frac{mx_2+nx_1}{m+n}$ $y=\frac{my_2+ny_1}{m+n}$ $=\frac{-2\times 5+3\times 1}{-2+3}=-7$ $=\frac{-2\times 3+3\times 1}{-2+3}=-3$ The coordinates of point are $(-7,-3)$	1 Mark: A
5	$u=e^{-x}+1$ $\frac{du}{dx}=-e^{-x}$ $du=-e^{-x}dx$ Also $u=e^{-x}+1$ or $e^{-x}=u-1$ $\int\frac{e^{-2x}\,dx}{e^{-x}+1}=\int\frac{e^{-x}\times e^{-x}\,dx}{e^{-x}+1}$ $=\int\frac{-(u-1)\,du}{u}$ $=\int\left(\frac{1}{u}-1\right)du$ $=\log_e u-u+c$ $=\log_e(e^{-x}+1)-(e^{-x}+1)+c$ $=\log_e(e^{-x}+1)-e^{-x}+c$	1 Mark: C

6	Domain: $-1 \le \frac{3x}{2} \le 1$ or $-\frac{2}{3} \le x \le \frac{2}{3}$. Range: $0 \le y \le \pi$	1 Mark: A
7	$\int\left(\cos^2 x + \sec^2 x\right)dx = \int\left(\frac{1}{2}(1+\cos 2x) + \sec^2 x\right)dx$ $= \frac{1}{2}x + \frac{1}{4}\sin 2x + \tan x + c$	1 Mark: C
8	$y = Vt\sin\alpha - \frac{1}{2}gt^2$ $\dot{y} = V\sin\alpha - gt$ Maximum height when $\dot{y} = 0$ $0 = V\sin\alpha - gt$ $t = \frac{V\sin\alpha}{g}$ Maximum height $h = V\sin\alpha \times \frac{V\sin\alpha}{g} - \frac{1}{2}g \times \left(\frac{V\sin\alpha}{g}\right)^2$ $= \frac{V^2\sin^2\alpha}{2g}$	1 Mark: C
9	With no restrictions there are 8 people Arrangements = $= (n-1)!$ $= 7!$ $= 5040$ When the host and hostess sit next to each other. Arrangements = $= 2!(n-1)!$ $= 2!6!$ $= 1440$ Number of arrangements when host and hostess are separated. $= 5040 - 1440$ $= 3600$	1 Mark: C
10	$v = 2x + 3$ $v^2 = 4x^2 + 12x + 9$ $\frac{1}{2}v^2 = 2x^2 + 6x + \frac{9}{2}$ $a = \frac{d}{dx}\left(2x^2 + 6x + \frac{9}{2}\right)$ $= 4x + 6$ When $x = 4$ then $a = 22$	1 Mark: C

Section II: Free Response – Worked Solutions

Question 11

(a) Evaluate $\int_{-1}^{1} \frac{dx}{\sqrt{2-x^2}}$. *[3]*

$\int_{-1}^{1} \frac{dx}{\sqrt{2-x^2}}$

$= \left[\sin^{-1}\left(\frac{x}{\sqrt{2}}\right)\right]_{-1}^{1}$ ❶ uses standard integral correctly

$= \sin^{-1}\left(\frac{1}{\sqrt{2}}\right) - \sin^{-1}\left(\frac{-1}{\sqrt{2}}\right)$

$= \frac{\pi}{4} - \frac{-\pi}{4}$ ❶ correct inverse values

$= \frac{\pi}{2}$ ❶ correct answer

(b) For what values of x is $\frac{x}{x-2} < 2$ *?* *[3]*

Noting $x \neq 2$:

$\frac{x}{x-2}.(x-2)^2 < 2(x-2)^2$

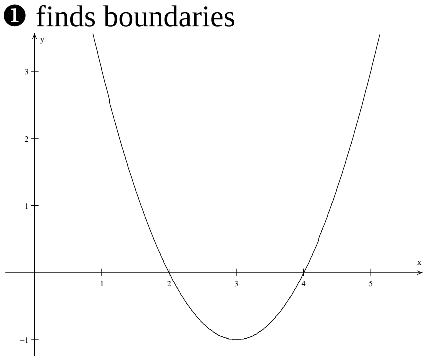
$x(x-2) < 2(x-2)^2$

$0 < 2(x-2)^2 - x(x-2)$

$< (x-2)[2(x-2) - x]$

$< (x-2)(x-4)$

❶ finds boundaries



❶ justifies required values

Hence $x < 2$ or $x > 4$ ❶ correct answer

(c) Differentiate $(1-x^2)\ln(x^2-1)$ *with respect to x.* *[2]*

Let $y = (1-x^2)\ln(x^2-1)$, then

$\frac{dy}{dx} = (1-x^2).\frac{2x}{(x^2-1)} + \ln(x^2-1).(-2x)$

❶ correct use of product & chain rules

$= -2x - 2x\ln(x^2-1)$

$(= -2x(1+\ln(x^2-1)))$ ❶ correct answer

(d) In how many ways can a committee of 2 men and 3 women be chosen from a group of 7 men and 9 women? *[1]*

${}^7C_2 \times {}^9C_3$

$= 21 \times 84$

$= 1764$ ❶ correct answer

(e) Let $f(x) = x^3 + 5x^2 + 17x - 10$. *The equation* $f(x) = 0$ *has only one real root.*

i. Show that the root lies between 0 and 2. *[1]*

$f(0) = 0^3 + 5.0^2 + 17.0 - 10$
$= -10$
< 0

$f(2) = 2^3 + 5.2^2 + 17.2 - 10$
$= 52$
> 0

❶ justification correct

Hence the root lies between 0 and 2.

ii. Use one application of Newtons Method with an initial estimate of $x_0 = 1$ *to find a better approximation of the root (to 2 decimal places).* *[2]*

$f'(x) = 3x^2 + 10x + 17$

$f'(1) = 3.1^2 + 10.1 + 17$
$= 30$

$f(1) = 1^3 + 5.1^2 + 17.1 - 10$
$= 13$

Hence

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$$

$= 1 - \frac{13}{30}$ ❶ values and formula correct

$= \frac{17}{30}$

$= 0.57$ ❶ correct answer

(f) Evaluate $\int_0^1 \frac{2x}{2x+1}\,dx$ *using the substitution* $u = 2x + 1$. *[3]*

$u = 2x + 1$

$du = 2dx$

$dx = \frac{1}{2}du$

$x = \frac{u-1}{2}$

$x = 0, u = 1$

$x = 1, u = 3$

❶ set-up values correct

Then

$$\int_0^1 \frac{2x}{2x+1}\,dx$$

$= \int_1^3 \frac{u-1}{u} \cdot \frac{1}{2}\,du$ ❶ change of variable correct

$$= \frac{1}{2}\int_1^3 1 - \frac{1}{u}\, du$$

$$= \frac{1}{2}\left[u - \ln u\right]_1^3$$

$$= \frac{1}{2}\left[\left(3 - \ln 3\right) - \left(1 - \ln 1\right)\right]$$

$$= \frac{1}{2}\left(2 - \ln 3\right)$$

$(= 1 - \frac{1}{2}\ln 3)$ ❶ correct answer

Question 12

(a) *Two points* $P\left(2ap, ap^2\right)$ *and* $Q\left(2aq, aq^2\right)$ *lie on the parabola* $x^2 = 4ay$.
The general tangent at any point on the parabola with parameter t is given by $y = tx - at^2$ *(do NOT prove this).*

i. *Find the co-ordinates of the point of intersection* T *of the tangents to the parabola at P and Q.* [2]

