GOSFORD HIGH SCHOOL

2008
YEAR 12 TRIAL HIGHER SCHOOL CERTIFICATE

MATHEMATICS EXTENSION 2

General Instructions:

- Reading time – 5minutes
- Working time – 3 hours
- Write using black or blue pen.
- Board-approved calculators may be used
- A table of standard integrals is provided at the back of this paper
- All necessary working should be shown in every question.

Total marks: - 120

- Attempt Questions 1 -8
- All questions are of equal value.

Question 1. (15 marks) use a **SEPARATE** writing booklet **Marks**

a) Show that $x^2 \sin x$ is an odd function and hence find $\int_{-1}^{1} \pi - x^2 \sin x \, dx$ **2**

b) Find $\int_{-1}^{0} \frac{1}{x^2 + 2x + 2} \, dx$ **3**

c) Find $\int \frac{4x^2 - 3x - 4}{x^3 + x^2 - 2x} \, dx$ **3**

d) Using the substitution $u = a \sin \theta$ find $\int \sqrt{a^2 - u^2} \, du$ where a is a constant and $|u| < a$. **3**

e) Find $\int e^{2x} \sin 3x \, dx$ **4**

Question 2. (15 marks) use a **SEPARATE** writing booklet

a) If $z = 3 + 4i$ and $\omega = 1 + i$ find in the form $a + ib$

i) $z + \omega$ **1**

ii) $z\omega$ **1**

iii) $\bar{z}$ **1**

iv) $|z|$ **1**

v) $\frac{z}{\omega}$ **2**

b) Find two numbers whose sum is 4 and whose product is 8. **2**

Marks

c) If two complex numbers Z_1 and Z_2 are such that $|Z_1 + Z_2| = |Z_1 - Z_2|$ prove that $\frac{Z_1}{Z_2}$ is a pure imaginary number. **3**

d) **i)** If $z = \cos\theta + i\sin\theta$ show $\cos n\theta = \frac{z^n + z^{-n}}{2}$ and $\sin n\theta = \frac{z^n - z^{-n}}{2i}$ **2**

ii) Hence or otherwise prove $\sin 4\theta + \sin 2\theta = 2\sin 3\theta \cos\theta$. **2**

Question 3. (15 marks) use a **SEPARATE** writing booklet

a) **i)** Graph the function $f(x) = 3 - |x - 1|$. **2**

ii) Use your answer to part (i) to do neat, separate sketches of the following.

α) $y = 3 - f(x)$ **1**

β) $y = \frac{1}{f(x)}$ **2**

γ) $y^2 = f(x)$ **2**

b) Show that $\lim_{\theta \to 0} \frac{1 - \cos\theta}{\theta} = 0$ **2**

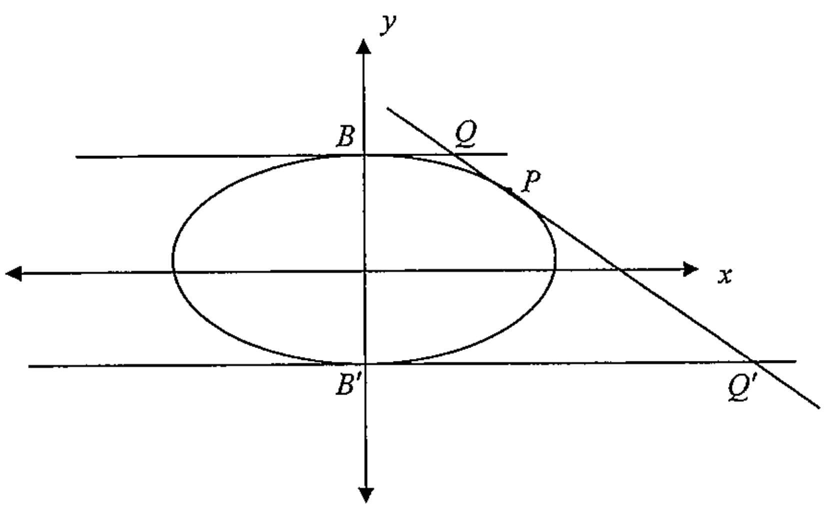
c) The quadratic equation $z^2 + (1 + i)z + k = 0$ has a root of $1 - 2i$. Find, in the form $a + ib$, the value of k and the other root of the equation. **3**

