GOSFORD HIGH SCHOOL

2006
YEAR 12 TRIAL HIGHER SCHOOL CERTIFICATE

MATHEMATICS EXTENSION 2

General Instructions:

- Reading time – 5minutes
- Working time – 3 hours
- Write using black or blue pen.
- Board-approved calculators may be used
- A table of standard integrals is provided at the back of this paper
- All necessary working should be shown in every question.

Total marks: - 120

- Attempt Questions 1 -8
- All questions are of equal value.

Question 1

a) Find $\int \frac{dx}{\sqrt{x^2+16}}$. 1.

b) (i) Find real numbers a and b such that $\frac{x+5}{x^2-2x-3} \equiv \frac{a}{x-3} + \frac{b}{x+1}$ 2.

(ii) Hence evaluate $\int_4^5 \frac{x+5}{x^2-2x-3}dx$. 2.

c) Use the substitution $t = \tan\frac{x}{2}$ to evaluate $\int_0^{\frac{\pi}{2}} \frac{dx}{1+\sin x}$. 3.

d) Use integration by parts to find $\int_0^{\frac{\pi}{4}} x\sec^2 x dx$. 3.

e) Show that $\int_1^2 (2-x)\sqrt{(x-1)^3}\,dx = \frac{4}{35}$ 4.

Question 2.

a) If $Z = \frac{1-7i}{3+4i}$ find:

i) Z in the form $a + ib$ 1.

ii) $Z\overline{Z}$ 1.

iii) $|Z|$ 1.

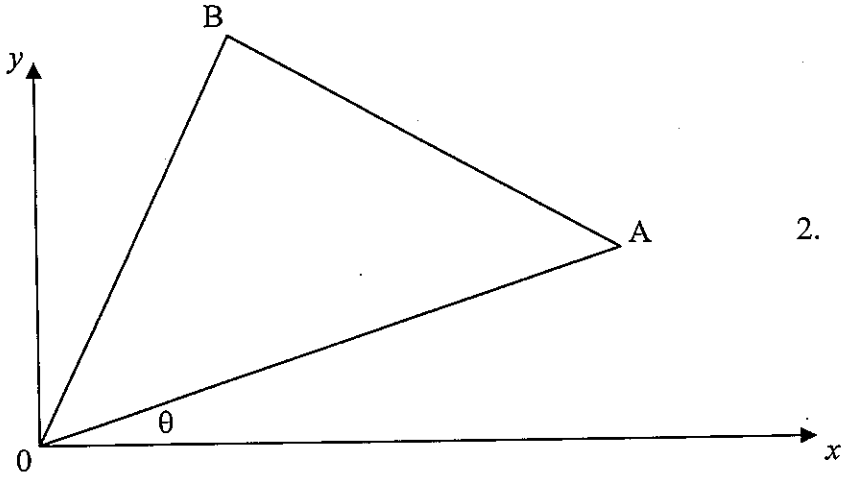
iv) $arg\ Z$ 1.

b) Find $\sqrt{-8+6i}$ 2.

c) Sketch on an argand diagram the locus of the point $P(Z)$ such that

$\arg(Z-1) - \arg(Z-i) = \frac{\pi}{4}$. 2.

d) i) Show that $\cos(\frac{\pi}{3}+\theta)+i\sin(\frac{\pi}{3}+\theta)=\frac{1}{2}(1+i\sqrt{3})(\cos\theta+i\sin\theta)$ 2.



2.

Let OAB be an equilateral triangle on an Argand diagram where OA = 1.
The point A represents the complex number Z, where $Z = cis\theta$.

ii) Find the complex number represented by B in terms of Z. 2.

iii) The triangle is now rotated about O through $\frac{\pi}{3}$ radians in an anticlockwise direction to become triangle $OA'B'$. Find the complex numbers represented by the points A' and B' in terms of Z. 3.

Question 3.

a) $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ is an ellipse.

i) Give the coordinates of S and S' the foci. 1.

ii) If PSP' is the latus rectum give the coordinates of P. 2.

iii) Find the equation of the tangent to the ellipse at the point P. 3.

iv) If the tangent at P meets the minor axis at M prove that the line joining M to the other focus is parallel to the normal at P. 3.

b) The area enclosed by the curve $y=(x+3)^2$ and the line $y=9$ is rotated about the y axis. Using cylindrical shells find the volume formed. 3.

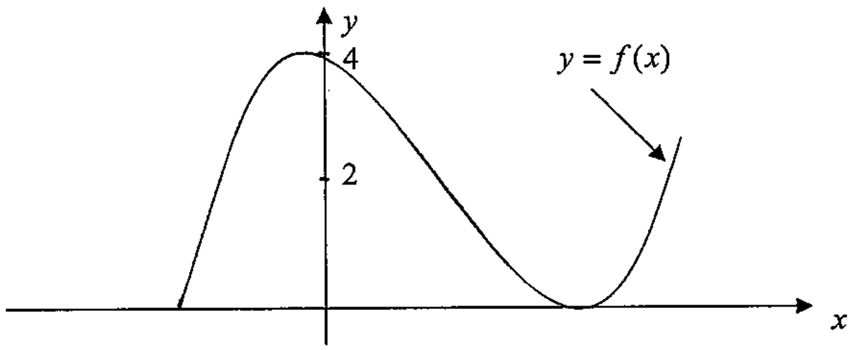
c)Suppose that a lake is stocked initially with 100 fish and that the fish population (P) satisfies hereafter the differential equation

$$\frac{dP}{dt} = k\sqrt{P} \quad (k \text{ a constant}).$$

If after 6 months there are 169 fish in the lake, how many fish will be in the lake after 1 year? 3.

Question 4.

a)



The above graph represents the curve $y = f(x)$. Copy the graph onto your answer sheet and draw an accurate sketch of the graph $y^2 = f(x)$ 2.

b) For the curve $y^2 = x^4(4+x)$

i) Find the x and y intercepts. 2.

ii) By implicit differentiation show that if $y^2 = x^4(4+x)$

then $$\frac{dy}{dx} = \frac{5x^4 + 16x^3}{2y}$$ 2.

iii) Find any stationary points for the curve. 3.