Tangents are

$y = px - ap^2$

$y = qx - aq^2$ and solving simultaneously:

$$0 = px - qx - ap^2 + aq^2$$

$$\left(q - p\right)x = a\left(q^2 - p^2\right)$$

$$= a\left(q + p\right)\left(q - p\right)$$

$$x = a\left(p + q\right) \qquad \text{since } p \neq q$$

❶ correct value for *x*

$$y = p\left[a\left(p + q\right) - ap^2\right]$$

$$= ap^2 + apq - ap^2$$

$$\therefore \quad = apq$$

❶ correct value for *y*

ii. *You are given that the tangents at P and Q intersect at an angle of* 45°. *Show that* [1]

$$p - q = 1 + pq$$

$\tan\theta = \dfrac{m_1 - m_2}{1 + m_1 m_2}$, with $\theta = 45,\ m_1 = p,\ m_2 = q$:

$$\tan 45 = \frac{p - q}{1 + pq}$$

$$1 = \frac{p - q}{1 + pq}$$

$p - q = 1 + pq$ ❶ correct use of formula and correct algebra to result

iii. *By evaluating the expression* $x^2 - 4ay$, *or otherwise, find the locus of the point T when the tangents at P and Q meet as described in part ii above.* [2]

$$x^2 - 4ay$$

$$= \left[a\left(p + q\right)\right]^2 - 4a\left(apq\right)$$

$$= a^2\left(p + q\right)^2 - 4a^2 pq$$

$$= a^2\left(p^2 + 2pq + q^2\right) - 4a^2 pq$$

$$= a^2\left(p^2 + 2pq + q^2 - 4pq\right)$$

$$= a^2\left(p^2 - 2pq + q^2\right)$$

$$= a^2\left(p - q\right)^2$$

From $x^2 = 4ay := a^2\left(1 + pq\right)^2$ since $p - q = 1 + pq$ ❶

$= a^2\left(1+2pq+p^2q^2\right)$

$= a^2+2a^2pq+a^2p^2q^2$

$= a^2+2a(apq)+(apq)^2$

$= a^2+2ay+y^2$ since $y = apq$

Thus $x^2-4ay = a^2+2ay+y^2$, leading to the equation of the locus of *T* being

$x^2 = a^2+6ay+y^2$ ❶

(b) For the function given by $f(x) = -1+\sqrt{x+4}$

i. State the domain for the function $f(x)$. *[1]*

$x \geq -4$

ii. Find the inverse function $f^{-1}(x)$ *for the given function* $f(x)$. *[2]*

Consider $y = -1+\sqrt{x+4}$: swapping *x* and *y* gives

$x = -1+\sqrt{y+4}$

$x+1 = \sqrt{y+4}$

$(x+1)^2 = y+4$ ❶ swapping and squaring correct

$y = (x+1)^2-4$

$y = x^2+2x-3$ ❶ correct answer

iii. Find the restrictions on the domain and range for $f^{-1}(x)$ *to be the inverse function of* $f(x)$. *[1]*

For $f(x)$, with $x \geq -4$ this leads to $y \geq -1$. For $f^{-1}(x)$, these reverse, so the restrictions on $f^{-1}(x)$ are $x \geq -1$ and $f^{-1}(x) \geq -4$ ❶ correct answer

(c) Prove by Mathematical Induction that n^3+2n *is divisible by 3, for all positive integer n.* *[3]*

To prove $n^3+2n = 3N$, where *N* is an integer:

Let $n = 1$:

n^3+2n

$= 1^3+2.1$

$= 3$ which is divisible by 3, hence the statement is true for $n = 1$.

Assume true for $n = k$

i.e. assume $k^3+2k = 3M$, M an integer

or $k^3 = 3M-2k$ ❶ initial value and assumption correct

Then show true for $n = k+1$

i.e. show $(k+1)^3+2(k+1) = 3Q$, Q an integer

$LHS = (k+1)^3+2(k+1)$

$= k^3+3k^2+3k+1+2k+2$

$= k^3+3k^2+5k+3$

$= 3M-2k+3k^2+5k+3$ by assumption ❶

$$= 3M + 3k^2 + 3k + 3$$
$$= 3\left(M + k^2 + k + 1\right)$$
$$= 3Q \qquad Q \text{ an integer.}$$

Hence, as true for $n = 1$, by the principle of mathematical induction, the statement is true for all integer *n*. ❶ correct resolution

