Girraween High School

2015 Year 12 Trial Higher School Certificate

Mathematics Extension 2

General Instructions

- Reading time – 5 minutes
- Working time – 3 hours
- Write using black or blue pen
  Black pen is preferred
- Board-approved calculators may be used
- A table of standard integrals is provided at the back of this paper
- In Questions 11–16, show relevant mathematical reasoning and/or calculations

Total marks – 100

Section I

10 marks

- Attempt Questions 1–10
- Allow about 15 minutes for this section

Section II

90 marks

- Attempt Questions 11–16
- Allow about 2 hours and 45 minutes for this section

- For **Section II: Questions 11 – 16** MUST be returned in clearly marked ***separate* sections.**
- On each page of your answers, clearly write:
  - the **QUESTION** being answered
  - **YOUR NAME**
  - your **Mathematics TEACHER'S NAME.**
- Start each new question on a **NEW PAGE.**
- You may ask for extra pieces of paper if you need them.

Multiple choice: Questions 1 – 10: Colour in the correct answer on your multiple choice answer sheet.

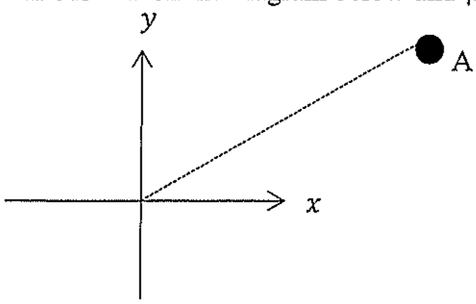
Question 1

If $z = 1 + 2i$ and $w = 3i - 4$ then $z\overline{w} =$

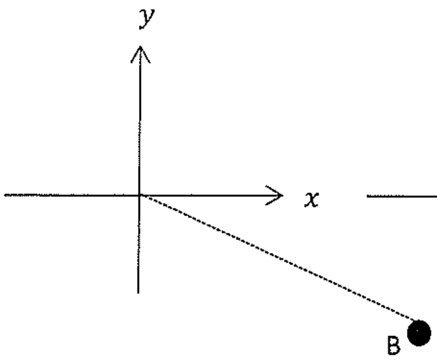
(A) $-2 + 11i$ (B) $2 - 11i$ (C) $2 - 3i$ (D) $4 + i$

Question 2

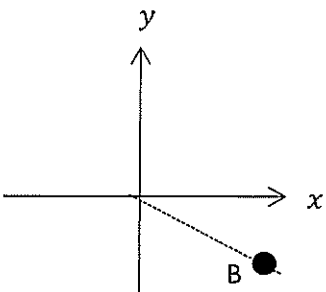
If $\overrightarrow{OA} = z$ on the diagram below and $|z| > 1$ then $\overrightarrow{OB} = \frac{1}{z}$ could be



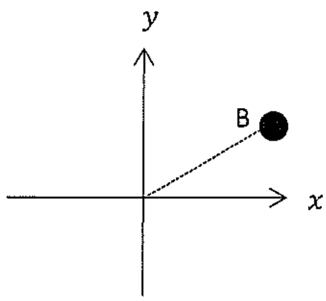
(A)



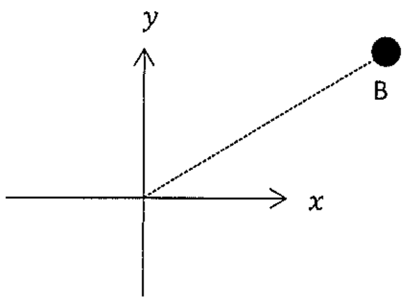
(B)



(C)



(D)



Question 3 In modulus and argument form, $\sqrt{5} - i\sqrt{15}$ is

(A) $2\sqrt{5}\ cis - \frac{\pi}{3}$ (B) $2\sqrt{5}\ cis - \frac{\pi}{6}$ (C) $2\sqrt{5}\ cis\ \frac{\pi}{6}$ (D) $2\sqrt{5}\ cis\ \frac{\pi}{3}$

Question 4

If $\alpha, \beta, and\ \gamma$ are the roots of the polynomial equation $24x^3 - 14x^2 - 11x + 6 = 0$ then the polynomial equation with roots $\frac{1}{\alpha}, \frac{1}{\beta}\ and\ \frac{1}{\gamma}\ is$

(A) $6x^3 - 11x^2 - 14x + 24 = 0$ (B) $6x^3 - 5x^2 - 22x + 24 = 0$

(C) $24x^3 - 11x^2 - 14x - 6 = 0$ (D) $6x^3 + 5x^2 - 22x + 24 = 0$

Question 5

The directices of the hyperbola $\frac{y^2}{9} - \frac{x^2}{16} = 1$ are

(A) $x = \pm\frac{9}{5}$ (B) $y = \pm\frac{9}{5}$ (C) $y = \pm 5$ (D) $x = \pm 5$

Question 6

The foci of the ellipse $\frac{(x-1)^2}{25}+\frac{(y+2)^2}{9}=1$ are

(A) $(\pm 4{,}0)$ (B) $(0,\pm 4)$ (C) $(1{,}2)\, and\ (1,-6)$ (D) $(5,-2)\, and\ (-3,-2)$

Question 7

$$\int \frac{\cos x}{\cos 2x - 2}.dx =$$

(A) $\frac{1}{\sqrt{3}} tan^{-1}(\sin x) + C$ (B) $\frac{1}{\sqrt{3}} tan^{-1}\left(\frac{\text{sinx}}{\sqrt{3}}\right)+C$ (C) $-\frac{1}{\sqrt{2}} tan^{-1}\left(\frac{\text{sinx}}{\sqrt{2}}\right)+C$

(D) $-\frac{1}{\sqrt{2}} tan^{-1}\left(\sqrt{2}\sin x\right)+C$

Question 8

The volume obtained by rotating the area enclosed by $y = 4 - x^2$ and the x axis about the line $y = 4$

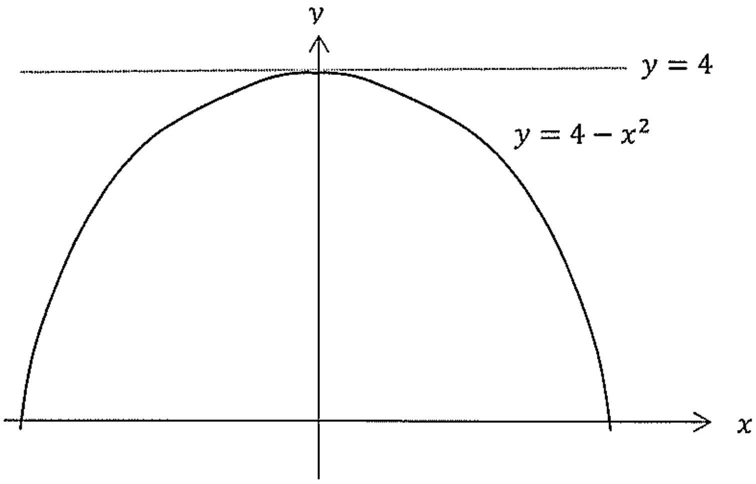


DIAGRAM NOT TO SCALE

would be obtained using the expression

(A) $\pi\int_{-2}^{2}(16 - 8x^2 + x^4).dx$ (B) $\pi\int_{-2}^{2}(16 - x^4).dx$

(C) $4\pi\int_0^4 (4-y)\sqrt{4-y}\,.dy$ (D) $4\pi\int_0^4 \sqrt{4-y}\,.dy$

Question 9

A particle is launched vertically upwards. It experiences air resistance which is proportional to the square of its velocity and gravity. Given up is positive, its acceleration could be expressed as:

(A) $\ddot{x} = -g - kv^2$ (B) $\ddot{x} = g + kv^2$ (C) $\ddot{x} = g - kv^2$ (D) $\ddot{x} = -g + kv^2$

Question 10

Taking into account only tension and gravity in the conical pendulum below, in which the radius of rotation is r , the height is h, the semi-vertical angle is α the tension in the string is T and particle has a mass of m kilograms, the angular velocity w can be obtained using

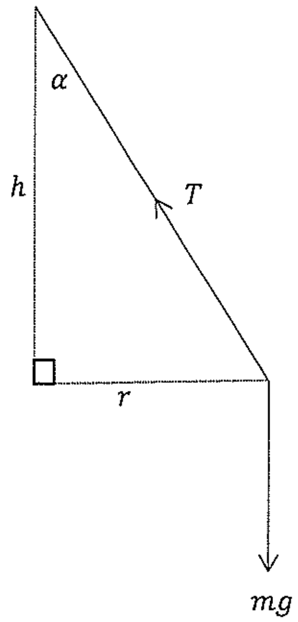


DIAGRAM NOT TO SCALE

(A) $w = \sqrt{\frac{h}{r}}$ (B) $w = \sqrt{\frac{r}{h}}$ (C) $w = \sqrt{\frac{g}{h}}$ (D) $w = \sqrt{\frac{h}{g}}$

Question 11 ***(15 marks) Show all necessary working on a separate page*** **Marks**

(a) If $z = -\sqrt{3} + i$ and $w = 1 + i$

(i) Find $\frac{z}{w}$ in Cartesian form. **2**

(ii) Convert both z and w to modulus/argument form. **3**

(iii) Use your answers to (i) and (ii) to find the exact value of $cos\,\frac{7\pi}{12}$. **1**

(b) (i) If $(x + iy)^2 = 7 - 24i, x, y$ real find the exact values of x and y. **3**

(ii) Hence solve the equation $2z^2 + 6z + (1 + 12i) = 0$. **2**

(c) Solve the polynomial equation $24x^4 + 172x^3 + 390x^2 + 225x - 125 = 0$ given that it has a triple root. **4**