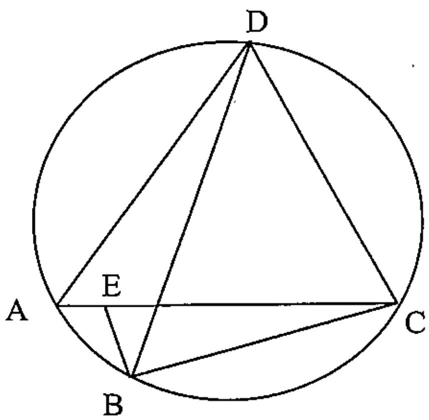
iv) Sketch the curve. 2.

v) By using the substitution $u^2 = 4 + x$, or otherwise, find the area of the loop of the curve. 4.

Question 5.

a) The base of a solid is the first quadrant area bounded by the line $4x+5y=20$ and the coordinate axes. Find the volume of the solid if every plane section perpendicular to the x axis is a semicircle. 3.

b)



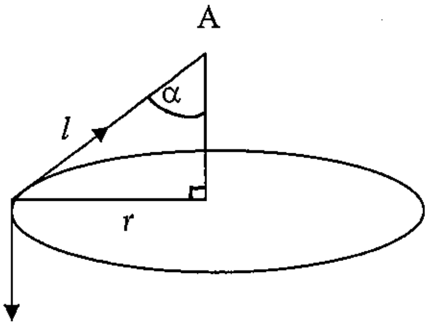
The figure shows a cyclic quadrilateral ABCD with diagonals AC and BD. E is a point on AC such that angle ABE equals angle DBC.

i) Prove that triangle ABE is similar to triangle DBC 2.

ii) Prove that triangle ABD is similar to triangle EBC. 2.

iii) Hence prove Ptolemy's theorem, which states $BA \times DC + AD \times BC = AC \times BD$ 3.

c)



i) A particle of mass m that is connected to a light string of length l to a fixed point A, describes with uniform speed a horizontal circle, radius r, whose centre is vertically below A. If the semi-vertical angle is α show that the tension (T) in the string is given by

$T = \dfrac{mg}{\cos\alpha}$ and that the linear velocity (v) is given by 3.

$v = \sqrt{rg\tan\alpha}$, where g is the force due to gravity.

ii) A string 60 cm long can sustain a mass of 10 kg vertically hung on it. Find the maximum semi-vertical angle and the number of revolutions per minute a particle of 5 kg mass can make before the string breaks. 2.

Question 6.

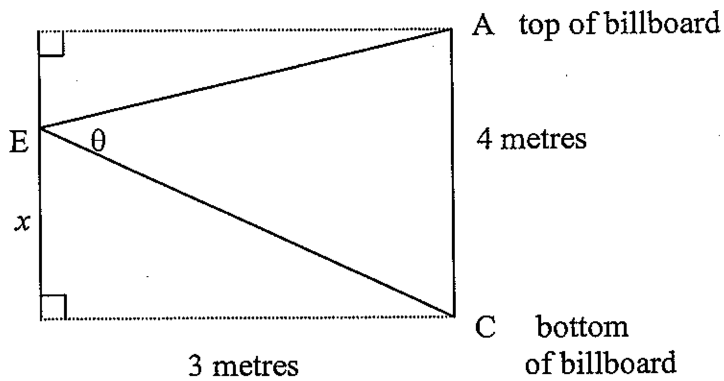
a) $P(x)$ is a polynomial of degree 4 with real coefficients.

i) Show that if the complex number α is one zero of $P(x)$ then its conjugate $\overline{\alpha}$ is also a zero of $P(x)$. 2.

ii) The complex number α satisfies $\mathrm{Im}(\alpha) \neq 0, \mathrm{Re}(\alpha) = a$ and $|\alpha| = r$. Show that if α is a zero of $P(x)$, then $P(x)$ has a factor $x^2 - 2ax + r^2$ over R, the field of real numbers. 2.

iii) α is a non-real double zero of $P(x) = x^4 - 8x^3 + 30x^2 - 56x + 49$. Factor $P(x)$ into irreducible factors over R and find the four roots of $x^4 - 8x^3 + 30x^2 - 56x + 49 = 0$ 4.

b)



i) Given $\sin(\alpha+\beta) = \sin\alpha\cos\alpha + \cos\alpha\sin\beta$ and
$\cos(\alpha+\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$
prove $\tan(\alpha+\beta) = \dfrac{\tan\alpha + \tan\beta}{1 - \tan\alpha\tan\beta}$ 2.

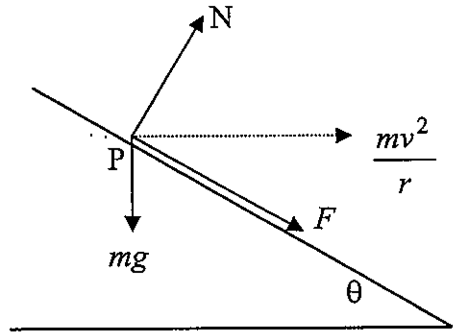
ii) A billboard AC, 4 metres high is to be positioned vertically and parallel to a highway and at a height which maximises $\tan\theta$, where θ is the angle subtended at E. E represents the eyes of passengers on the top deck of passing double decker buses. The billboard must be 3 metres from the passengers. Show that : 2.

$$\tan\theta = \frac{12}{9 - 4x + x^2} \quad \text{where } x \text{ is the distance DE.}$$

iii) Find how far below the eyelevel of passengers the base of the billboard must be for θ to be a maximum. 3.

Question 7.

a)



The diagram shows the forces acting on a point P which is moving on a banked circular track. The point P has a mass m and is moving in a horizontal circle of radius r with uniform speed v. The track is inclined at an angle θ to the horizontal. The point experiences a normal reaction force N from the track, a vertical force of magnitude mg due to gravity and a force due to friction F so that the net force on the particle is a force of magnitude $\frac{mv^2}{r}$ directed towards the centre of the horizontal circle. By resolving the forces at P into their horizontal and vertical components show that:

$$F = \frac{mv^2}{r}\cos\theta - mg\sin\theta$$ 3.

b) i) Show that the normal to the hyperbola $xy = c^2$ at the point $P(ct, \frac{c}{t})$ has the equation $y = t^2x + \frac{c}{t} - ct^3$. 2.

ii)If the normal at P meets the line $y = x$ at N, and the tangent at P meets $y = x$ at T, find the coordinates of N and T. 2.

iii)If O is the origin, prove that $OT.ON = 4c^2$ 3.

c) i) Divide x^3 by $(x+1)$ 2.

ii) Find $\int \frac{dx}{x^{\frac{1}{2}} + x^{\frac{1}{3}}}$ 3.