Marks

d) The region under the curve $y = \sin x$, bounded by the x axis and the ordinate $x = \frac{\pi}{2}$ is rotated about the y axis. By using the method of cylindrical shells find the volume of the solid generated. **3**

Question 4. (15 marks) use a **SEPARATE** writing booklet

a)



i) Show that $P(a\cos\theta, b\sin\theta)$ lies on the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$. **1**

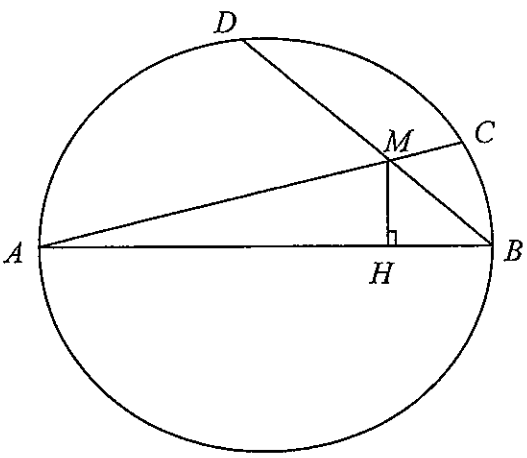
ii) This ellipse meets the y axis at B and B'. The tangents at B and B' to the ellipse meets the tangent at P at the points Q and Q' respectively. Prove that $BQ.B'Q' = a^2$ **3**

(you may assume the equation of the tangent at P is given by $\frac{x\cos\theta}{a}+\frac{y\sin\theta}{b}=1$)

Marks

b) A particle of mass 12 kg rests on a smooth horizontal table, and it is attached by a string 1.2 metres long to a fixed point on the table. If the particle describes a horizontal circle at 3.6 *m/s* find the tension in the string. **2**

c)



AB is the diameter of a circle. Chords *AC* and *BD* intersect at *M*. *H* is a point on *AB* such that *MH* is perpendicular to *AB*.

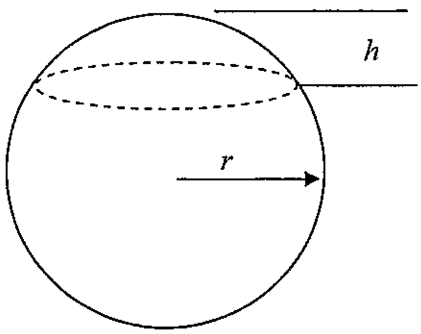
i) Prove that triangle *ABC* is similar to triangle *AMH*. **2**

ii) Show that $AB.AH = AC.AM$. **1**

iii) Prove that $AB^2 = AC.AM + BD.BM$ **3**

Marks

d)



The figure shows a "spherical segment" of height h cut off from a sphere of radius r by a horizontal plane. Show that its volume is

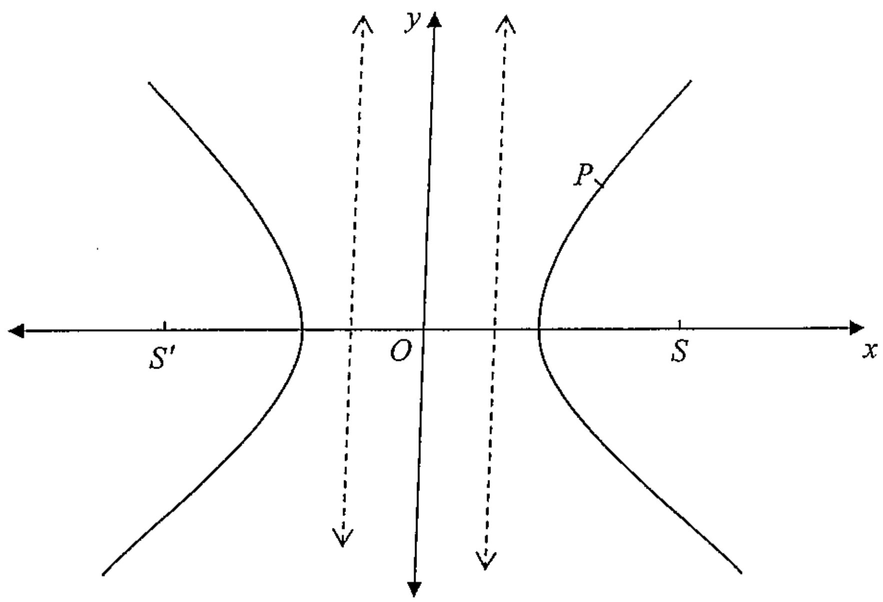
$$V = \frac{1}{3}\pi h^2(3r - h)$$ **3**