Examination continues on the next page

Question 12 *(15 marks) Show all necessary working on a separate page* **Marks**

(a) Find

(i) $\int 3xcosx.dx$ **2**

(ii) $\int \frac{1}{1+sin2x}.dx$ **2**

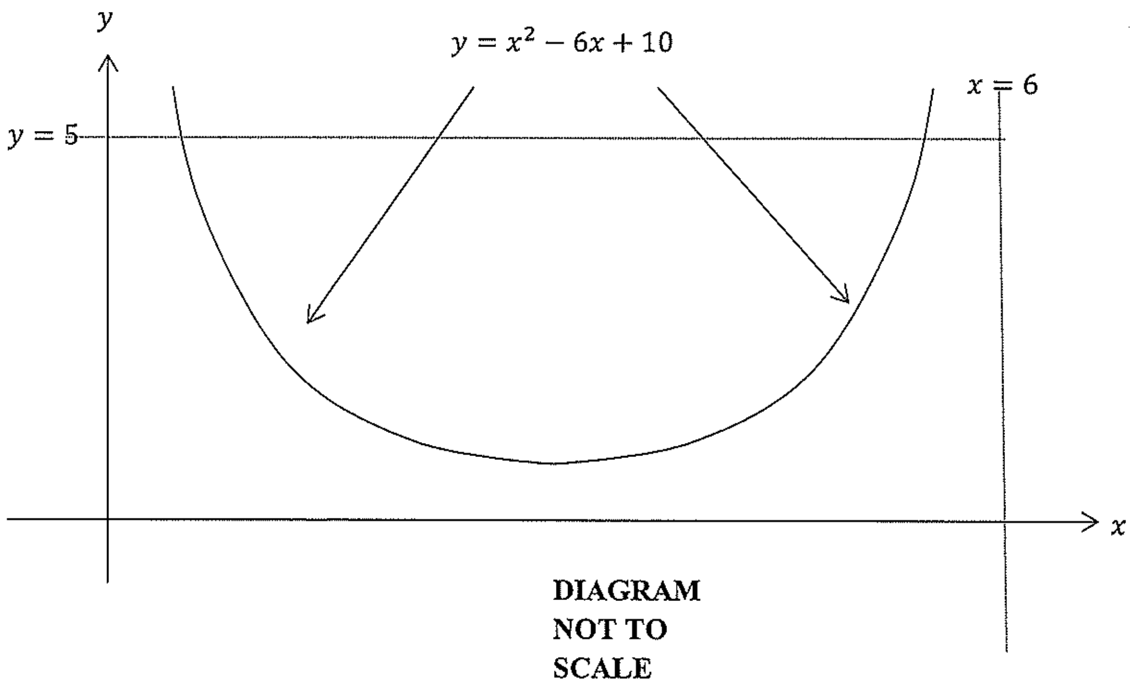
(iii) $\int_0^1 \frac{e^{2x}}{e^{4x}+1}.dx$ **3**

(b) (i) Express $\frac{7x^2-11x+7}{(3x-1)(x^2+4)}$ in the form $\frac{A}{3x-1}+\frac{Bx+C}{x^2+4}$ **3**

(ii) Hence find $\int \frac{7x^2-11x+7}{(3x-1)(x^2+4)}$ .dx **1**

(c)The area enclosed by the curve $y = x^2 - 6x + 10$ and the line $y = 5$ **4**

is rotated around the line $x = 6$ *(See diagram)*



Using the method of cylindrical shells, find the volume of the solid of revolution formed by this.

Examination continues on the next page

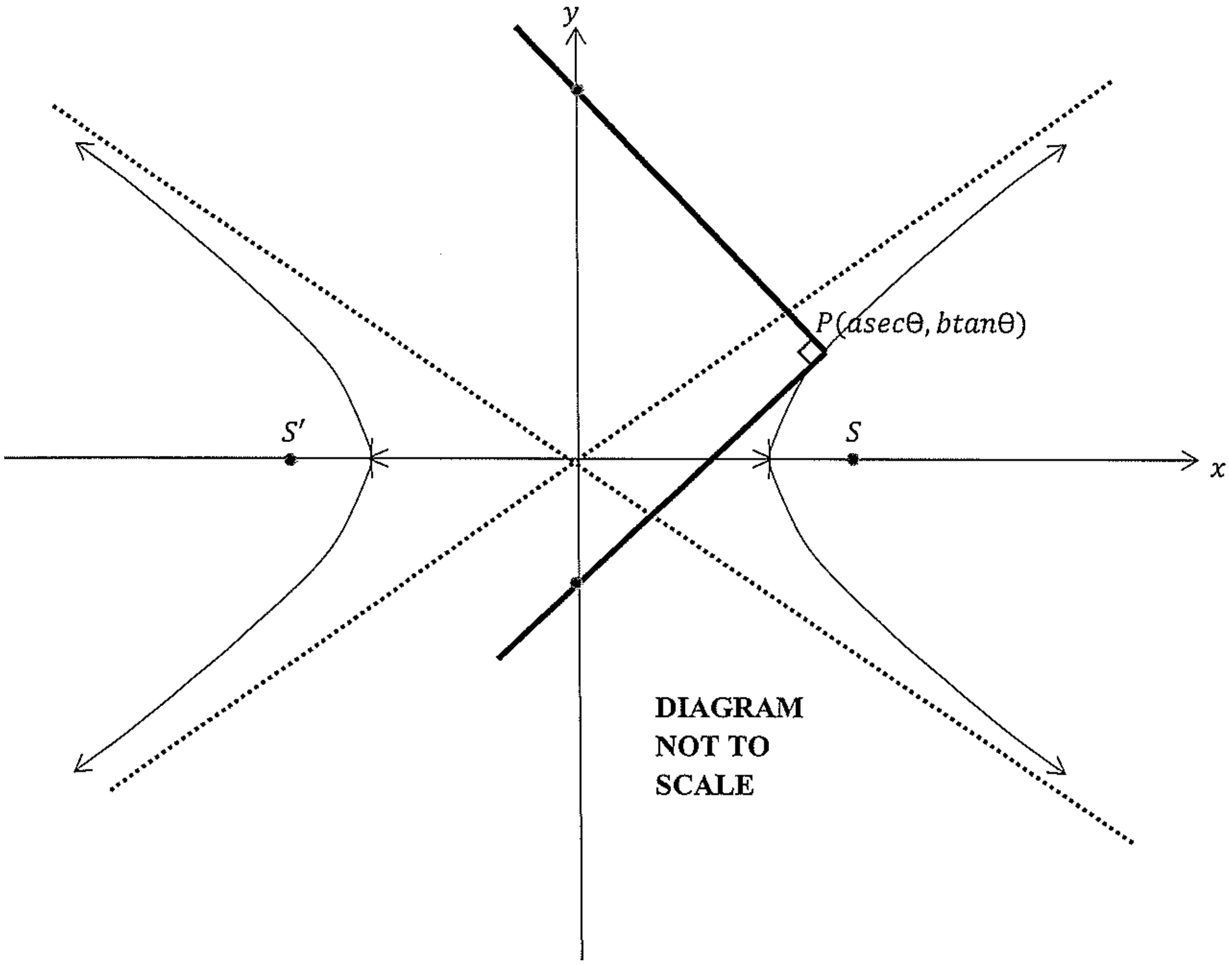
Question 13 ***(15 marks) Show all necessary working on a separate page*** **Marks**

(a) (i) Show that the equations of the tangent and normal to the hyperbola **3**

$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ at the point $P(a\sec\Theta, b\tan\Theta)$ are $bx\sec\Theta - ay\tan\Theta = ab$

and $by\sec\Theta + ax\tan\Theta = (a^2 + b^2)\sec\Theta\tan\Theta$ respectively.

(ii) The tangent and normal cut the y axis at M and N respectively *(see diagram).* **5**



Show that MN is a diameter of the circle $MSNS'$.

(b) A circular bend at a velodrome (bicycle track) with a radius of 100 metres is banked so that a cyclist riding at $14m/s$ experiences no friction. A cyclist and bicycle with a combined mass of $80kg$ are riding around the track. Letting $g = 9.8m/s^2$.

(i) By resolving forces in the vertical and horizontal directions or otherwise show that the curve is banked at an angle of $11^o19'$ to the horizontal. **1**

Question (13)(b) continues on the next page

(ii) The cyclist increases her pace to $15m/s$. Show that the lateral friction she experiences is given by $F = 180\cos 11^o19 - 80g\sin 11^o19$ and find the value of this friction in Newtons. **3**

(iii)The maximum value of this friction is 0.1 times the normal force. What is the slowest speed the cyclist can ride at without slipping down the slope? **3**

Question 14 ***(15 marks) Show all necessary working on a separate page***

(a) If $I_n = \int x(lnx)^n . dx$

(i) Show that $I_n = \frac{1}{2}x^2(lnx)^n - \frac{1}{2}nI_{n-1}$ **2**

(ii) Hence find the value of $\int_1^2 x(lnx)^2 . dx$ **3**

(b) $A(a,0), P(acos\Theta, bsin\Theta)$ and $Q(acos\alpha, bsin\alpha)$ are located on the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ so that $\angle PAQ = 90^0$. *(see diagram)* **5**