(d) For $0 \le \theta \le 2\pi$ *, find all the solutions of* $\sin 2\theta = -\cos\theta$. *[3]*

$$2\sin\theta\cos = -\cos\theta$$

$$2\sin\theta\cos + \cos\theta = 0$$

$$\cos\theta\left(2\sin\theta + 1\right) = 0$$ ❶ correct factors

Hence

$\cos\theta = 0$ $\qquad$ $2\sin\theta + 1 = 0$

$\theta = \frac{\pi}{2}, \frac{3\pi}{2}$ ❶ $\qquad$ $\sin\theta = \frac{-1}{2}$

$\theta = \frac{7\pi}{6}, \frac{11\pi}{6}$ ❶

Question 13:

(a) *The rate at which a body warms in air is proportional to the difference in temperature T of the body and the constant temperature A of the surrounding air. This rate is given by the differential equation*

$$\frac{dT}{dt} = k\left(T - A\right)$$

where t is the time in minutes and k is a constant.

i. *Show that* $T = A + A_0 e^{kt}$*, where* A_0 *is a constant, is a solution of this equation.* [1]

$T = A + A_0 e^{kt}$ ①

$\frac{dT}{dt} = kA_0 e^{kt}$

But from ①, $TA_0 e^{kt} = T - A$

$\therefore \frac{dT}{dt} = k\left(T - A\right)$, hence $T = A + A_0 e^{kt}$ is a solution to the equation.

❶ correct resolution

ii. *A cold body, initially at* 5° *C, warms to* 10° *C in 20 minutes. The air temperature around the body is* 25° *C. Find the temperature of the body after a further 40 minutes have elapsed. Give your answer to the nearest degree.* [3]

$A = 25,$ hence $T = 25 + A_0 e^{kt}$

When $t = 0, T = 5$, hence

$5 = 25 + A_0 e^{k\times 0}$

$-20 = A_0$

$\therefore T = 25 - 20e^{kt}$ ❶ correct resolution of initial constants

Now when $t = 10, T = 10$ gives

$10 = 25 - 20e^{20k}$

$-15 = -20e^{20k}$

$e^{20k} = \frac{3}{4}$

$20k = \ln\left(\frac{3}{4}\right)$

$k = \frac{1}{20}\ln\left(\frac{3}{4}\right)$ ❶ correct resolution of *k*

Then, when $t = 60$:

$T = 25 - 20e^{60\times\frac{1}{20}\ln\left(\frac{3}{4}\right)}$

$= 25 - 20e^{\ln\left(\frac{3}{4}\right)^3}$

$= 25 - 20.\left(\frac{3}{4}\right)^3$

$\doteq 16.5625$

$\doteq 17^\circ$ ❶ correct answer

(b) *The acceleration of a particle moving in a straight line is given by*

$$\frac{d^2x}{dt^2} = 2x - 3$$

where x is the displacement, in metres, from the origin O and t is the time in seconds. Initially the particle is at rest at $x = 4$.

i. *If the velocity of the particle is v m/s, show that* [1]

$$v^2 = 2\left(x^2 - 3x - 4\right)$$

$$\frac{d^2x}{dt^2} = 2x - 3$$

$$\frac{d\left(\frac{1}{2}v^2\right)}{dx} = 2x - 3$$

$$\frac{1}{2}v^2 = \int 2x - 3dx$$

$$= x^2 - 3x + c$$

At $x = 4, v = 0$:

$$0 = 4^2 - 3.4 + c$$

$$c = -4$$

$$\therefore \frac{1}{2}v^2 = x^2 - 3x - 4$$

or $v^2 = 2x^2 - 6x - 8$ ❶ correct answer

ii. *Show that the particle does not pass through the origin.* [1]

At $x = 0$:

$v^2 = -8$, but this is impossible, hence the particle does not pass through the origin.
❶ correct answer with justification

iii. *Determine the position of the particle when* $v = 10$. *Justify your answer.* [2]

When $v = 10$:

$$10^2 = 2x^2 - 6x - 8$$

$$100 = 2x^2 - 6x - 8$$

$$0 = x^2 - 3x - 54$$

$= \left(x - 9\right)\left(x + 6\right)$ ❶ correct solutions

Hence, $x = 9$ or $x = -6$, but the particle starts at $x = 4$ and never passes through the origin ($x = 0$), so $x = -6$ is not an acceptable answer.