Question 8.

a) Show $\ddot{x} = v\dfrac{dv}{dx}$ 2.

b) i) A particle of mass m is projected vertically upwards under gravity (g), the air resistance to the motion being $\dfrac{mgv^2}{a^2}$ when the speed is v, where a is a constant.

Show that during the upward motion of the particle

$$v\frac{dv}{dx} = -\frac{g}{a^2}(a^2 + v^2)$$ 2.

where x is the upward vertical displacement.

ii) show that the greatest height reached, given the speed of projection u, is: $\dfrac{a^2}{2g}\ln(1 + \dfrac{u^2}{a^2})$. 3.

c) A polynomial of degree n is given by $P(x) = x^n + ax - b$. It is given that the polynomial has a double root at $x = \alpha$.

i) Find the derived polynomial $P'(x)$ and show that $\alpha^{n-1} = -\dfrac{a}{n}$. 2.

ii) Show that $\left(\dfrac{a}{n}\right)^n + \left(\dfrac{b}{n-1}\right)^{n-1} = 0$ 4.

iii)Hence deduce that the double root is $\dfrac{bn}{a(n-1)}$ 2.

STANDARD INTEGRALS

$$\int x^n \, dx = \frac{1}{n+1} x^{n+1}, \; x \neq 0 \text{ if } n < 0$$

$$\int \frac{1}{x} \, dx = \ln x, \; x > 0$$

$$\int e^{ax} \, dx = \frac{1}{a} e^{ax}, \; a \neq 0$$

$$\int \cos ax \, dx = \frac{1}{a} \sin ax, \; a \neq 0$$

$$\int \sin ax \, dx = -\frac{1}{a} \cos ax, \; a \neq 0$$

$$\int \sec^2 ax \, dx = \frac{1}{a} \tan ax, \; a \neq 0$$

$$\int \sec ax \tan ax \, dx = \frac{1}{a} \sec ax, \; a \neq 0$$

$$\int \frac{1}{a^2 + x^2} \, dx = \frac{1}{a} \tan^{-1} \frac{x}{a}, \; a \neq 0$$

$$\int \frac{1}{\sqrt{a^2 - x^2}} \, dx = \sin^{-1} \frac{x}{a}, \; a > 0, \; -a < x < a$$

$$\int \frac{1}{\sqrt{x^2 - a^2}} \, dx = \ln\left(x + \sqrt{x^2 - a^2}\right), \; x > a > 0$$

$$\int \frac{1}{\sqrt{x^2 + a^2}} \, dx = \ln\left(x + \sqrt{x^2 + a^2}\right)$$

NOTE: $\ln x = \log_e x, \; x > 0$

2006 Ext 2. Trial: Solutions

Ques 1.

a) $\displaystyle\int \frac{dx}{\sqrt{x^2+16}}$

$= \ln\left(x+\sqrt{x^2+16}\right)$

(from standard integrals)

b) (i) $\dfrac{x+5}{x^2-2x-3} \equiv \dfrac{a}{x-3} + \dfrac{b}{x+1}$

$\dfrac{x+5}{(x-3)(x+1)} \equiv \dfrac{a}{x-3} + \dfrac{b}{x+1}$

$x+5 = a(x+1) + b(x-3)$

let $x=3$: $8 = 4a$

$2 = a$

let $x=-1$: $4 = -4b$

$-1 = b$

$\therefore \dfrac{x+5}{x^2-2x-3} \equiv \dfrac{2}{x-3} - \dfrac{1}{x+1}$

ii) $\displaystyle\int_4^5 \frac{x+5}{x^2-2x-3}\,dx$

$= \displaystyle\int_4^5 \frac{2}{x-3} - \frac{1}{x+1}\,dx$

$= \Big[2\ln(x-3) - \ln(x+1)\Big]_4^5$

$= \left[\ln\left(\dfrac{(x-3)^2}{x+1}\right)\right]_4^5$

$= \ln\frac{4}{6} - \ln\frac{1}{5}$

$= \ln\left(\frac{10}{3}\right)$

c) $\displaystyle\int_0^{\pi/2} \frac{dx}{1+\sin x}$

$t = \tan \frac{x}{2}$

$x=\frac{\pi}{2}$: $t=1$; $x=0$, $t=0$

$= \displaystyle\int_0^1 \frac{1}{1+\frac{2t}{1+t^2}} \cdot \frac{2}{1+t^2}\,dt$

$= \displaystyle\int_0^1 \frac{2}{t^2+2t+1}\,dt$

$= \displaystyle\int_0^1 \frac{2}{(t+1)^2}\,dt$

$= \left[\dfrac{-2}{t+1}\right]_0^1$

$= -1 - (-2)$

$= 1.$

d) $\displaystyle\int_0^{\pi/4} x.\sec^2 x\,dx$

$= \displaystyle\int_0^{\pi/4} x.\ \frac{d}{dx}(\tan x)\,dx$

$= \Big[x\tan x\Big]_0^{\pi/4} - \displaystyle\int_0^{\pi/4} \tan x\,dx$

$= \pi/4 - \displaystyle\int_0^{\pi/4} \frac{\sin x}{\cos x}\,dx$

$= \pi/4 + \Big[\ln(\cos x)\Big]_0^{\pi/4}$

$= \pi/4 + \ln\left(\frac{1}{\sqrt{2}}\right) - \ln 1$

$= \pi/4 - \frac{1}{2}\ln 2.$

e) $\displaystyle\int_1^2 (2-x)\sqrt{(x-1)^3}\,dx$

$= \displaystyle\int_1^2 2(x-1)^{3/2}dx - \int_1^2 x(x-1)^{3/2}\,dx$

$= \left[\dfrac{4(x-1)^{5/2}}{5}\right]_1^2$

let $u = x-1$

$\frac{du}{dx} = 1$

$du = dx$

$x=1$, $u=0$

$x=2$, $u=1$

$= \frac{4}{5} - \int_0^1 (u+1)u^{3/2}\,du$

$= \frac{4}{5} - \int_0^1 u^{5/2} + u^{3/2}\,du$

$= \frac{4}{5} - \left[\frac{2}{7}u^{7/2} + \frac{2}{5}u^{5/2}\right]_0^1$

$= \frac{4}{5} - \left(\left(\frac{2}{7} + \frac{2}{5}\right) - 0\right)$

$= \frac{4}{35}$

Ques 2

a) i) $\frac{1-7i}{3+4i}$

$= \frac{1-7i}{3+4i} \times \frac{3-4i}{3-4i}$

$= \frac{3-4i-21i-28}{9+16}$

$= \frac{-25-25i}{25}$

$= -1-i$

ii) $z\bar{z}$

$= (-1-i)(-1+i)$

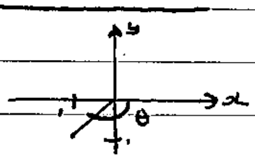
$= 2.$

iii) $|z|$

$= \sqrt{1^2+1^2}$

$= \sqrt{2}.$

iv)