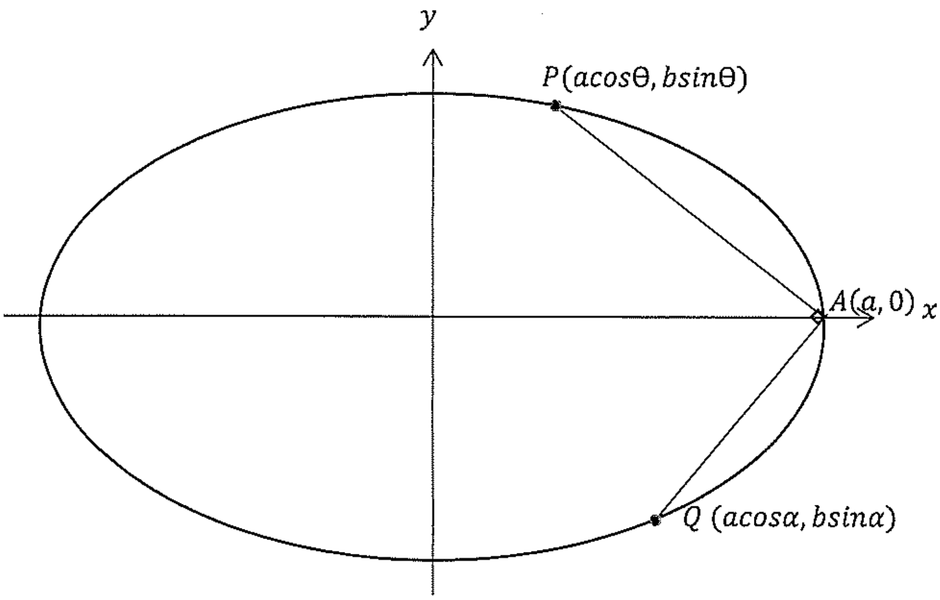


DIAGRAM NOT TO SCALE

Show that $tan\frac{\alpha}{2}tan\frac{\Theta}{2} = -\frac{b^2}{a^2}$

Question (14) continues on the next page

Question (14) continued **Marks**

(c)Two particles A and B of mass M and m respectively with $M > m$ are attached to each end of a light, inelastic string. Particle A hangs directly below a ring at O and particle B rotates in a circle with velocity vm/s at the end of a part of the string which is at an angle of α to the vertical *(See diagram)*

The tension is the same through the whole string.

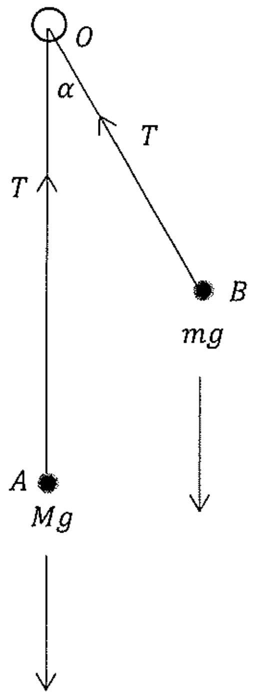


DIAGRAM NOT TO SCALE

(i) Show that $cos\alpha = \frac{m}{M}$. **2**

(ii) Find an expression for the radius of rotation in terms of M, m, v and g (but NOT α). **3**

Examination continues on the next page

Question 15 ***(15 marks) Show all necessary working on a separate page*** **Marks**

(a) The area enclosed within the circle $x^2 + y^2 = 4$ is rotated around the line $x = 5$ *(See diagram)*

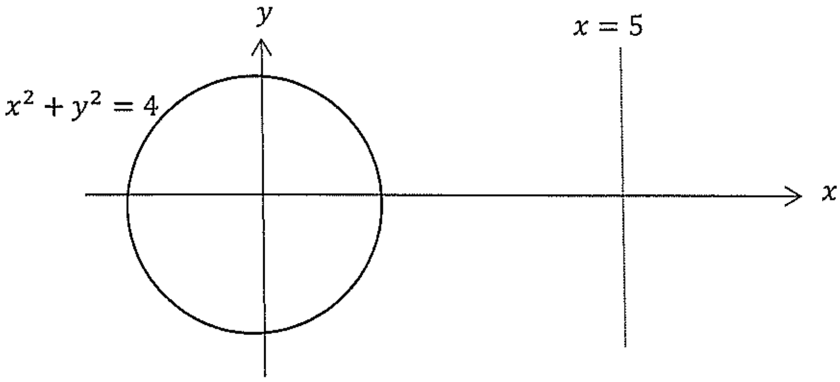


DIAGRAM NOT TO SCALE

(i) By taking strips *perpendicular* to the axis of rotation show that $\delta V = 20\pi\sqrt{4 - y^2}.\delta y$ **2**

(ii) Find the volume of the *torus* formed. **3**

(b) A wedge of wood has a circular cross-section at one end and a vertical straight edge at the other end The circle has a radius of $40cm$, the straight edge is $80cm$ long and the length of the wood *perpendicular* to each end is $3m$.
(See diagram)

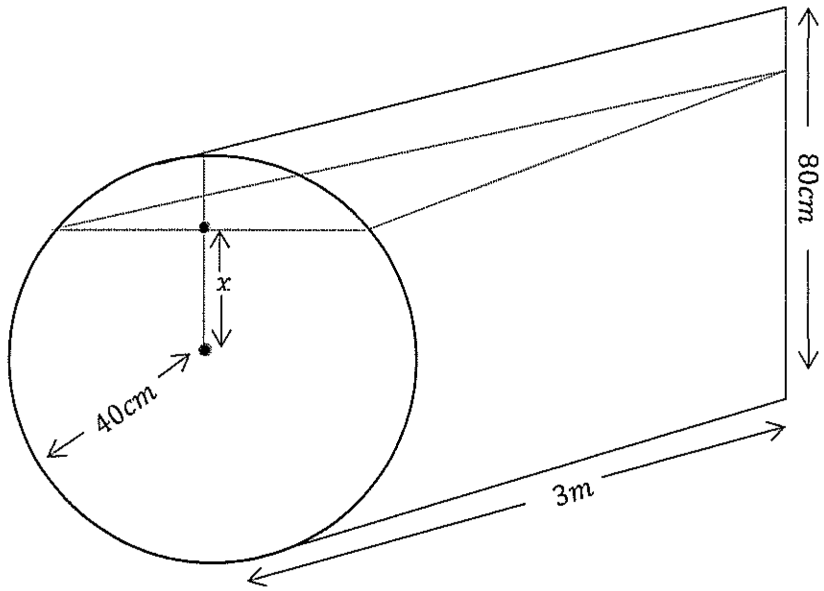


DIAGRAM NOT TO SCALE

(i) Given that $x\ cm$ is the distance of each triangular slice vertically up from the centre of the circle, show that the volume of each horizontal triangular slice is given by **2**
$\delta V = 300\sqrt{1600 - x^2}.\delta x$

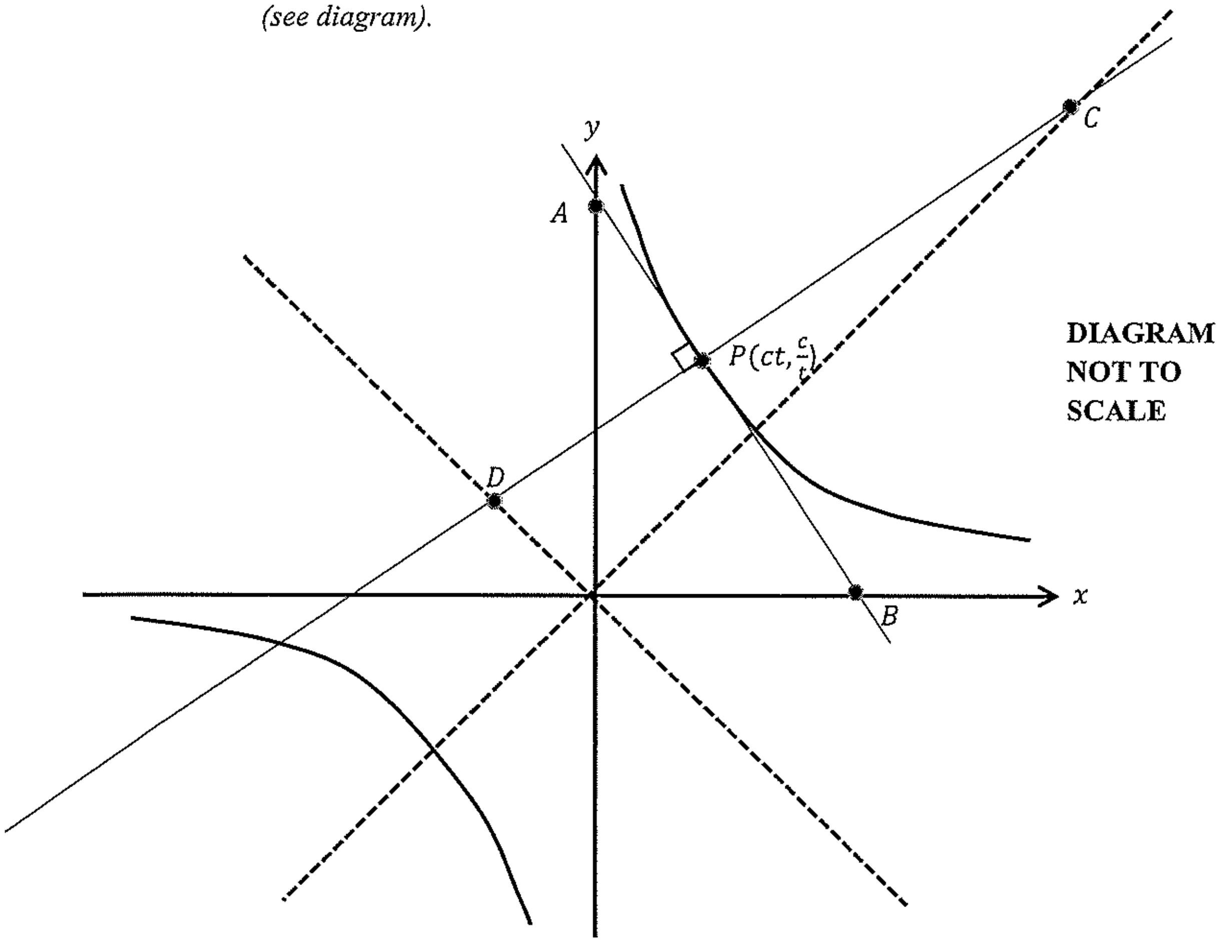
(ii) Find the exact volume of the wedge in cm^3. **3**

Question (15) continues on the next page

Question (15) continued **Marks**

(c)The point $P(ct, \frac{c}{t})$ lies on the rectangular hyperbola $xy = c^2$ where $t \neq \pm 1$. The tangent to the hyperbola at P intersects the coordinate axes at A and B.The normal to the hyperbola at P intersects the *axes of symmetry* of the hyperbola at C and D.