$\therefore$ The position of the particle when $v = 10$ is $x = 9$. ❶ justification correct

(c) *For the graph of* $y = \frac{2x+1}{x-1}$

i. *Find the horizontal asymptote of the graph.* [1]

$$y = \frac{2x+1}{x-1}\sqrt{b^2 - 4ac}$$

$$= \frac{2x - 2 + 3}{x - 1}$$

$$= \frac{2\left(x-1\right)+3}{x-1}$$

$$= 2 + \frac{3}{x-1}$$

$$y - 2 = \frac{3}{x-1}$$

$$3 = \left(x-1\right)\left(y-2\right)$$

Now, as $\left(x-1\right)\left(y-2\right) \neq 0$, then

$x \neq 1, y \neq 2$ gives a horizontal asymptote at $y = 2$. ❶ asymptote correct

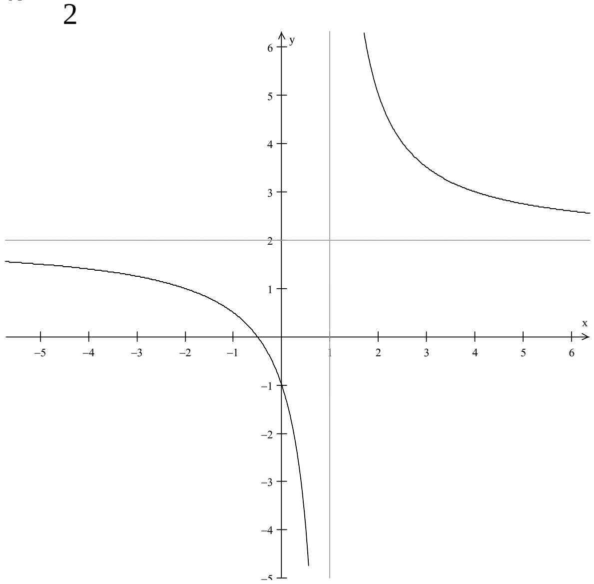
ii. *Without the use of calculus, sketch the graph of* $y = \frac{2x+1}{x-1}$, *showing the asymptote found in part (i).* [2]

Noting asymptotes at $y = 2$ and $x = 1$, and intercepts of $x = 0$;

$$3 = (0-1)(y-2)$$
$$y = -1$$

$y = 0$;

$$3 = (x-1)(0-2)$$
$$x = \frac{-1}{2}$$



❶ asymptotes, ❶ branches/intercepts correct

(d) *The velocity v m/s of a particle moving in simple harmonic motion along the x-axis is given by*

$$v^2 = 8 + 2x - x^2$$

i. *Between which two points is the particle oscillating?* [1]

$$v^2 = 8 + 2x - x^2$$
$$= (4-x)(2+x)$$

Thus at $x = -2$ and $x = 4$, $v = 0$.

∴ The particle oscillates between $x = -2$ and $x = 4$. ❶ values correct

ii. *What is the amplitude of the motion?* [1]

Amplitude is $\frac{4 - ^{-}2}{2}$ ❶ correct value

$$= 3$$

iii. *Find the acceleration of the particle is terms of x.* [1]

$$v^2 = 8 + 2x - x^2$$

$\frac{1}{2}v^2 = 4 + x - \frac{1}{2}x^2$, then

$$a = \frac{d\left(\frac{1}{2}v^2\right)}{dx}$$
$$= \frac{d}{dx}\left(4 + x - \frac{1}{2}x^2\right)$$
$$= 1 - x$$
$$= -(x-1)$$