Question 5. (15 marks) use a **SEPARATE** writing booklet

a) Sketch the graph of $y = \dfrac{2 + x - x^2}{(x-1)^2}$ clearly showing any turning points and any asymptotes. **4**

b) When a certain polynomial is divided by $x + 1$ and $x - 3$ the respective remainders are 6 and -2. Find the remainder when this polynomial is divided by $x^2 - 2x - 3$. **3**

c) **Marks**



$P(x_1, y_1)$ is a point on the rectangular hyperbola $x^2 - y^2 = a^2$.
S and S' are the foci.

i) Show that the eccentricity is $\sqrt{2}$ **1**

ii) Using the focus directrix definition or otherwise show that

$SP = \sqrt{2}x_1 - a$ and that $S'P = \sqrt{2}x_1 + a$ **2**

iii) Show that $SP.S'P = OP^2$, where O is the origin. **2**

d) According to one cosmological theory, there were equal amounts of the two uranium isotopes ^{235}U and ^{238}U at the creation of the universe in the "big bang." At present there are $137.7\,^{238}U$ atoms for each atom of ^{235}U. Using the half-lives 4.51 billion years for ^{238}U and 0.71 billion years for ^{235}U, calculate the age of the universe. **3**

Marks

Question 6. (15 marks) use a **SEPARATE** writing booklet

a) Consider the equation $z^3 + mz^2 + nz + 6 = 0$, where m and n are real. It is known that $1 - i$ is a root of the equation.

i) Find the other two roots of the equation. **2**

ii) Find the values of m and n. **2**

b) A railway track has been constructed around a circular curve of radius 500 metres. The distance across the track between the rails is 1.5 metres and the outer rail is 0.1 metres above the inner rail. The train travels on the track at a speed of $v_0\ m/s$ which eliminates any sideways force on the wheels.

i) Draw a diagram showing all the forces on the train. **1**

ii) Show that $v_0^2 = 500g\tan\theta$, where θ is the angle the track makes with the horizontal. **2**

iii) Taking $g = 9.8\ \text{m/s}^2$ calculate v_0. **1**

If the train travels on the track at a speed v where $v > v_0$.

iv) State which rail exerts a lateral force on the wheel at the point of contact. **1**

v) Draw a diagram showing all the forces on the train. **1**

vi) Show that the lateral force, F, exerted by the rail on the wheel is given by :

$$F = \frac{mv^2}{500}\cos\theta - mg\sin\theta,$$ where m is the mass of the train. **3**

vii) Deduce that F is one fifth the weight of the train when $v = 2v_0$. **2**

Marks

Question 7. (15 marks) use a **SEPARATE** writing booklet

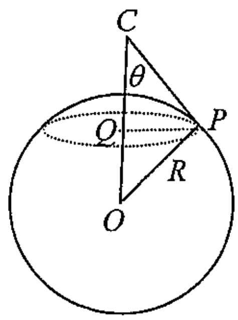
a) The polynomial $P(x) = x^4 - 6x^3 + 13x^2 - ax - b$ has two double zeros α and β. Find a and b. **3**

b) A particle of mass m kg is projected vertically upwards from the ground with a velocity u m/s in a medium whose resistance is given by mkv^2 Newtons, where v is the speed at that instant (in m/s) and k is a positive constant.