(see diagram).



(i) Given that the equation of the tangent to the hyperbola at P is $x + t^2y = 2ct$, find the coordinates of A and B. **1**

(ii) Given that the equation of the normal to the hyperbola at P is $t^3x - ty = c(t^4 - 1)$ find the coordinates of C and D. **1**

(iii) Show that $PA = PB = PC = PD$ and state why $ACBD$ is a square. **3**

Examination continues on the next page

Question 16 *(15 marks) Show all necessary working on a separate page* **Marks**

(a) A particle with mass m is fired vertically upwards from the Earth's surface at Um/s. Ignoring air resistance, the particle is under the influence of gravity, which is inversely proportional to the square of the distance of the particle from the centre of Earth. At the Earth's surface the force of gravity acting on the particle is mg. If the Earth's radius is R:

(i) Show that $v^2 = \frac{2gR^2}{x} + U^2 - 2gR$ and find the *escape velocity* for Earth in m/s if $R = 6366km$ and $g = 9.8m/s^2$. **3**

(ii) If $U^2 = gR$ show that the particle reaches a height of R *above* the Earth's surface. **1**

(iii) Also if $U^2 = gR$ show that the time taken to reach a height of R *above* the Earth's surface is given by **3**

$$t = \int_R^{2R} \frac{1}{\sqrt{\frac{2gR^2}{x} - gR}} . dx$$

and find this time in terms of R and g.

(b) Let α be a root of the polynomial equation $x^4 + Ax^3 + Bx^2 + Ax + 1 = 0$ where $(2 + B)^2 \neq 4A^2$.

(i) Show that α can not equal 1 or -1. **1**

(ii) Show that $\frac{1}{\alpha}$ is also a root of $x^4 + Ax^3 + Bx^2 + Ax + 1 = 0$. **1**

(iii) Show that if both α and $\frac{1}{\alpha}$ are both *multiple* roots of $x^4 + Ax^3 + Bx^2 + Ax + 1 = 0$ then $4B = 8 + A^2$. **2**

(c) (i) Using DeMoivre's theorem, show that $s3\theta = 4cos^3\theta - 3cos\theta$. **1**

(ii)Show that $x = 2\sqrt{3}\cos$ is a solution to $x^3 - 9x = 9$ if $cos3\theta = \frac{\sqrt{3}}{2}$. **1**

(iii)Solve $x^3 - 9x = 9$ giving your solutions to four decimal places. **2**

Here endeth the examination!!!

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Trial Exam 2015 [Girraween HS]

Solutions/Marking Scheme

Q. (1) B (2) B (3) A (4) A (5) B (6) D (7) D (8) C or B (9) A (10) C

(11)(a)(i) $\dfrac{z}{w}$

$= \dfrac{-\sqrt{3}+i}{1+i} \times \dfrac{(1-i)}{(1-i)}$ 1

$= \dfrac{-\sqrt{3}+i\sqrt{3}+i+1}{2}$

$= \left(\dfrac{1-\sqrt{3}}{2}\right) + i\left(\dfrac{1+\sqrt{3}}{2}\right)$ 1

2

(ii) $z = 2\operatorname{cis}\dfrac{5\pi}{6}$ 1

$w = \sqrt{2}\operatorname{cis}\dfrac{\pi}{4}$ 1

3

(iii) $\dfrac{z}{w} = \dfrac{2\operatorname{cis}\frac{5\pi}{6}}{\sqrt{2}\operatorname{cis}\frac{\pi}{4}}$

$= \sqrt{2}\operatorname{cis}\dfrac{7\pi}{12}$ 1

Equating real parts of (i) & (iii)

$\dfrac{1-\sqrt{3}}{2} = \sqrt{2}\cos\dfrac{7\pi}{12}$

$\dfrac{1-\sqrt{3}}{2\sqrt{2}} = \cos\dfrac{7\pi}{12}$

(11)(b)(i) $(x+iy)^2 = 7-24i$

$x^2+2ixy-y^2 = 7-24i$

$x^2-y^2 = 7$ (1)

$2xy = -24$

$\therefore y = -\dfrac{12}{x}$ (2)

Sub. (2) in (1):

$x^2 - \left(-\dfrac{12}{x}\right)^2 = 7$

$x^2 - \dfrac{144}{x^2} = 7$

$x^4 - 144 = 7x^2$

$x^4 - 7x^2 - 144 = 0$

$(x^2-16)(x^2+9) = 0$

$x = \pm 4$ as x real.

If $x = 4$, $y = -\dfrac{12}{4} = -3$.

$x = -4$, $y = 3$.

Solutions to $(x+iy)^2 = 7-24i$ are

$x = \pm 4$, $y = \mp 3$.

(ii) $2z^2+6z+(1+12i) = 0$.

$\Delta = b^2 - 4ac$

$= 6^2 - 4\times 2\times(1+12i)$

$= 28-96i$

$= 4(7-24i)$

$\therefore z = \dfrac{-b\pm\sqrt{b^2-4ac}}{2a}$

$= \dfrac{-6\pm\sqrt{4(7-24i)}}{2\times 2}$

$= \dfrac{-6\pm 2(4-3i)}{4}$

$z = \dfrac{1-3i}{2}, \dfrac{-7+3i}{2}$.

Q. (11)(c) $P(x) = 24x^4 + 172x^3 + 390x^2 + 225x - 125 = 0$
Has TRIPLE

$P'(x)$: $96x^3 + 516x^2 + 780x + 225 = 0$ has DOUBLE.
$P''(x)$ $288x^2 + 1032x + 780 = 0$ has single 1
$24x^2 + 86x + 65 = 0$ has single root. 1

Root must be $\frac{p}{q}$ → p a factor of 125 & 65 → = 5.
→ q a factor of 24 → = 1 2 3 4 6 12 24.

$P(5) = 47\,250$ → not a root.
$P(-5) = -12\,375$ → not a root.
$P\left(\frac{5}{2}\right) = 6\,500$ → not a root.
$P\left(-\frac{5}{2}\right) = 0$ 1
$\Rightarrow (2x+5)^3$ is a factor of $P(x)$.

4

By $\alpha\beta\gamma\delta = \frac{e}{a}$

$\left(-\frac{5}{2}\right)^3 \times \delta = -\frac{125}{24}.$

$-\frac{125}{8}\delta = -\frac{125}{24}.$

$\delta = \frac{1}{3}.$ 1

Solutions to $P(x) = 0$ are
$-\frac{5}{2}, -\frac{5}{2}, -\frac{5}{2}, \frac{1}{3}.$

Q. (12)(a)(i) $\int 3x\cos x \, .dx$ $\quad u = 3x \quad v = \sin x$

$u' = 3 \quad v' = \cos x$

By $\int u.\frac{dv}{dx}.dx = uv - \int v.\frac{du}{dx}.dx$

$\int 3x\cos x \, .dx = 3x\sin x - \int 3\sin x \, .dx$ 1 $\underline{2}$

$= 3x\sin x + 3\cos x + C.$ 1

(ii) $\int \frac{1}{1+\sin 2x} .dx$

Letting $t = \tan x$

$\frac{dt}{dx} = \sec^2 x$

$= 1 + t^2.$

$\therefore dx = \frac{dx}{dt}.dt$

$= \frac{1}{1+t^2}.dt$ 1

$\therefore \int \frac{1}{1+\sin 2x} .dx$

$= \int \frac{1}{1+\frac{2t}{1+t^2}} . \frac{1}{1+t^2} .dt$ 1 $\underline{2}$

$= \int \frac{1}{1+t^2+2t} .dt$

$= \int \frac{1}{(1+t)^2} .dt$

$= -\frac{1}{1+t} + C$

$= -\frac{1}{1+\tan x} + C$ 1

Q.(12)(a)(iii) $\displaystyle\int_0^1 \frac{e^{2x}}{e^{4x}+1}\,dx$

$u = e^{2x} \quad du = 2e^{2x}\,dx$

$= \frac{1}{2}\displaystyle\int_0^1 \frac{1}{e^{4x}+1}\cdot 2e^{2x}\,dx$

$= \frac{1}{2}\displaystyle\int_{u=e^0}^{u=e^{2\cdot 1}} \frac{1}{u^2+1}\,1.du$

$= \frac{1}{2}\left[\tan^{-1}(u)\right]_1^{e^2}$

$= \frac{1}{2}\left[\tan^{-1}(e^2) - \tan^{-1}(1)\right]$ 3

$= \frac{1}{2}\left[\tan^{-1}(e^2) - \frac{\pi}{4}\right]$

$\doteqdot 0.3254$ (4SF) 1

(b)(i) $\dfrac{7x^2-11x+7}{(3x-1)(x^2+4)} \equiv \dfrac{A}{3x-1} + \dfrac{Bx+C}{x^2+4}$

$\therefore 7x^2-11x+7 \equiv A(x^2+4)+(Bx+C)(3x-1)$ (1)

Sub. in $x=\frac{1}{3}$ in (1)

$\frac{37}{9} = \frac{37}{9}A$

$\therefore A = 1.$ 3

Sub. $x=0$, $A=1$ in (1)

$7 = 4 - C$

$\therefore -3 = C$

Sub. $x=1$, $A=1$, $C=-3$ in (1):