❶ correct acceleration

iv. Find the period of oscillation. *[1]*

$\ddot{x} = -1^2 \left(x - 1 \right)$

So $n = 1$, and period

$T = \dfrac{2\pi}{n}$

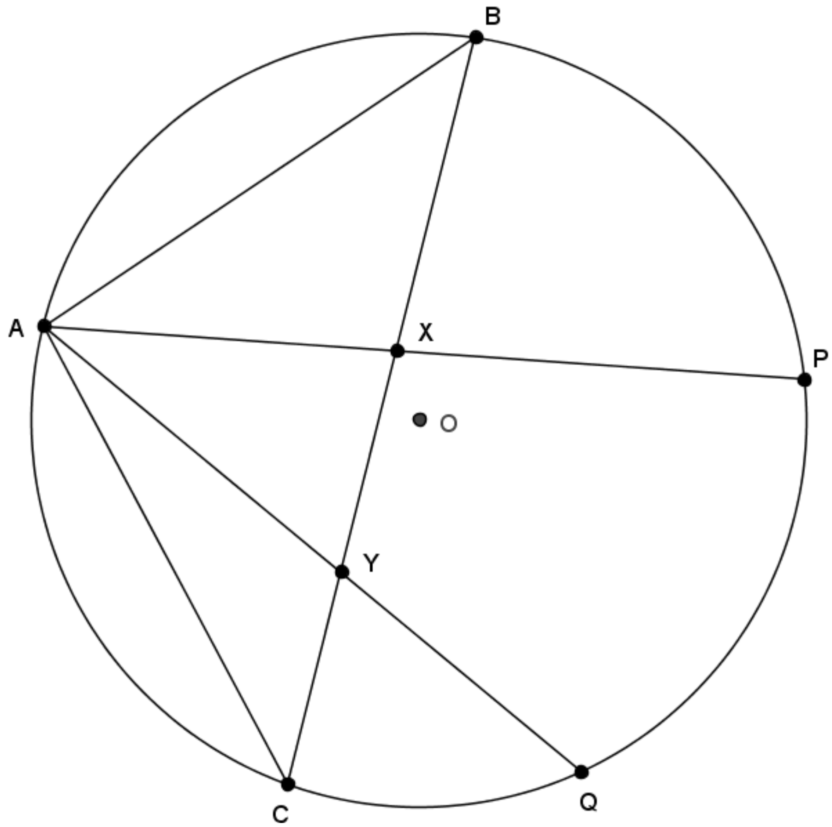
$= 2\pi$ secs. ❶ correct value

Question 14:

(a) *Let ABPQC be a circle such that AB=AC, AP meets BC at X and AQ meets BC at Y, as shown below. Let* $\angle BAP = \alpha$ *and* $\angle ABC = \beta$.

i. *Copy the diagram into your writing booklet, marking the information given above, and state why* $\angle AXC = \alpha + \beta$. *[1]*

i) Well done, although most students did it the long way



$\angle AXC$ is the external angle to ΔABX, which is equal to the opposite interior angles. ❶ correct reason

ii. *Prove* $\angle BQP = \alpha$. *[1]*

Construction: join *BQ*, *PQ*.

$\angle BQP = \angle BAP$ (angles in same segment on arc *BP*) ❶ correct reason

$= \alpha$

ii) Well done, although students who used "angles subtended by the same arc" should also write "at the circumference."

iii. *Prove* $\angle BQA = \beta$. *[1]*

$\angle BQA = \angle BCA$ (angles in same segment on arc *AB*)

$\angle BCA = \angle ABC$ (ΔABC isosceles, given *AB*=*BC*)

$= \beta$

Hence $\angle BQA = \beta$ ❶ correct reason

iii) This was poorly done. Only a handful of students knew the rule "equal chords subtend equal angles ath the circumference".

iv. *Prove the quadrilateral PQYX is cyclic.* *[2]*

$\angle AXC = \alpha + \beta$ (from (i))