$\arg z = -\frac{3\pi}{4}$

b) $\sqrt{-8+6i}$

let $a+ib = \sqrt{-8+6i}$

$a^2-b^2+2iab = -8+6i$

$\therefore\ a^2-b^2 = -8$ ---(1)

$2ab = 6$

$ab = 3$ ---(2)

from (2) $b = \frac{3}{a}$; sub into (1)

$a^2 - \frac{9}{a^2} = -8$

$a^4 - 9 = -8a^2$

$a^4 + 8a^2 - 9 = 0$

$(a^2+9)(a^2-1) = 0$

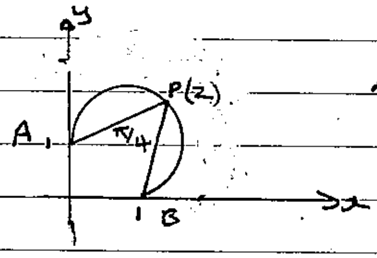
$a^2 = -9$ or $a^2 = 1$

no soln. $a = \pm 1$

$\therefore\ b = \pm 3.$

$\therefore\ \sqrt{-8+6i} = 1+3i$ or $-1-3i.$

c)



Locus is arc APB.

d) i) $\cos(\frac{\pi}{3}+\theta) + i\sin(\frac{\pi}{3}+\theta) = \frac{1}{2}(1+i\sqrt{3})(\cos\theta + i\sin\theta)$

L.H.S

$= \cos\frac{\pi}{3}\cos\theta - \sin\frac{\pi}{3}\sin\theta + i\sin\frac{\pi}{3}\cos\theta + i\cos\frac{\pi}{3}\sin\theta$

$= \frac{1}{2}\cos\theta - \frac{\sqrt{3}}{2}\sin\theta + \frac{i\sqrt{3}}{2}\cos\theta + \frac{i}{2}\sin\theta$

$= \frac{1}{2}(\cos\theta - \sqrt{3}\sin\theta + i\sqrt{3}\cos\theta + i\sin\theta)$

$= \frac{1}{2}(\cos\theta + i\sin\theta + \sqrt{3}(i\cos\theta - \sin\theta))$

$= \frac{1}{2}(\cos\theta + i\sin\theta + i\sqrt{3}(\cos\theta + i\sin\theta))$

$= \frac{1}{2}(1+i\sqrt{3})(\cos\theta + i\sin\theta)$

$=$ RHS

Let A be the point $\cos\theta + i\sin\theta \equiv z$

ii) For B. mod $= 1$ and.

arg $= \theta + \frac{\pi}{3}$ (ΔOAB isosceles)

$\therefore$ B is the point $\cos(\theta+\frac{\pi}{3}) + i\sin(\theta+\frac{\pi}{3})$

$= \frac{1}{2}(1+i\sqrt{3})(\cos\theta + i\sin\theta)$

$= \frac{1}{2}(1+i\sqrt{3})z$

iii) When ΔOAB is rotated through $\pi/3$ A' will coincide with B

Hence $A' = B = (1+i\sqrt{3})z$

Now B' will have a modulus of 1 and an argument of $2\pi/3$.

$\therefore B' = \cos(2\pi/3 + \theta) + i\sin(2\pi/3+\theta)$

$= -\frac{1}{2}\cos\theta - \frac{\sqrt{3}}{2}\sin\theta + i\left(\frac{\sqrt{3}}{2}\cos\theta - \frac{1}{2}\sin\theta\right)$

$= -\frac{1}{2}\cos\theta - \frac{\sqrt{3}}{2}\sin\theta + \frac{i\sqrt{3}}{2}\cos\theta - \frac{i}{2}\sin\theta$

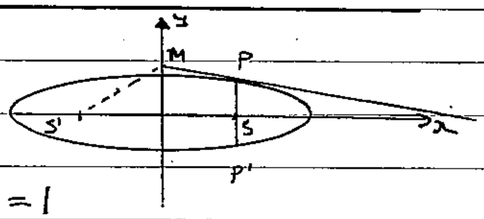
$= -\frac{1}{2}(\cos\theta + i\sin\theta) + \frac{i\sqrt{3}}{2}(\cos\theta + i\sin\theta)$

$= \frac{1}{2}(i\sqrt{3}-1)(\cos\theta + i\sin\theta)$

$= \frac{1}{2}(i\sqrt{3}-1)z$

Ques 3

a)



$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

i) $S(ae, 0)$, $S'(-ae, 0)$

ii) $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ ---- (1)

$x = ae$ ---- (2)

Sub (2) into (1)

$\frac{a^2e^2}{a^2} + \frac{y^2}{b^2} = 1$

$e^2 + \frac{y^2}{b^2} = 1$

$\frac{y^2}{b^2} = 1 - e^2$

$= 1 - \left(1 - \frac{b^2}{a^2}\right)$

$\frac{y^2}{b^2} = \frac{b^2}{a^2}$

$y^2 = \frac{b^4}{a^2}$

$y = \pm\frac{b^2}{a}$

$\therefore P\left(ae, \frac{b^2}{a}\right)$

iii) $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

$\frac{2x}{a^2} + \frac{2y}{b^2}\cdot\frac{dy}{dx} = 0$

$\frac{dy}{dx} = \frac{-b^2x}{a^2y}$

at $\left(ae, \frac{b^2}{a}\right)$: $\frac{dy}{dx} = \frac{-b^2 \cdot ae}{a^2 \cdot b^2/a}$