i) Prove that the time taken to reach the highest point is $\frac{1}{\sqrt{kg}}\tan^{-1}(u\sqrt{\frac{k}{g}})$ seconds where $g\ m/s^2$ is the acceleration due to gravity. **3**

ii) Prove that the greatest height reached is $\frac{1}{2k}\ln(1+\frac{ku^2}{g})$ metres. **3**

c)



A particle P of mass 2 kg at the end of a string of length $l = 1.3$ metres is suspended from C a point vertically above the highest point of a smooth sphere centre O radius $R = 1.3$ metres. P describes a horizontal circle of radius $PQ = 0.5$ metres on the surface of the sphere.

i) If T is the tension in the string CP and N is the reaction of the surface of the sphere exerted on P show that:

$$\frac{mv^2}{r} = (T - N)\sin\theta \qquad \text{and} \qquad mg = (T + N)\cos\theta$$ **3**

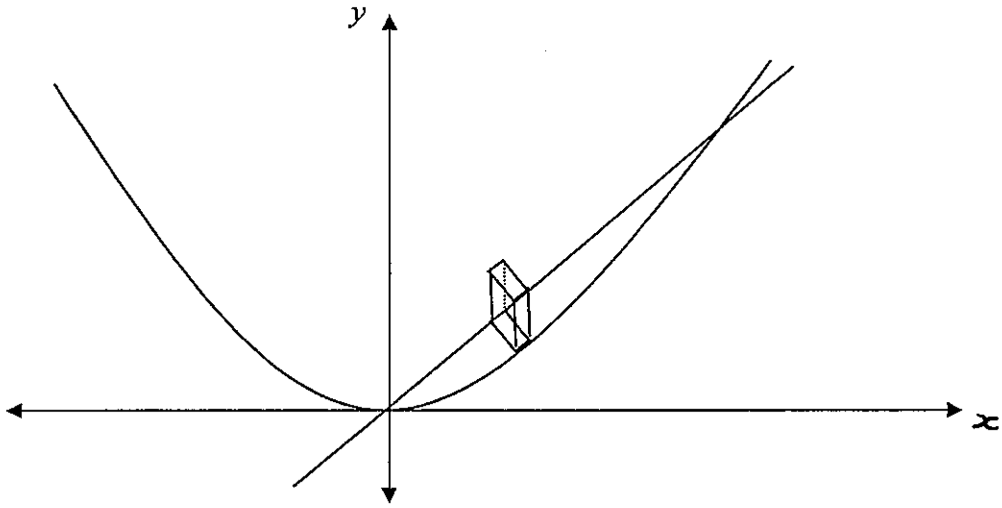
ii) If there is no force exerted by the particle on the sphere find the velocity of P. (take $g = 9.8\ m/s^2$) **3**

Question 8. (15 marks) use a **SEPARATE** writing booklet **Marks**

a) Show that the derivative of $y = x^{x+1}$ is $\left(1 + \frac{1}{x} + \ln x\right)x^{x+1}$. **2**

b) Indicate on an Argand diagram the locus of the point P representing Z when $\arg\left(\frac{Z+1}{Z-i}\right) = 0$. **2**

c)



The base of a solid is the region in the first quadrant bounded by the graphs of $y = x$ and $y = x^2$. Each cross section perpendicular to the line $y = x$ is a square. Find the volume of the solid. **4**

d)

i) Show that $\cos[(n-1)\theta + \theta] - \cos[(n-1)\theta - \theta] = -2\sin(n-1)\theta \sin\theta$. **2**

ii) If $U_n = \int \cos n\theta . \operatorname{cosec}\theta \, d\theta$, prove that $U_n - U_{n-2} = \frac{2\cos(n-1)\theta}{n-1}$. **2**

iii) Hence or otherwise prove that $\int_0^{\frac{\pi}{2}} \frac{\cos 2\theta - \cos 8\theta}{\sin\theta} d\theta = \frac{142}{105}$. **3**