$\angle PQA = \angle BQP + \angle BQA$

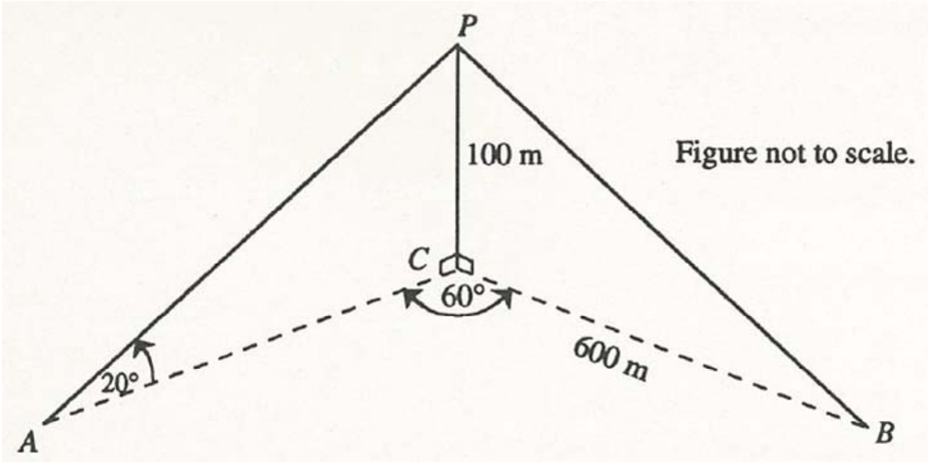
$= \alpha + \beta$ (from (ii) and (iii)) ❶ use of previous parts correctly

$\therefore \angle AXC = \angle PQA$

$\therefore$ *PQXY* is a cyclic quadrilateral (external $\angle$ eq. opp. int. $\angle$) ❶ correct reason

iv) Well done, but again most students did it the long way.

(b) Two yachts, A and B, subtend an angle of 60° *at the base C of a cliff C.*



From yacht A, the angle of elevation to point P, 100 m vertically above C, is 20°. *Yacht B is 600m from C.*

i. Calculate length AC. *[1]*

In ΔACP:

$$\tan 20 = \frac{100}{AC}$$

$$AC = \frac{100}{\tan 20}$$

$$= 274.7477...$$

$\approx 275m$ (to nearest m) ❶ correct answer

ii. Calculate the distance between the two yachts. *[2]*

In ΔABC:

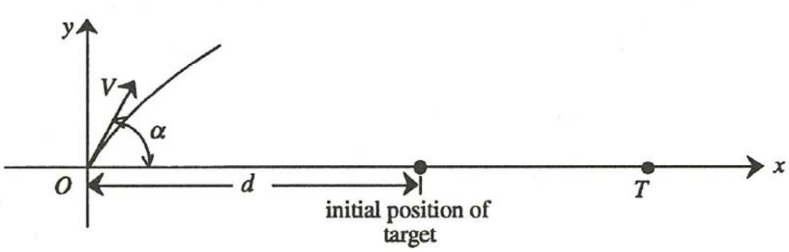
$$AB^2 = AC^2 + BC^2 - 2.AC.BC.\cos 60$$

$= AC^2 + 600^2 - 2 \times AC \times 600 \times \frac{1}{2}$, then using calculator memory for AC, ❶ correct substitutions

$$AB = 520.2284$$

$\approx 520m$ (to nearest m) ❶ correct answer

(c) A projectile, with initial speed V_0 *m/s, is fired at an angle of elevation* α *from the origin at O towards a target T, which is moving away from O along the x-axis.*



You may assume that the projectiles trajectory is defined by the equations

$$x = Vt\cos\alpha \quad y = Vt\sin\alpha - \frac{1}{2}gt^2$$

Very well done. Only a handful of students lost any marks. The most common errors were:

$$\tan 20 = \frac{100}{AC}$$

$$\therefore AC = 100\tan 20$$

or using the incorrect trig ratio.

where x and y are the horizontal and vertical displacements of the projectile in metres at time t seconds after firing, and where g is the acceleration due to gravity.

i. *Show that the projectile is above the x-axis for a total of* $\dfrac{2V\sin\alpha}{g}$ *seconds.* *[1]*

i) Well done.