$= -e$

$\therefore$ equation of the tangent

$y - \frac{b^2}{a} = -e(x - ae)$

$y - \frac{b^2}{a} = -ex + ae^2$

$y = -ex + ae^2 + \frac{b^2}{a}$

iv) for M $x = 0$

$y = ae^2 + \frac{b^2}{a}$

$\therefore M\left(0, ae^2 + \frac{b^2}{a}\right)$

gradient $MS' = \frac{ae^2 + \frac{b^2}{a} - 0}{0 - (-ae)}$

$= \frac{ae^2 + \frac{b^2}{a}}{ae}$

$= \frac{a\left(1 - \frac{b^2}{a^2}\right) + \frac{b^2}{a}}{ae}$

$= \frac{a - \frac{b^2}{a} + \frac{b^2}{a}}{ae}$

$= \frac{a}{ae}$

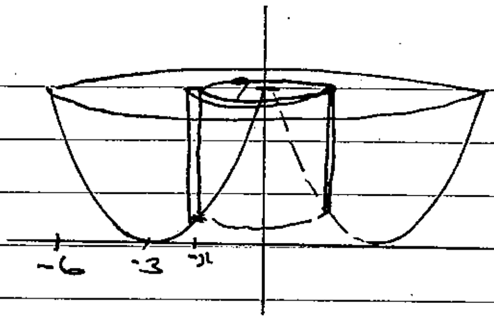
$= \frac{1}{e}$

Now Slope of the normal at P

$= \frac{1}{e}$ (slope tangent $= -e$)

$\therefore$ normal at P $\parallel$ to MS'

b)



Volume of a shell $= 2\pi r h$

$= 2\pi x y \, \Delta x$

$= 2\pi x (9 - y) \Delta x$

$\therefore \; V = \sum_{x=-6}^{0} 2\pi x (9-y) \Delta x$

$\therefore \; V = \lim_{\Delta x \to 0} \sum_{x=-6}^{0} 2\pi x (9-y) \Delta x$

$= 2\pi \int_{-6}^{0} x(9-y)\, dx$

$= 2\pi \int_{-6}^{0} x\left(9 - (x+3)^2\right) dx$

$= 2\pi \int_{-6}^{0} x\left(9 - x^2 - 6x - 9\right) dx$

$= 2\pi \int_{-6}^{0} -x^3 - 6x^2 \, dx$

$= -2\pi \left[\frac{x^4}{4} + 2x^3 \right]_{-6}^{0}$

$= -2\pi \left[-108 \right]$

$= 216\pi$ cubic units

c) $\frac{dP}{dt} = k\sqrt{P}$

$\frac{dP}{dt} = k P^{1/2}$

$\frac{dt}{dP} = \frac{1}{k} P^{-1/2}$

$t = \frac{2}{k} P^{1/2} + c$

$kt = 2\sqrt{P} + c_1$

at $t = 0$, $P = 100$

$0 = 2\sqrt{100} + c_1$

$-20 = c_1$

$kt = 2\sqrt{P} - 20$

$t = 6$, $P = 169$

$6k = 2\sqrt{169} - 20$

$6k = 26 - 20$

$6k = 6$

$k = 1$

$\therefore \; t = 2\sqrt{P} - 20$

$\therefore \; P = \left(\frac{t+20}{2}\right)^2$

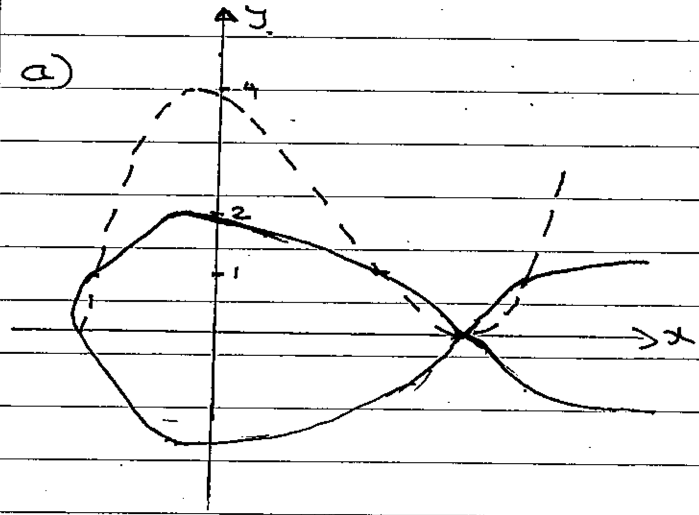
at $t = 12$. $P = \left(\frac{12+20}{2}\right)^2$

$= 256.$

$\therefore$ after one year there will be 256 fish in the lake.

Question 4.

a)



b) i) $y^2 = x^4(4+x)$

'x' intercepts : $y = 0$

$x^4(4+x) = 0$

$x = 0, -4$

'y' intercept : $x = 0$

$y^2 = 0$

$y = 0.$

ii) $y^2 = x^4(4+x)$

ie $y^2 = 4x^4 + x^5$

$2y \frac{dy}{dx} = 16x^3 + 5x^4$

$\frac{dy}{dx} = \frac{16x^3 + 5x^4}{2y}.$

iii) Stationary points: $\frac{dy}{dx} = 0$

$$\frac{5x^4 + 16x^3}{2y} = 0$$

$\therefore 5x^4 + 16x^3 = 0$

$x^3(5x+16) = 0$

$x = 0, \quad x = \frac{16}{5}.$

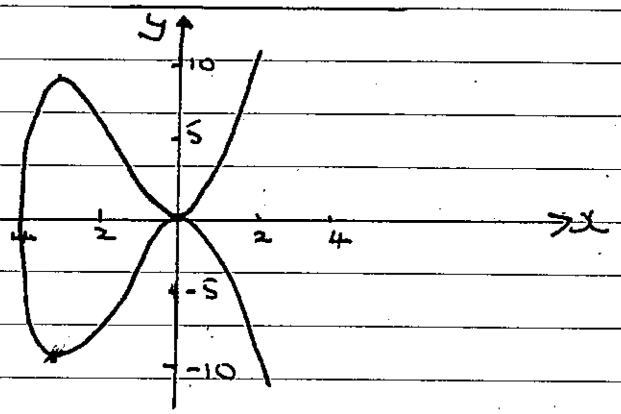
at $x = 0$; $x = \frac{-16}{5}$

$y = 0, \quad y^2 \doteqdot 83.9$

$y = \pm 9.2$

$\therefore$ S.P. $(0,0)$, $(-3.2, 9.2)$, $(3.2, -9.2)$

iv)



v) $y^2 = x^4(4+x)$

$A = 2\int_{-4}^{0} y\, dx.$

$= 2\int_{-4}^{0} x^2\sqrt{4+x}\, dx$

Let $u^2 = 4+x$ at $x=-4$, $u=0$

$x = u^2 - 4$ at $x=0$, $u=2$.