End of paper

STANDARD INTEGRALS

$$\int x^n\,dx = \frac{1}{n+1}x^{n+1}, \quad n \neq -1; \quad x \neq 0, \text{ if } n < 0$$

$$\int \frac{1}{x}\,dx = \ln x, \quad x > 0$$

$$\int e^{ax}\,dx = \frac{1}{a}e^{ax}, \quad a \neq 0$$

$$\int \cos ax\,dx = \frac{1}{a}\sin ax, \quad a \neq 0$$

$$\int \sin ax\,dx = -\frac{1}{a}\cos ax, \quad a \neq 0$$

$$\int \sec^2 ax\,dx = \frac{1}{a}\tan ax, \quad a \neq 0$$

$$\int \sec ax\,\tan ax\,dx = \frac{1}{a}\sec ax, \quad a \neq 0$$

$$\int \frac{1}{a^2+x^2}\,dx = \frac{1}{a}\tan^{-1}\frac{x}{a}, \quad a \neq 0$$

$$\int \frac{1}{\sqrt{a^2-x^2}}\,dx = \sin^{-1}\frac{x}{a}, \quad a > 0, \quad -a < x < a$$

$$\int \frac{1}{\sqrt{x^2-a^2}}\,dx = \ln\left(x+\sqrt{x^2-a^2}\right), \quad x > a > 0$$

$$\int \frac{1}{\sqrt{x^2+a^2}}\,dx = \ln\left(x+\sqrt{x^2+a^2}\right)$$

NOTE : $\ln x = \log_e x, \quad x > 0$

EXT 2. TRIAL 2008 SOLUTIONS

Q.1)

a) $x^2 \sin x$

odd if $f(a) = -f(-a)$

$f(a) = a^2 \sin a$

$f(-a) = (-a)^2 \sin(-a)$

$= a^2 \times -\sin a$

$= -a^2 \sin a$

$= -f(a)$

$\therefore$ function is odd.

$\int_{-1}^{1} \pi - x^2 \sin x \, dx$

$= \int_{-1}^{1} \pi \, dx - \int_{-1}^{1} x^2 \sin x \, dx$

$= [\pi x]_{-1}^{1} - 0$

$= \pi - (-\pi)$

$= 2\pi$

b) $\int_{-1}^{0} \frac{1}{x^2+2x+2} dx$

$= \int_{-1}^{0} \frac{1}{x^2+2x+1+1}$

$= \int_{-1}^{0} \frac{1}{(x+1)^2+1}$

$= \left[\tan^{-1}(x+1)\right]_{-1}^{0}$

$= \tan^{-1}1 - \tan^{-1}0$

$= \frac{\pi}{4}$

c) $\int \frac{4x^2-3x-4}{x^3+x^2-2x} dx$

Now $\frac{4x^2-3x-4}{x^3+x^2-2x} = \frac{4x^2-3x-4}{x(x+2)(x-1)}$

Let

$\frac{4x^2-3x-4}{x(x+2)(x-1)} = \frac{a}{x} + \frac{b}{x+2} + \frac{c}{x-1}$

$4x^2-3x-4 = a(x+2)(x-1) + bx(x-1) + cx(x+2)$

Let $x=1$: $-3 = 3c$

$-1 = c$

Let $x=0$: $-4 = -2a$

$2 = a$

Let $x=-2$: $18 = 6b$

$3 = b$

$\therefore \int \frac{4x^2-3x-4}{x^3+x^2-2x} dx = \int \frac{2}{x} + \frac{3}{x+2} - \frac{1}{x-1} dx$