The particle returns to the *x*-axis when *y*=0. Hence

$$0 = Vt\sin\alpha - \frac{1}{2}gt^2$$

$= t\left(V\sin\alpha - \frac{1}{2}gt\right)$, and so *t*=0 or $V\sin\alpha - \frac{1}{2}gt = 0$, which leads to (❶ correct solving of quadratic)

$$\frac{1}{2}gt = V\sin\alpha$$

$t = \dfrac{2V\sin\alpha}{g}$, thus the particle is above the *x*-axis for $\dfrac{2V\sin\alpha}{g}$ seconds as reqd.

ii. *Show that the horizontal range of the projectile is* $\dfrac{2V^2\sin\alpha\cos\alpha}{g}$ *metres.* *[1]*

ii) Well done.

The horizontal range is is the value of *x* for *t* found in (*i*), i.e.

$$x = V.\left(\frac{2V\sin\alpha}{g}\right)\cos\alpha$$

$= \dfrac{2V^2\sin\alpha\cos\alpha}{g}$ as reqd. ❶ correct substitution

iii. *At the instant the projectile is fired, the target T is d metres from O and it is moving away at a constant speed of u m/s.*
Suppose that the projectile hits the target when fired at an angle of elevation α. *Show that* *[3]*

$$u = V\cos\alpha - \frac{gd}{2V\sin\alpha}$$

iii) Mixed results. Many students didn't derive $x = ut + d$ or it's equivalent.

For the target, $\dfrac{dx}{dt} = u$, hence

$$x = \int u\,dt$$

$= ut + c$, and at $t = 0, x = d$, so $c = d$,

$\therefore x = ut + d$ ① ❶ derives target equation correctly

The projectile therefore hits the target after time $\dfrac{2V\sin\alpha}{g}$ (from part *i*) when

$x = \dfrac{2V^2\sin\alpha\cos\alpha}{g}$ (from part *ii*).

Thus, substituting these values in ① gives:

$\dfrac{2V^2\sin\alpha\cos\alpha}{g} = u\left(\dfrac{2V\sin\alpha}{g}\right) + d$ ❶ substitutes correct values

$$2V^2 \sin\alpha \cos\alpha = 2Vu \sin\alpha + gd$$

$$2Vu \sin\alpha = 2V^2 \sin\alpha \cos\alpha - gd$$

$$u = \frac{2V^2 \sin\alpha \cos\alpha}{2V \sin\alpha} - \frac{gd}{2V \sin\alpha}$$

$$= V \cos\alpha - \frac{gd}{2V \sin\alpha}$$ ❶ correct algebra to required result

iv. *Suppose the projectile is fired at an angle of* $\alpha = \frac{\pi}{4}$. *Deduce that the projectile will not hit the target if* *[2]*

$$u > \frac{\sqrt{2}\left(V^2 - gd\right)}{2V}.$$

If $\alpha = \frac{\pi}{4}$, then the maximum range of the projectile is $x_{\max} = \frac{V^2 \sin 2\alpha}{g}$ reached

$$= \frac{V^2}{g}$$

in time $t = \frac{2V \sin\alpha}{g}$

The target then must move beyond $x_{\max}$ in this same time, i.e.

$ut + d > \frac{V^2}{g}$ ❶ derives condition for a miss correctly

$\therefore u\left(\frac{2V \sin\alpha}{g}\right) > \frac{V^2}{g} - d$, and with $\sin\alpha = \sin\frac{\pi}{4} = \frac{1}{\sqrt{2}}$

$$\therefore u\left(\frac{2V}{g\sqrt{2}}\right) > \frac{V^2}{g} - d$$

$$u\left(\frac{2V}{g\sqrt{2}}\right) > \frac{V^2}{g} - d$$

$$u > \left(\frac{V^2}{g} - d\right).\left(\frac{g\sqrt{2}}{2V}\right)$$

$$> \frac{\sqrt{2}\left(V^2 - gd\right)}{2V}$$ as reqd. ❶ correct algebra to required result.

iv) Most students could substitute $\alpha = \frac{\pi}{4}$ to arrive at RHS of inequality.