$3 = 1\times 5 + (B-3)\times 2$

$-2 = 2(B-3)$

$2 = B.$

$\therefore A=1,\ B=2,\ C=-3.$

p.5

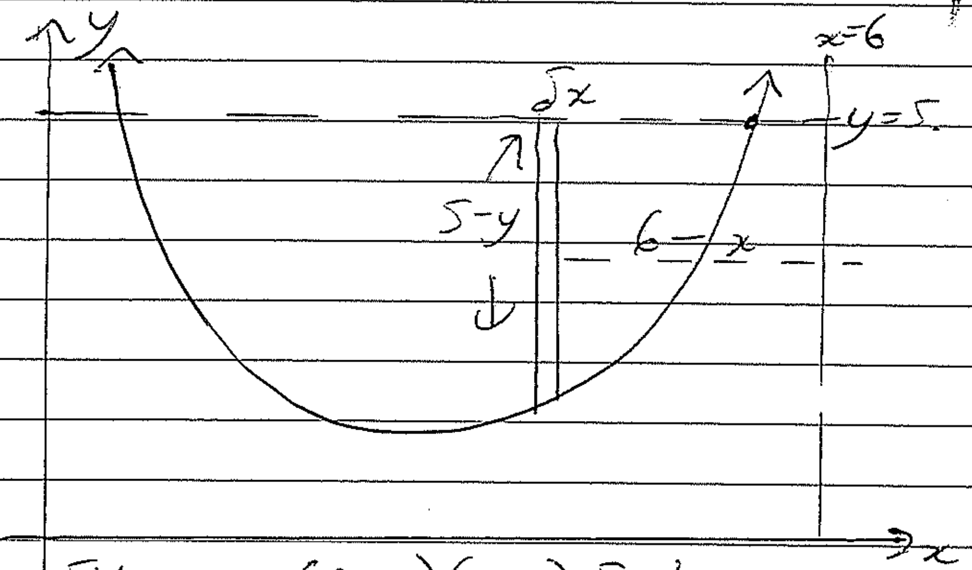
(12)(b)(ii) $\int \frac{7x^2 - 11x + 7}{(3x-1)(x^2+4)}\,dx$

$= \int \frac{1}{3x-1} + \frac{2x}{x^2+4} - \frac{3}{x^2+4}\,dx$ 1

$= \frac{1}{3}\ln(3x-1) + \ln(x^2+4) - \frac{3}{2}\tan^{-1}\left(\frac{x}{2}\right) + C$

$= \ln\left[\sqrt[3]{3x-1}\,(x^2+4)\right] - \frac{3}{2}\tan^{-1}\left(\frac{x}{2}\right) + C$

(c)



$\delta V = 2\pi(6-x)(5-y)\,\delta x$ 1

As $y = x^2 - 6x + 10$

$5 - y = 5 - (x^2 - 6x + 10)$

$= 6x - x^2 - 5$

$\therefore \delta V = 2\pi(6-x)(6x - x^2 - 5)\,\delta x$

$= 2\pi[x^3 - 12x^2 + 41x - 30]\,\delta x$ 1

$\therefore V = \lim_{\delta x \to 0} 2\pi \sum_{x=1}^{x=5} (x^3 - 12x^2 + 41x - 30)\,\delta x$

Letting $\delta x \to 0$

$V = 2\pi \int_1^5 (x^3 - 12x^2 + 41x - 30)\,dx$

$= 2\pi\left[\frac{1}{4}x^4 - 4x^3 + \frac{41}{2}x^2 - 30x\right]_1^5$

$= 2\pi\left[\frac{75}{4} - -\frac{53}{4}\right]$

$= 2\pi \times 32$

$= 64\pi$ c.u. 1

4

p.6

Q (13)(a) (i) $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1.$

$$\frac{2x}{a^2} - \frac{2y}{b^2}\cdot\frac{dy}{dx} = 0$$

$$\frac{dy}{dx} = \frac{b^2x}{a^2y}$$

At $P(a\sec\theta, b\tan\theta)$

$$\frac{dy}{dx} = \frac{b^2 a\sec\theta}{a^2 \tan\theta}$$

3

m of tangent $= \frac{b\sec\theta}{a\tan\theta}.$ 1

m of normal $= -\frac{a\tan\theta}{b\sec\theta}.$

∴ Equation of tangent at P:

$y - b\tan\theta = \frac{b\sec\theta}{a\tan\theta}(x - a\sec\theta)$

$ay\tan\theta - ab\tan^2\theta = bx\sec\theta - ab\sec^2\theta$

$ab(\sec^2\theta - \tan^2\theta) = bx\sec\theta - ay\tan\theta.$

$ab = bx\sec\theta - ay\tan\theta$

or $bx\sec\theta - ay\tan\theta = ab.$ 1

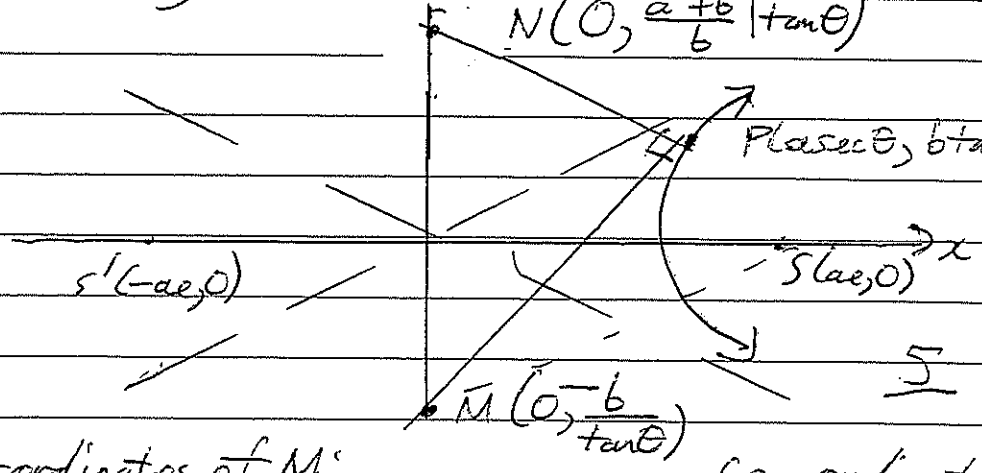
Equation of normal at P:

$y - b\tan\theta = \frac{-a\tan\theta}{b\sec\theta}(x - a\sec\theta)$

$by\sec\theta - b^2\sec\theta\tan\theta = -ax\tan\theta + a^2\sec\theta\tan\theta$

∴ $by\sec\theta + ax\tan\theta = (a^2+b^2)\sec\theta\tan\theta$ 1

(ii)



5

Co-ordinates of M:

$-ay\tan\theta = ab$

$y = \frac{-b}{\tan\theta}$ 1

Co-ordinates of N:

$by\sec\theta = (a^2+b^2)\sec\theta\tan\theta.$

$y = \left(\frac{a^2+b^2}{b}\right)\tan\theta.$ 1

PTO →

(13)(c)(ii) $m\,MS$

$$= \frac{\frac{b}{\tan\theta}}{ae}$$

$$= \frac{b}{ae\tan\theta}.$$

$$m\,NS = \frac{-\frac{(a^2+b^2)\tan\theta}{b}}{ae}$$

$$= -\frac{(a^2+b^2)\tan\theta}{aeb}.$$

$$m MS \times m NS = \frac{\cancel{b}}{ae\cancel{\tan\theta}} \times \frac{-(a^2+b^2)\cancel{\tan\theta}}{ae\cancel{b}}.$$

$$= \frac{-(a^2+b^2)}{a^2e^2}$$

$$= \frac{-(a^2+b^2)}{a^2\left(1+\frac{b^2}{a^2}\right)}$$

$$= \frac{-(a^2+b^2)}{(a^2+b^2)}$$

$$= -1.$$

As $m\,MS \times m\,NS = -1$, $MS \perp NS$

$\rightarrow$ S must be on a semicircle with MN as diameter

[$\angle$ in semicircle $= 90^\circ$].

Similarly

$$m\,MS' = \frac{-b}{ae\tan\theta} \quad \& \quad m\,NS' = \frac{(a^2+b^2)\cancel{\tan\theta}}{aeb}$$

$m\,MS' \times m\,NS'$

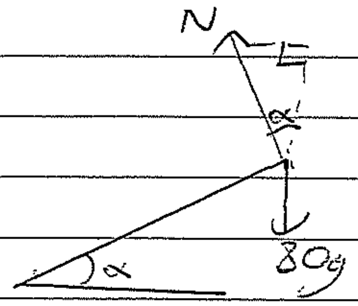
$$= \frac{-(a^2+b^2)}{a^2e^2}$$

$$= -1.$$

$\therefore$ S′ must also be on the circle [$\angle$ in semicircle $= 90^\circ$].

Q.(13)(b) p.8

(i)



Resolving horizontally: $N\sin\alpha = \frac{80\times 14^2}{100}$ (1)

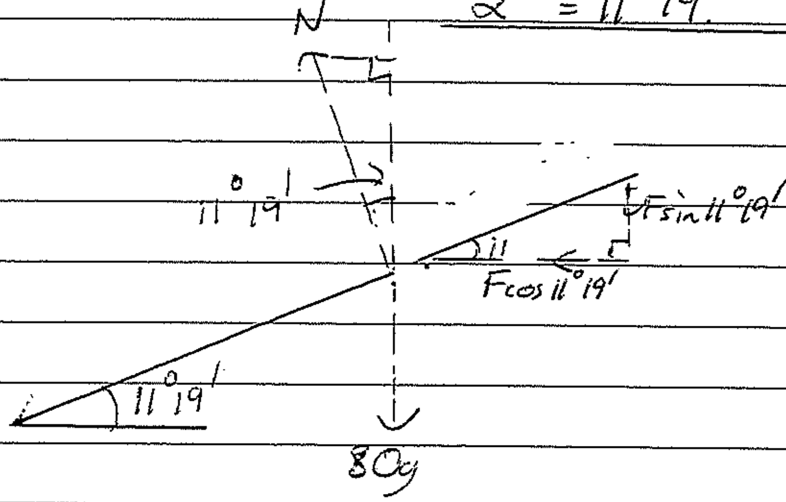
vertically: $N\cos\alpha = 80g$ (2)

Show for mark.