$\frac{dx}{du} = 2u$

$dx = 2u\,du$

$= 2\int_0^2 (u^2-4)^2 u \cdot 2\,du$

$= 4\int_0^2 (u^4 - 8u^2 + 16) u\, du$

$= 4\int_0^2 u^5 - 8u^3 + 16u\, du$

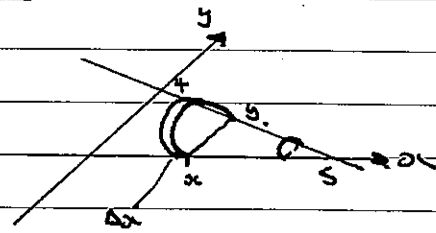
$= 4\left[\frac{u^6}{6} - 2u^4 + 8u^2\right]_0^2$

$= 4\left[10^{2}/_{3} - 0\right]$

$= 42^{2}/_{3}$ Square units.

Ques 5

a)



Volume of slice $= \frac{\pi}{2}\left(\frac{y}{2}\right)^2 \Delta x$

$\therefore V \doteqdot \sum_{x=0}^{5} \frac{\pi}{8} y^2 \Delta x$

$V = \lim_{\Delta x \to 0} \sum_{x=0}^{5} \frac{\pi}{8} y^2\, dx$

$= \frac{\pi}{8}\int_0^5 \left(\frac{20-4x}{5}\right)^2 dx$

$= \frac{\pi}{8}\int_0^5 \left(4 - \frac{4}{5}x\right)^2 dx$

$= \frac{\pi}{8}\left[\frac{(4 - \frac{4}{5}x)^3}{3 \times ^{-4}/_{5}}\right]_0^5$

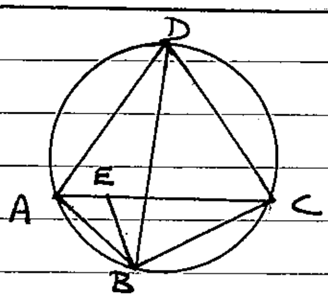
$= \frac{\pi}{8}\left[\frac{5(4 - \frac{4}{5}x)^3}{-12}\right]_0^5$

$= \frac{\pi}{8}\left[0 - \left(\frac{320}{-12}\right)\right]$

$= \frac{\pi}{8} \times \frac{320}{12}$

$= \frac{10\pi}{3}$ cubic units.

b)



i) $\angle ABE = \angle DBC$ given

$\angle BAE = \angle BDC$. angles at the circumference standing on the same arc are equal.

$\therefore \triangle ABE \;|||\; \triangle DBC$ (A.A.A.)

ii) let $\angle DAC = \alpha$

$\therefore \angle DBC = \alpha$ angle at circumference standing on same arc

$\because \angle ABE = \alpha$ given

let $\angle BAE = \beta$

$\therefore \angle DAB = \alpha + \beta$

also $\angle BEC = \alpha + \beta$ exterior $\angle$ of $\triangle ABE$

$\therefore \angle DAB = \angle BEC$

$\angle ADB = \angle ECB$ angle at the circumference standing on same arc.

$\therefore \triangle ABD \;|||\; \triangle EBC$ (A.A.A.)

iii) As $\triangle ABE \;|||\; \triangle DBC$

$$\therefore \frac{AB}{BD} = \frac{AE}{DC}$$

ie $AB \times CD = BD \times AE$

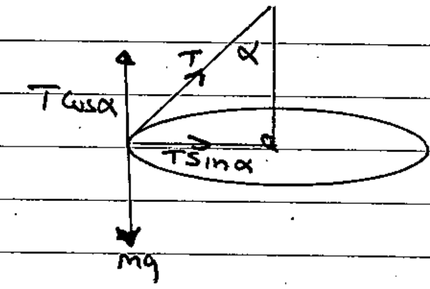
Also as $\triangle ABD \;|||\; \triangle EBC$

$$\therefore \frac{BC}{DB} = \frac{EC}{AD}$$

Now:

$$AB \times CD + BC \times AD = BD \times AE + BD \times EC$$
$$= BD\,(AE + EC)$$
$$= BD \times AC$$

c) (i)



resolving forces horizontally and vertically

$$T\sin\alpha = \frac{mv^2}{r} \quad \text{---(1)}$$

$$T\cos\alpha = mg \quad \text{---(2)}$$

from (2) $T = \dfrac{mg}{\cos\alpha}$

(1) $\div$ (2) $\tan\alpha = \dfrac{v^2}{rg}$

$$\therefore v^2 = rg\tan\alpha$$
$$v = \sqrt{rg\tan\alpha}$$

ii) If string can sustain a mass of 10kg $\Rightarrow$ maximum tension in the string $= Mg = 10 \times 10$

$= 100$ newtons.

Now from (i)

$$\cos\alpha = \frac{mg}{T}$$
$$= \frac{5 \times 10}{100}$$
$$= 0.5$$
$$\therefore \alpha = 60°$$

Also $T\sin\alpha = mr\omega^2$

$= ml\omega^2\sin\alpha$

$T = ml\omega^2$

$$\therefore \omega^2 = \frac{T}{ml}$$
$$= \frac{100}{5 \times 0.6}$$
$$= 33.3$$

$\therefore \omega = 5.77$ rad/sec.

$= 0.9$ rev/sec

$= 54$ rev/...

Ques 6

a) (i) let $P(x) = a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4$

with $a_0, a_1, a_2, a_3, a_4 \in \mathbb{R}$.

if α is a zero then $P(\alpha) = 0$

ie $a_0 + a_1\alpha + a_2\alpha^2 + a_3\alpha^3 + a_4\alpha^4 = 0$

$\overline{a_0 + a_1\alpha + a_2\alpha^2 + a_3\alpha^3 + a_4\alpha^4} = 0$

$\bar{a}_0 + \bar{a}_1(\bar{\alpha}) + \bar{a}_2(\overline{\alpha^2}) + \bar{a}_3(\bar{\alpha})^3 + \bar{a}_4(\overline{\alpha^4}) = 0$

$a_0 + a_1(\bar{\alpha}) + a_2(\bar{\alpha})^2 + a_3(\bar{\alpha})^3 + a_4(\bar{\alpha})^4 = 0$

$\therefore P(\bar{\alpha}) = 0$

ii) $\alpha, \bar{\alpha}$ are zeros of $P(x)$

$\Rightarrow (x-\alpha)(x-\bar{\alpha})$ is a factor of $P(x)$ $(\alpha \neq \bar{\alpha})$

But $(x-\alpha)(x-\bar{\alpha}) = x^2 - (\alpha+\bar{\alpha})x + \alpha\bar{\alpha}$

$= x^2 - 2\mathrm{Re}(\alpha)x + |\alpha|^2$

$= x^2 - 2ax + r^2$

$\therefore x^2 - 2ax + r^2$ is a factor of $P(x)$

iii) α zero of multiplicity 2

$\Rightarrow P(\alpha) = P'(\alpha) = 0$

$\Rightarrow P(\bar{\alpha}) = P'(\bar{\alpha}) = 0$ (from above)

$\therefore \bar{\alpha}$ is also a zero of multiplicity 2.