$= 2\ln x + 3\ln(x+2) - \ln(x-1) + c$

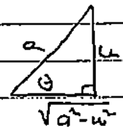
$= \ln\left(\frac{x^2(x+2)^3}{x-1}\right) + c$

d) $\int \sqrt{a^2-u^2} \, du$

$u = a\sin\theta$

$\frac{du}{d\theta} = a\cos\theta$

$du = a\cos\theta \, d\theta$



$= \int \sqrt{a^2 - a^2\sin^2\theta} \cdot a\cos\theta \, d\theta$

$= a\int \sqrt{a^2(1-\sin^2\theta)} \cos\theta \, d\theta$

$= a\int a\cos\theta \cdot \cos\theta \, d\theta$

$= a^2 \int \cos^2\theta \, d\theta$

$= \frac{a^2}{2} \int \cos 2\theta + 1 \, d\theta$

$= \frac{a^2}{2}\left[\frac{1}{2}\sin 2\theta + \theta\right]$

$= \frac{a^2}{2}\left[\sin\theta\cos\theta + \theta\right]$

$= \frac{a^2}{2}\left[\frac{u}{a} \cdot \frac{\sqrt{a^2-u^2}}{a} + \sin^{-1}\left(\frac{u}{a}\right)\right]$

$= \frac{1}{2}\left(u\sqrt{a^2-u^2} + a^2\sin^{-1}\left(\frac{u}{a}\right)\right) + c$

e) $\int e^{2x} \sin 3x \, dx$

Let

$I = \int \frac{d}{dx}\left(\frac{1}{2}e^{2x}\right) \sin 3x \, dx$

$= \frac{1}{2}e^{2x}\sin 3x - \int \frac{1}{2}e^{2x} \cdot 3\cos 3x \, dx$

$= \frac{1}{2}e^{2x}\sin 3x - \frac{3}{2}\int e^{2x} \cdot \cos 3x \, dx$

$= \frac{1}{2}e^{2x}\sin 3x - \frac{3}{2}\int \frac{d}{dx}\left(\frac{1}{2}e^{2x}\right) \cdot \cos 3x \, dx$

$= \frac{1}{2}e^{2x}\sin 3x - \frac{3}{2}\left[\frac{1}{2}e^{2x} \cdot \cos 3x - \int \frac{1}{2}e^{2x} - 3\sin 3x \, dx\right]$

$= \frac{1}{2}e^{2x}\sin 3x - \frac{3}{4}e^{2x}\cos 3x - \frac{9}{4}\int e^{2x}\sin 3x \, dx$

$= \frac{1}{2}e^{2x}\sin 3x - \frac{3}{4}e^{2x}\cos 3x - \frac{9}{4} I$

$\therefore \frac{13}{4} I = \frac{1}{2}e^{2x}\sin 3x - \frac{3}{4}e^{2x}\cos 3x$

$I = \frac{4}{13}\left(\frac{1}{2}e^{2x}\sin 3x - \frac{3}{4}e^{2x}\cos 3x\right)$

$= \frac{e^{2x}}{13}\left(2\sin 3x - 3\cos 3x\right) + c$

Q2)

a) $z = 3+4i$, $w = 1+i$

i) $z + w$
$= 4+5i$

ii) zw
$= (3+4i)(1+i)$
$= 3+3i+4i-4$
$= -1+7i$

iii) $\bar{z}$
$= 3-4i$

iv) $|z|$
$= \sqrt{3^2+4^2}$
$= 5$

v) $\frac{z}{w} = \frac{3+4i}{1+i} \times \frac{1-i}{1-i}$

$= \frac{3-3i+4i+4}{2}$

$= \frac{7+i}{2}$

$= \frac{7}{2} + \frac{i}{2}$.

b) Let the numbers be a, b

$a+b = 4$ ---(1)

$ab = 8$ ---(2)

(1) $\Rightarrow b = 4-a$

Sub into (2)