$\tan\alpha = \frac{14^2}{100g}$

$\alpha = 11^\circ 19'$

(ii)



Resolving horizontally: $N\sin 11^\circ 19' + F\cos 11^\circ 19' = \frac{80\times 15^2}{100}$

$\therefore N\sin 11^\circ 19' + F\cos 11^\circ 19' = 180$ (1) $\times\cos 11^\circ 19' = (3)$

Resolving vertically: $N\cos 11^\circ 19' - F\sin 11^\circ 19' = 80g$ (2) $\times\sin 11^\circ 19' = (4)$

$N\sin 11^\circ 19'\cos 11^\circ 19' + F\cos^2 11^\circ 19' = 180\cos 11^\circ 19'$ (3)

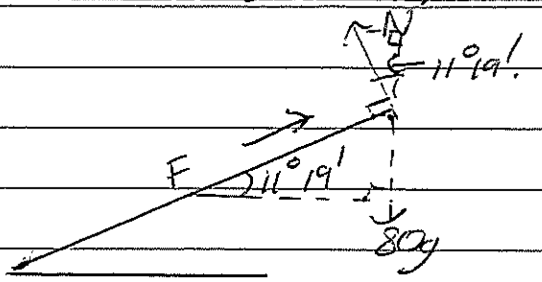
$N\cos 11^\circ 19'\sin 11^\circ 19' - F\sin^2 11^\circ 19' = 80g\sin 11^\circ 19'$ (4)

$F(\cos^2 11^\circ 19' + \sin^2 11^\circ 19') = 180\cos 11^\circ 19' - 80g\sin 11^\circ 19'$

$\therefore F = 180\cos 11^\circ 19' - 80g\sin 11^\circ 19'$

$F = 22.65$ Newtons.

Q.(13)(b)(iii) p. 9



Note: $F = 0.1N$.

Resolving vertically: $N\cos 11°19' + 0.1N\sin 11°19' = 80g$.

$$\therefore N = \frac{80g}{\cos 11°19' + 0.1\sin 11°19'}$$ 1

$\div 10$

$$\& F = \frac{8g}{\cos 11°19' + 0.1\sin 11°19'}$$

Resolving horizontally:

$$N\sin 11°19' - 0.1N\cos 11°19' = \frac{80 \times v^2}{100.}$$ 1

$$\therefore N\left(\sin 11°19' - 0.1\cos 11°19'\right) = \frac{4v^2}{5.}$$ 3

$\times \frac{5}{4}$ $\times \frac{5}{4}$

$$\frac{5}{4}\left(\sin 11°19' - 0.1\cos 11°19'\right) \times \frac{80g}{\cos 11°19' + 0.1\sin 11°19'} = v^2$$

$9.807.. = v.$

Slowest speed before going down slope

$= 9.807 m/s.$ 1

(14)(a)(i) $I_n = \int x[\ln x]^n . dx$ $\quad u = [\ln x]^n \quad v = \frac{1}{2}x^2$

$u' = \frac{n[\ln x]^{n-1}}{x} \quad v' = x$

By $\int u \frac{dv}{dx} . dx = uv - \int v . \frac{du}{dx} . dx$

$\int x[\ln x]^n . dx$

$= \frac{1}{2}x^2[\ln x]^n - \int \frac{n}{2} x [\ln x]^{n-1} . dx$

$\underline{I_n = \frac{1}{2}x^2[\ln x]^n - \frac{n}{2} I_{n-1}}$.

(ii) $I_0 = \int_1^2 x . dx$

$= [\frac{1}{2}x^2]_1^2$

$= \frac{3}{2}$.

$I_1 = [\frac{1}{2}x^2 \ln x]_1^2 - \frac{1}{2} I_0$

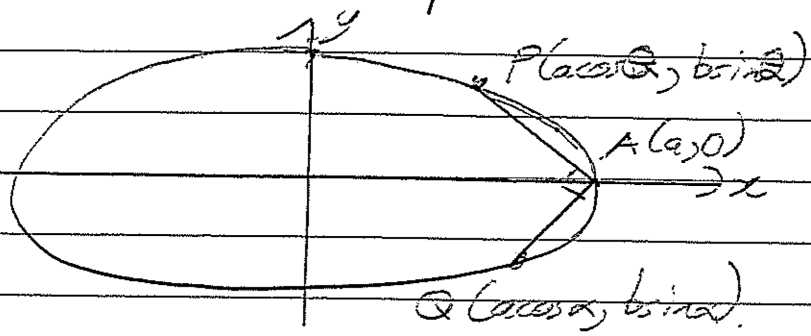
$= 2\ln 2 - \frac{3}{4}$.

$I_2 = [\frac{1}{2}x^2(\ln x)^2]_1^2 - \frac{2}{2} I_1$

$= 2[\ln 2]^2 - 2\ln 2 + \frac{3}{4}$.

$\underline{\doteqdot 0.3246}$ [4SF].

Q. (14)(b)



$$mPA = \frac{-b\sin\theta}{a - a\cos\theta} \qquad mQA = \frac{-b\sin\alpha}{a(1-\cos\alpha)}$$

$$= \frac{-b\sin\theta}{a(1-\cos\theta)}$$

If $\angle PAQ = 90^\circ$, $mPA \times mQA = -1$.

$$\therefore \frac{-b\sin\theta}{a(1-\cos\theta)} \times \frac{-b\sin\alpha}{a(1-\cos\alpha)} = -1.$$

$$\frac{b^2\sin\theta\sin\alpha}{a^2(1-\cos\theta)(1-\cos\alpha)} = -1.$$

$$\therefore -\frac{b^2}{a^2} = \frac{(1-\cos\theta)(1-\cos\alpha)}{\sin\theta\sin\alpha}.$$

Letting $t_1 = \tan\frac{\theta}{2}$, $t_2 = \tan\frac{\alpha}{2}$.

$$-\frac{b^2}{a^2} = \frac{\left(1-\frac{1-t_1^2}{1+t_1^2}\right)\left(1-\frac{1-t_2^2}{1+t_2^2}\right)}{\frac{2t_1}{(1+t_1^2)} \times \frac{2t_2}{(1+t_2^2)}}$$

$$= \frac{1 - \frac{1-t_2^2}{1+t_2^2} - \frac{1-t_1^2}{1+t_1^2} + \frac{(1-t_1^2)(1-t_2^2)}{(1+t_1^2)(1+t_2^2)}}{\frac{4t_1t_2}{(1+t_1^2)(1+t_2^2)}}$$

× numerator & denominator by $(1+t_1^2)(1+t_2^2)$

Q. (14)(b) [cont].

$$-\frac{b^2}{a^2} = \frac{(1+t_1^2)(1+t_2^2) - (1-t_2^2)(1+t_1^2) - (1-t_1^2)(1+t_2^2) + (1-t_1^2)(1-t_2^2)}{4t_1t_2}$$

$$= \frac{1+t_2^2+t_1^2+t_1^2t_2^2 - (1+t_1^2-t_2^2-t_1^2t_2^2) - (1+t_2^2-t_1^2-t_1^2t_2^2) + (1-t_2^2-t_1^2+t_1^2t_2^2)}{4t_1t_2}$$

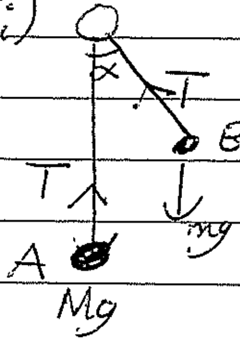
$$= \frac{4t_1^2t_2^2}{4t_1t_2}$$

$$-\frac{b^2}{a^2} = t_1t_2 \qquad 1 \qquad 5$$

$$-\frac{b^2}{a^2} = \tan\left(\frac{\theta}{2}\right)\tan\left(\frac{\alpha}{2}\right)$$

as required.

(c) (i)



Resolving vertically at B

$T\cos\alpha = mg \quad (1)$

at A: $T = Mg \quad (2) \qquad 1$

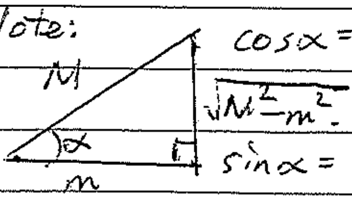
2

$(1) \div (2) \quad \cos\alpha = \frac{m}{M} \qquad 1$

(ii) Resolving horizontally at B:

$T\sin\alpha = \frac{mv^2}{r} \quad (1)$

Note:



$\cos\alpha = m/M$

$\sqrt{M^2-m^2} \qquad 1$

$\sin\alpha = \frac{\sqrt{M^2-m^2}}{M} \quad (2)$

& $T = Mg \quad (3)$

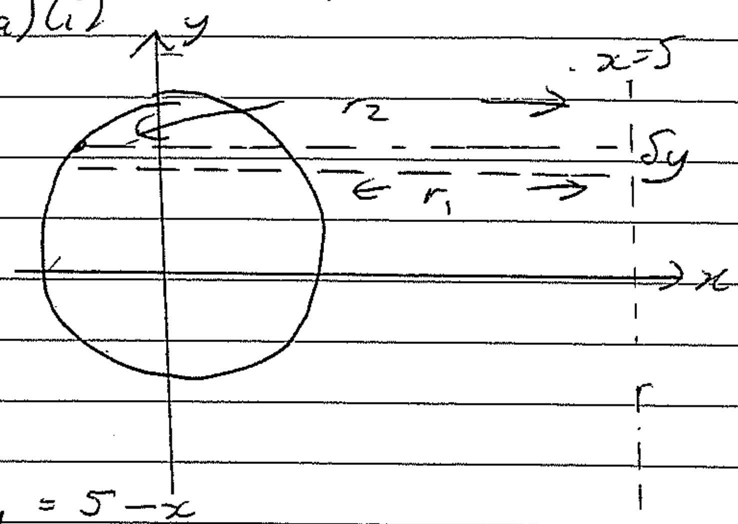
Sub. (2) & (3) in (1):

$\frac{\cancel{M}g\sqrt{M^2-m^2}}{\cancel{M}} = \frac{mv^2}{r} \qquad 1$

$g\sqrt{M^2-m^2} = \frac{mv^2}{r}$

$r = \frac{mv^2}{g\sqrt{M^2-m^2}} \qquad 1 \qquad 3$

Q. (15)(a)(i)



$r_1 = 5 - x$

$r_2 = 5 + x$

As $x^2 + y^2 = 4$.