$\therefore x^4 - 8x^3 + 30x^2 - 56x + 49 = 0$

has roots $\alpha, \alpha, \bar{\alpha}, \bar{\alpha}$

then $2(\alpha + \bar{\alpha}) = 8$ $(-b/a)$

ie $4a = 8$

$a = 2$.

also $(\alpha\bar{\alpha})^2 = 49$ (e/a)

$(r^2)^2 = 49$

$r^4 = 49$

$r = \sqrt{7}$

$\Rightarrow \alpha = 2 \pm i\sqrt{7}$

$\therefore P(x) = (x^2 - 4x + 7)^2$

and roots of $P(x) = 0$ are

$2+i\sqrt{7},\ 2+i\sqrt{7},\ 2-i\sqrt{7},\ 2-i\sqrt{7}$.

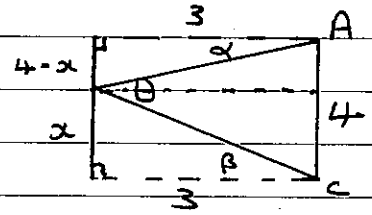
b) i) $\tan(\alpha+\beta) = \dfrac{\sin(\alpha+\beta)}{\cos(\alpha+\beta)}$

$= \dfrac{\sin\alpha\cos\beta + \cos\alpha\sin\beta}{\cos\alpha\cos\beta - \sin\alpha\sin\beta}$

$= \dfrac{\dfrac{\sin\alpha}{\cos\alpha} + \dfrac{\sin\beta}{\cos\beta}}{1 - \dfrac{\sin\alpha\sin\beta}{\cos\alpha\cos\beta}}$ $(\div \cos\alpha\cos\beta)$

$= \dfrac{\tan\alpha + \tan\beta}{1 - \tan\alpha\tan\beta}$.

ii)



from the diagram:

$\tan\alpha = \dfrac{4-x}{3}$, $\tan\beta = \dfrac{x}{3}$

Now $\theta = \alpha + \beta$. (alternate ∠'s in parallel lines)

$\therefore \tan\theta = \tan(\alpha+\beta)$

$= \dfrac{\tan\alpha + \tan\beta}{1 - \tan\alpha\tan\beta}$

$= \dfrac{\dfrac{4-x}{3} + \dfrac{x}{3}}{1 - \dfrac{(4-x)}{3}\cdot\dfrac{x}{3}}$

$= \dfrac{\frac{4}{3}}{1 - \frac{x(4-x)}{9}}$

$= \dfrac{12}{9 - x(4-x)}$

$= \dfrac{12}{9 - 4x + x^2}$

iii) $\tan\theta = \dfrac{12}{9 - 4x + x^2}$

$\theta = \tan^{-1}\left(\dfrac{12}{9 - 4x + x^2}\right)$

$$\frac{d\theta}{dx} = \frac{1}{1+\left(\frac{12}{x^2-4x+9}\right)^2} \times \frac{-12(2x-4)}{(x^2-4x+9)^2}$$

$$= \frac{-24(x-2)}{(x^2-4x+9)^2+12}$$

Stationary pt. when $\frac{d\theta}{dx} = 0$

$-24(x-2) = 0$

$x = 2.$

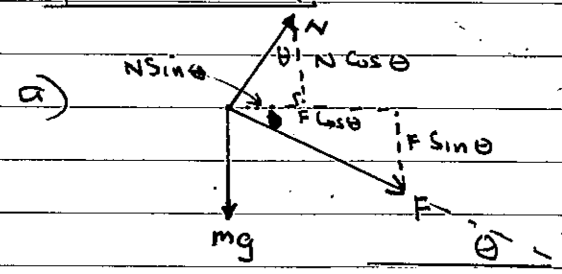
$0 < f'(1), \quad f'(3) < 0$

$\therefore$ max when $x = 2$.

$\therefore$ θ max when the billboard is 2m below the eye level.

Question 7

a)



Resolving forces.

Vertically,

$N\cos\theta - F\sin\theta - mg = 0 \quad \cdots(1)$

$N\sin\theta + F\cos\theta = \frac{mv^2}{r} \quad \cdots(2)$

(1) $\times \sin\theta$.

$N\sin\theta\cos\theta - F\sin^2\theta = mg\sin\theta \quad \cdots(3)$

(2) $\times \cos\theta$

$N\sin\theta\cos\theta + F\cos^2\theta = \frac{mv^2\cos\theta}{r} \quad \cdots(4)$

(4) $-$ (3) $F(\sin^2\theta + \cos^2\theta) = \frac{mv^2\cos\theta}{r} - mg\sin\theta$

$$F = \frac{mv^2\cos\theta}{r} - mg\sin\theta.$$

b) i) $xy = c^2$

$y = \frac{c^2}{x}$

$\frac{dy}{dx} = \frac{-c^2}{x^2}$

at $x = ct$: $\frac{dy}{dx} = \frac{-c^2}{c^2t^2}$

$= -\frac{1}{t^2}$

$\therefore$ gradient normal $= t^2$

$\therefore$ equation of the normal

$y - \frac{c}{t} = t^2(x - ct)$

$y - \frac{c}{t} = t^2x - ct^3$

$y = t^2x + \frac{c}{t} - ct^3.$

ii) for N: $y = t^2x + \frac{c}{t} - ct^3 \quad --(1)$

$y = x \quad ---(2)$

equating (1) and (2) $x = t^2x + \frac{c}{t} - ct^3$

$tx(1-t^2) = c - ct^4$

$x = \frac{c(1-t^4)}{t(1-t^2)}$

$= \frac{c(1+t^2)}{t}$

$\therefore N \left(\frac{c(1+t^2)}{t}, \frac{c(1+t^2)}{t}\right)$

For T. $y = -\frac{x}{t^2} + \frac{2c}{t} \quad \cdots(3)$

$y = x \quad ----(4)$

equating (3) & (4) $x = -\frac{x}{t^2} + \frac{2c}{t}$

$x(t^2+1) = 2ct$

$x = \frac{2ct}{t^2+1}$

$\therefore T \left(\frac{2ct}{t^2+1}, \frac{2ct}{t^2+1}\right)$

$OT.ON = \sqrt{\left(\frac{c(1+t^2)}{t}\right)^2 + \left(\frac{c(1+t^2)}{t}\right)^2} \cdot \sqrt{\left(\frac{2ct}{1+t^2}\right)^2 + \left(\frac{2ct}{1+t^2}\right)^2}$