$x^2 = 4 - y^2$

$x = \pm\sqrt{4-y^2}$

$\therefore r_1 = 5 - \sqrt{4-y^2}$, $r_2 = 5 + \sqrt{4-y^2}$] 1

$\delta V = \pi [r_2^2 - r_1^2] \, \delta y$ 2

$= \pi [(5+\sqrt{4-y^2})^2 - (5-\sqrt{4-y^2})^2] \, \delta y$

$= \pi [20\sqrt{4-y^2}] \, \delta y$] 1

$\delta V = 20\pi\sqrt{4-y^2} \, \delta y$ as required.

(ii) $V = \lim_{\delta y \to 0} 20\pi \sum_{y=-2}^{y=2} \sqrt{4-y^2} \, \delta y$ 1

Letting $\delta y \to 0$

$V = 20\pi \underbrace{\int_{-2}^{2} \sqrt{4-y^2} \, dy}$

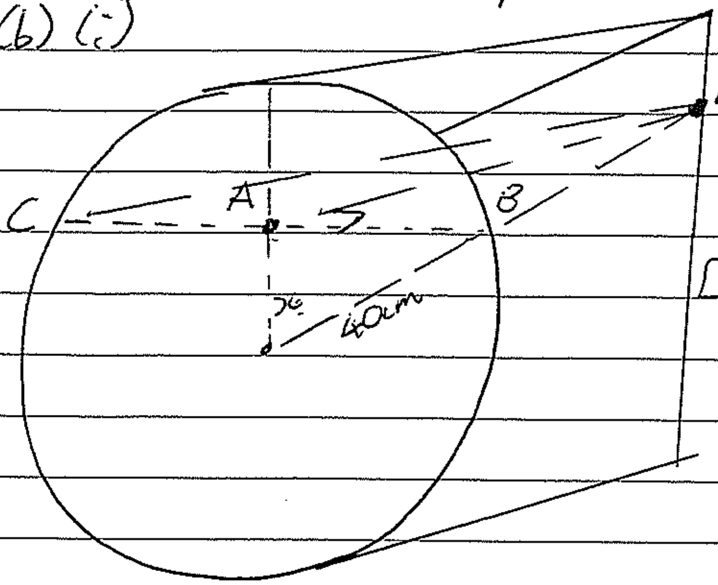
$\Rightarrow$ As $\int_{-2}^{2} \sqrt{4-y^2} \, dy$ gives area of semicircle, $r = 2$.

$= 2\pi$

$V = 20\pi \times 2\pi$ 1

$V = 40\pi^2$ square units. 1

(15)(b) (i)



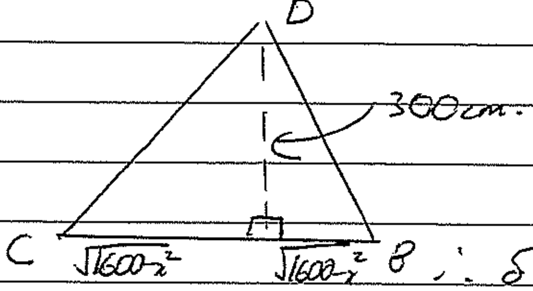
Distance AB

$= \sqrt{40^2 - x^2}$

$= \sqrt{1600 - x^2}$ 1

[By Pythagoras].

Area $\triangle CBD$ $= \frac{1}{2}\ B\ h$



$= \frac{1}{2} \times 2\sqrt{1600 - x^2} \times 300$

$= 300\sqrt{1600 - x^2}$ 2

$\therefore\ \delta V = 300\sqrt{1600 - x^2}\ .\ \delta x$. 1

(ii) $V = 2 \times \lim_{\delta x \to 0} \sum_{x=0}^{x=40} 300\sqrt{1600 - x^2}\ .\ \delta x$ [as $x = 40$ to $x = 0$ only gives V of top half of shape].

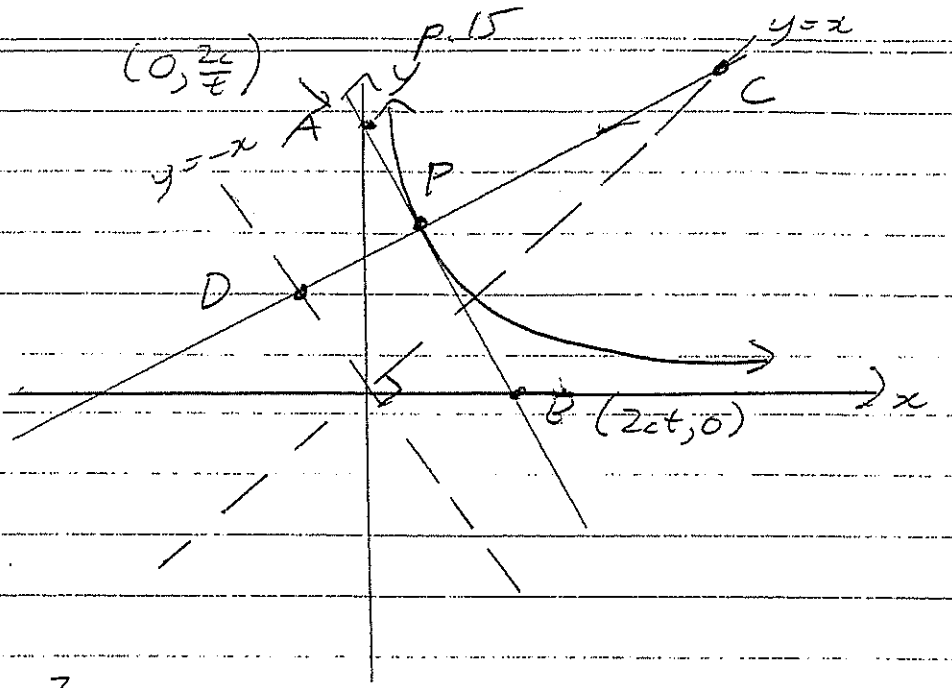
Letting $\delta x \to 0$

$V = 600 \int_0^{40} \sqrt{1600 - x^2}\ .\ dx$ 3

$= 600 \times \frac{1}{4} \times \pi \times 40^2$ [as $\int_0^{40} \sqrt{1600 - x^2}\ .\ dx$ gives area of $\frac{1}{4}$ circle].

$V = 240\ 000\pi\ cm^3$. 1

Q.(15)(c)



(i) $x + t^2 y = 2ct$

Finding A:

$t^2 y = 2ct$

$y = \frac{2c}{t}$

$\underline{A = (0, \frac{2c}{t})}$

Finding B:

$x = 2ct$

$\underline{B = (2ct, 0)}$

(ii) $t^3 x - ty = c(t^4 - 1)$

C: $t^3 x - tx = c(t^2-1)(t^2+1)$ (as $y=x$)

$t(t^2-1)x = c(t^2-1)(t^2+1)$

$x = \frac{c(t^2+1)}{t}$

$\underline{C = \left(\frac{c(t^2+1)}{t}, \frac{c(t^2+1)}{t}\right)}$

D: $t^3 x + tx = c(t^2-1)(t^2+1)$ (as $y=-x$)

$t(t^2+1)x = c(t^2-1)(t^2+1)$

$x = \frac{c(t^2-1)}{t}$

$\underline{D = \left(\frac{c(t^2-1)}{t}, -\frac{c(t^2-1)}{t}\right)}$.

(iii) Showing $PA = PB$:

Midpoint $AB = \left(\frac{2ct}{2}, \frac{2c}{2t}\right)$

$= \left(ct, \frac{c}{t}\right)$

$= P$

$\therefore PA = PB$ as P is midpoint AB.

Showing $PC = PD$

Midpoint CD

$= \left(\frac{c[t^2+1+t^2-1]}{2t}, \frac{c(t^2+1-t^2+1)}{2t}\right)$

$= \left(\frac{2ct^2}{2t}, \frac{2c}{2t}\right)$

$= \left(ct, \frac{c}{t}\right)$

$= P$

$\therefore PC = PD$ as P is midpoint CD.

PTO $\rightarrow$

Q.(15)(c)(iii) [cont]. Showing $PA = PC$

$$PA = \sqrt{(ct - 0)^2 + \left(\frac{c}{t} - \frac{2c}{t}\right)^2}$$

$$= \sqrt{c^2t^2 + \frac{c^2}{t^2}}$$

$$= \sqrt{\frac{c^2t^4 + c^2}{t^2}}$$

$$= \sqrt{\frac{c^2(t^4+1)}{t^2}}$$

$$= \frac{c}{t}\sqrt{t^4+1}.$$

$$PC = \sqrt{\left[\frac{c(t^2+1)}{t} - ct\right]^2 + \left[\frac{c(t^2+1)}{t} - \frac{c}{t}\right]^2}$$

$$= \sqrt{\left(\frac{ct^2 + c - ct^2}{t}\right)^2 + \left(\frac{ct^2 + c - c}{t}\right)^2}$$

$$= \sqrt{\left(\frac{c}{t}\right)^2 + \left(\frac{ct^2}{t}\right)^2}$$

$$= \sqrt{\frac{c^2}{t^2} + \frac{c^2t^4}{t^2}}$$

$$= \frac{c}{t}\sqrt{1 + t^4}$$

$$= PA.$$

3

As $PA = PC$

& $PA = PB$, $PC = PD$

$PA = PB = PC = PD$.