$= \sqrt{\frac{2c^2(1+t^2)^2}{t^2}} \cdot \sqrt{\frac{8c^2t^2}{(1+t^2)^2}}$

$= \frac{\sqrt{2}\,c(1+t^2)}{t} \times \frac{2\sqrt{2}\,ct}{1+t^2}$

$= 4c^2$

c) (i)

$$\begin{array}{r|l} & x^2 - x + 1 \\ x+1 & x^3 \\ & x^3 + x^2 \\ & -x^2 \\ & -x^2 - x \\ & x \\ & x+1 \\ & -1 \end{array}$$

$\therefore x^3 = (x+1)(x^2-x+1) - 1$

OR $\dfrac{x^3}{x+1} = x^2 - x + 1 - \dfrac{1}{x+1}$

ii) $\displaystyle\int \frac{dx}{x^{1/2} + x^{1/3}}$

let $x = u^6$

$\dfrac{dx}{du} = 6u^5$

$dx = 6u^5\,du.$

also $x^{1/2} = u^3$, $x^{1/3} = u^2$

$= \displaystyle\int \frac{6u^5}{u^3+u^2}\,du$

$= \displaystyle\int \frac{6u^3}{u+1}\,du$

$= 6\displaystyle\int \frac{u^3}{u+1}\,du$

$= 6\displaystyle\int u^2 - u + 1 - \frac{1}{u+1}\,du$

(from (i) above)

$= 6\left[\dfrac{u^3}{3} - \dfrac{u^2}{2} + u - \ln(u+1)\right]$

$= 2u^3 - 3u^2 + 6u - 6\ln(u+1) + c$

$= 2x^{1/2} - 3x^{1/3} + 6x^{1/6} - 6\ln(x^{1/6}+1) + c$

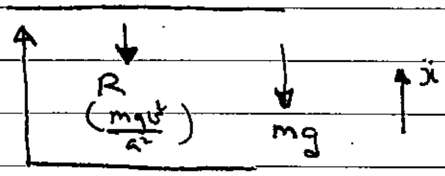
Ques 8

a) $\ddot{x} = \dfrac{dv}{dt}$

$= \dfrac{dv}{dx} \cdot \dfrac{dx}{dt}$

$= v\dfrac{dv}{dx}$

b) i)



$m\ddot{x} = -mg - R$

$\cancel{m\ddot{x} = -mg - \dfrac{mgv^2}{a^2}}$

$\ddot{x} = -g - \dfrac{gv^2}{a^2}$

$v\dfrac{dv}{dx} = -g\left(1 + \dfrac{v^2}{a^2}\right)$

$= -g\left(\dfrac{a^2+v^2}{a^2}\right)$

$= \dfrac{-g}{a^2}(a^2+v^2)$

ii) $v\dfrac{dv}{dx} = -\dfrac{g}{a^2}(a^2+v^2)$

$\dfrac{dv}{dx} = -\dfrac{g}{a^2}\left(\dfrac{a^2+v^2}{v}\right)$

$\dfrac{dx}{dv} = -\dfrac{a^2}{g}\left(\dfrac{v}{a^2+v^2}\right)$

$x = -\dfrac{a^2}{2g}\ln(a^2+v^2) + c$

at $t=0$, $x=0$, $v=u$.

$0 = -\dfrac{a^2}{2g}\ln(a^2+u^2)$

$c = \dfrac{a^2}{2g}\ln(a^2+u^2)$

$$x = -\frac{a^2}{2g}\ln(a^2+v^2) + \frac{a^2}{2g}\ln(a^2+u^2)$$

$$x = \frac{a^2}{2g}\ln\left(\frac{a^2+u^2}{a^2+v^2}\right)$$

for greatest height $v = 0$

$$x = \frac{a^2}{2g}\ln\left(\frac{a^2+u^2}{a^2}\right)$$

$$= \frac{a^2}{2g}\ln\left(1+\frac{u^2}{a^2}\right)$$

---

c) i) $P(x) = x^n + ax - b$

$P'(x) = nx^{n-1} + a$

if α is a double root

$P(\alpha) = P'(\alpha) = 0$

$P'(\alpha) = n\alpha^{n-1} + a$

$\therefore\ n\alpha^{n-1} + a = 0$

$$\alpha^{n-1} = \frac{-a}{n}.$$

---

ii) $P(\alpha) = 0$

$\alpha^n + a\alpha - b = 0$

but $a = -n\alpha^{n-1}$

$\alpha^n + (-n\alpha^{n-1})\alpha - b = 0$

$\alpha^n - n\alpha - b = 0$

$\alpha^n(1-n) = b$

$$\alpha^n = \frac{b}{1-n}$$

Now $(\alpha^n)^{n-1} = (\alpha^{n-1})^n$

$$\therefore \left(\frac{b}{1-n}\right)^{n-1} = \left(\frac{-a}{n}\right)^n$$

$$(-1)^{n-1}\left(\frac{b}{1-n}\right)^{n-1} = (-1)^n\left(\frac{a}{n}\right)^n.$$

$$(-1)^n\left(\frac{a}{n}\right)^n - (-1)^{n-1}\left(\frac{b}{1-n}\right)^{n-1} = 0$$

$$(-1)^n\left(\frac{a}{n}\right)^n + (-1)^n\left(\frac{b}{1-n}\right)^{n-1} = 0$$

$$\left(\frac{a}{n}\right)^n + \left(\frac{b}{1-n}\right)^{n-1} = 0$$

---

iii) $$\alpha = \frac{\alpha^n}{\alpha^{n-1}}$$

$$= \frac{\frac{b}{1-n}}{-\frac{a}{n}}$$

$$= \frac{b}{1-n} \times \frac{n}{-a}$$

$$= \frac{nb}{a(n-1)}$$