As $CD \perp AB$ [normal $\perp$ tangent].

$ACBD$ is a parallelogam [diagonals bisect each other]

a rectangle [diagonals $=$]

a rhombus [diagonals $\perp$]

$\therefore$ $ACBD$ is a SQUARE.

Q.(16)(a)(i).

$$mg = \frac{mk}{x^2} \text{ when } x = R.$$

$$\therefore mg = \frac{mk}{R^2}$$

$$gR^2 = k$$

$$\therefore \ddot{x} = -\frac{gR^2}{x^2}.$$

$$\frac{d}{dx}\left(\frac{1}{2}v^2\right) = -\frac{gR^2}{x^2}$$

$$\frac{d}{dx}\left(v^2\right) = -\frac{2gR^2}{x^2}.$$

$$v^2 = \frac{2gR^2}{x} + C \quad 1$$

3

As $v = U$ when $x = R$,

$$U^2 = \frac{2gR^2}{R} + C$$

$$U^2 = 2gR + C$$

$$\therefore U^2 - 2gR = C$$

$$\therefore v^2 = \frac{2gR^2}{x} + U^2 - 2gR. \quad 1$$

Note: As $\frac{2gR^2}{x} > 0$ for all positive x

if $U^2 > 2gR$ then

$v^2 = \frac{2gR^2}{x} + U^2 - 2gR > 0$ at all times.

$\Rightarrow \therefore$ ESCAPE velocity $= \sqrt{2gR}$

$= \sqrt{2 \times 9.8 \times 6\,366\,000}$ 1

$= 11\,170$ m/s (4SF)

(ii) $U^2 = gR$: $v^2 = \frac{2gR^2}{x} + gR - 2gR$

$$v^2 = \frac{2gR^2}{x} - gR$$

Q.(16) (a)(ii) [cont].

Particle $\Rightarrow v = 0$ Find x.

$$0 = \frac{2gR^2}{x} - gR.$$

$$gR = \frac{2gR^2}{x}$$

$$gRx = 2gR^2$$

$$x = 2R.$$

$\therefore$ $x = 2R$ is R ABOVE Earth's surface [as Earth's surface is $x = R$].

(iii) $$v^2 = \frac{2gR^2}{x} - gR$$

$$v = \sqrt{\frac{2gR^2}{x} - gR}$$

$$\frac{dx}{dt} = \sqrt{\frac{2gR^2}{x} - gR}$$

$$\frac{dt}{dx} = \frac{1}{\sqrt{\frac{2gR^2}{x} - gR}}$$

Time taken to get from $x = R$ to $x = 2R$ is

$$\int_R^{2R} \frac{1}{\sqrt{\frac{2gR^2}{x} - gR}}\,dx \quad \frac{\times x}{\times x}$$

$$= \int_R^{2R} \frac{x}{\sqrt{2gR^2x - gRx^2}}\,dx \quad (1)$$

$$= \frac{1}{\sqrt{gR}} \int_R^{2R} \frac{x}{\sqrt{2Rx - x^2}}\,dx$$

Note: $2Rx - x^2 = R^2 - (x-R)^2$.

Q.(16)(a)(iii) [cont]

$$t = \frac{1}{\sqrt{gR}} \int_{R}^{2R} \frac{x}{\sqrt{R^2 - (x-R)^2}} \, dx \quad (1)$$

Letting $x - R = R\sin\theta$

$\therefore \quad x = R(\sin\theta + 1)$

$dx = R\cos\theta \, d\theta.$

$2R = R(\sin\theta + 1) \qquad R = R(\sin\theta + 1)$

$2 = \sin\theta + 1 \qquad 1 = \sin\theta + 1$

$1 = \sin\theta \Rightarrow \theta = \frac{\pi}{2}. \qquad \sin\theta = 0 \Rightarrow \theta = 0$

$$\therefore \quad t = \frac{1}{\sqrt{gR}} \int_{0}^{\frac{\pi}{2}} \frac{R(\sin\theta + 1)}{\sqrt{R^2 - R^2\sin^2\theta}} \cdot R\cos\theta \, d\theta$$

$$= \frac{1}{\sqrt{gR}} \int_{0}^{\frac{\pi}{2}} \frac{R(\sin\theta + 1)}{R\cos\theta} \cdot R\cos\theta \, d\theta$$

$$= \frac{1}{\sqrt{gR}} \int_{0}^{\frac{\pi}{2}} R(\sin\theta + 1) \, d\theta \qquad 3$$

$$= \sqrt{\frac{R}{g}} \int_{0}^{\frac{\pi}{2}} \sin\theta + 1 \, d\theta$$

$$= \sqrt{\frac{R}{g}} \left[-\cos\theta + \theta \right]_{0}^{\frac{\pi}{2}}$$

$$= \sqrt{\frac{R}{g}} \left[-\cos\frac{\pi}{2} + \frac{\pi}{2} + \cos 0 - 0 \right]$$

$$= \sqrt{\frac{R}{g}} \left[\frac{\pi}{2} + 1 \right].$$

Time taken to reach height R above Earth's surface

$= \sqrt{\frac{R}{g}} \left[\frac{\pi}{2} + 1 \right]$ seconds.

Q.(16)(b)(i) If $\alpha = 1$,

$$1 + A + B + A + 1 = 0$$
$$2A + B = -2$$
$$2A = -(2+B)$$
$$4A^2 = (2+B)^2$$

which is not possible.

If $\alpha = -1$,

$$1 - A + B - A + 1 = 0$$
$$2 + B = 2A.$$
$$(2+B)^2 = 4A^2$$

which is not possible.

1

(ii) If α is a root then $\alpha^4 + A\alpha^3 + B\alpha^2 + A\alpha + 1 = 0$

Sub. $x = \frac{1}{\alpha}$ in equation:

$$\frac{1}{\alpha^4} + \frac{A}{\alpha^3} + \frac{B}{\alpha^2} + \frac{A}{\alpha} + 1$$
$$= \frac{1}{\alpha^4}\left(1 + A\alpha + B\alpha^2 + A\alpha^3 + \alpha^4\right)$$
$$= \frac{1}{\alpha^4} \times 0 \text{ as } \alpha \text{ is a root}$$
$$= 0$$

1

$\therefore$ $x = \frac{1}{\alpha}$ is also a root.

(iii) If α & $\frac{1}{\alpha}$ are both MULTIPLE roots

then $2\alpha + \frac{2}{\alpha} = -A$ (1) from roots 1 at a time.

$\alpha^2 + \frac{1}{\alpha^2} + 4 = B$ (2) from roots 2 at a time.

1

$\therefore$ $4B = 4\alpha^2 + \frac{4}{\alpha^2} + 16.$

2

$$8 + A^2$$
$$= 8 + \left(2\alpha + \frac{2}{\alpha}\right)^2$$
$$= 8 + \frac{4}{\alpha^2} + 8 + 4\alpha^2$$
$$= 4\alpha^2 + \frac{4}{\alpha^2} + 16 = 4B \text{ as required.}$$

1

Q.(16)(c)(i) $(\cos\theta + i\sin\theta)^3 = \cos 3\theta + i\sin 3\theta$ using DeMoivre

$\cos^3\theta + 3i\cos^2\theta\sin\theta - 3\cos\theta\sin^2\theta - i\sin^3\theta = \cos 3\theta + i\sin 3\theta$.

Equating real parts

$\cos 3\theta = \cos^3\theta - 3\cos\theta\sin^2\theta$

$= \cos^3\theta - 3\cos\theta(1-\cos^2\theta)$ 1

$= 4\cos^3\theta - 3\cos\theta$ as required.

(ii) If $x = 2\sqrt{3}\cos\theta \Rightarrow$ sub. in $x^3 - 9x = 9$

$\therefore (2\sqrt{3}\cos\theta)^3 - 9\times 2\sqrt{3}\cos\theta = 9$

$\therefore 24\sqrt{3}\cos^3\theta - 18\sqrt{3}\cos\theta = 9$

$\div 6\sqrt{3}$.

$4\cos^3\theta - 3\cos\theta = \frac{\sqrt{3}}{2}$. 1

$\cos 3\theta = \frac{\sqrt{3}}{2}$

$\therefore$ If $\cos 3\theta = \frac{\sqrt{3}}{2}$, $x = 2\sqrt{3}\cos\theta$ is a solution to $x^3 - 9x = 9$.

(iii) Solutions to $x^3 - 9x = 9$ are $x = 2\sqrt{3}\cos\theta$ where $\cos 3\theta = \frac{\sqrt{2}}{2}$.

If $\cos 3\theta = \frac{\sqrt{3}}{2}$

$3\theta = \frac{\pi}{6}, \frac{11\pi}{6}, \frac{13\pi}{6}, \frac{23\pi}{6}, \frac{25\pi}{6}, \frac{35\pi}{6}$.

$\theta = \frac{\pi}{18}, \frac{11\pi}{18}, \frac{13\pi}{18}, \frac{23\pi}{18}, \frac{25\pi}{18}, \frac{35\pi}{18}$. 1

2

$x = 2\sqrt{3}\cos\frac{\pi}{18} \doteqdot 3.4115 = \alpha$

or $x = 2\sqrt{3}\cos\frac{11\pi}{18} \doteqdot -1.1848 = \beta$ 1

or $x = 2\sqrt{3}\cos\frac{13\pi}{18} \doteqdot -2.2267 = \gamma$

Note: try $2\sqrt{3}\cos\frac{23\pi}{18} \doteqdot -2.2267\ldots$

$\rightarrow$ roots start to repeat

[& there can only be 3 roots for a cubic!]

$\therefore$ Roots of $x^3 - 9x = 9$ are

3.4115

−1.1848

& −2.2267. [4 decimal places.]