$a(4-a) = 8$

$4a - a^2 = 8$

$a^2 - 4a + 8 = 0$

$a = \frac{4 \pm \sqrt{16-32}}{2}$

$= \frac{4 \pm \sqrt{-16}}{2}$

$= \frac{4 \pm 4i}{2}$

$= 2 \pm 2i$

$\therefore$ 2 numbers are $2+2i$, $2-2i$

c) $|z_1+z_2| = |z_1-z_2|$

Let $z_1 = x_1+iy_1$, $z_2 = x_2+iy_2$

$\sqrt{(x_1+x_2)^2+(y_1+y_2)^2} = \sqrt{(x_1-x_2)^2+(y_1-y_2)^2}$

$2x_1x_2 + 2y_1y_2 = -2x_1x_2 - 2y_1y_2$

$4x_1x_2 + 4y_1y_2 = 0$

$x_1x_2 + y_1y_2 = 0$

$\frac{z_1}{z_2} = \frac{x_1+iy_1}{x_2+iy_2}$

$= \frac{x_1+iy_1}{x_2+iy_2} \times \frac{x_2-iy_2}{x_2-iy_2}$

$= \frac{x_1x_2 - ix_1y_2 + ix_2y_1 + y_1y_2}{x_2^2+y_2^2}$

$= \frac{x_1x_2+y_1y_2 + i(x_2y_1 - x_1y_2)}{x_2^2+y_2^2}$

but $x_1x_2+y_1y_2 = 0$

$= \frac{i(x_2y_1-x_1y_2)}{x_2^2+y_2^2}$

which is a pure imaginary number

d)

(i) let $z = \cos\theta + i\sin\theta$

$z^n = (\cos\theta + i\sin\theta)^n$

$= \cos n\theta + i\sin n\theta$ ---(1)

also $z^{-n} = \cos(-n)\theta + i\sin(-n)\theta$

$= \cos n\theta - i\sin n\theta$ ---(2)

(1)+(2) $z^n + z^{-n} = 2\cos n\theta$

$\cos n\theta = \frac{z^n+z^{-n}}{2}$

(1)-(2) $z^n - z^{-n} = 2i\sin n\theta$

$\sin n\theta = \frac{z^n - z^{-n}}{2i}$

ii) $\sin 4\theta + \sin 2\theta = 2\sin 3\theta\cos\theta$

R.H.S.

$= 2\left(\frac{z^3-z^{-3}}{2i}\right)\left(\frac{z+\frac{1}{z}}{2}\right)$

$= \frac{z^4+z^2-z^{-2}-z^{-4}}{2i}$

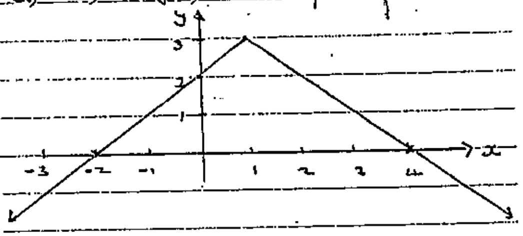
$= \frac{z^4-z^{-4}}{2i} + \frac{z^2-z^{-2}}{2i}$

$= \frac{2i\sin 4\theta}{2i} + \frac{2i\sin 2\theta}{2i}$

$= \sin 4\theta + \sin 2\theta$

Q3)

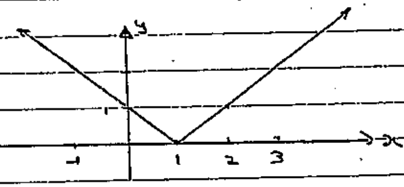
a) i) $f(x) = 3 - |x-1|$



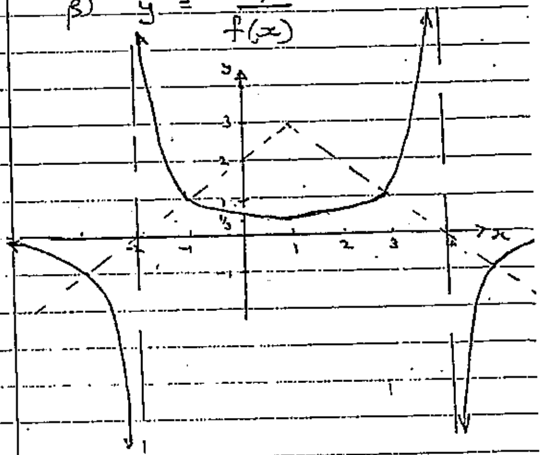
ii) α) $y = 3 - f(x)$

$= 3 - (3-|x-1|)$

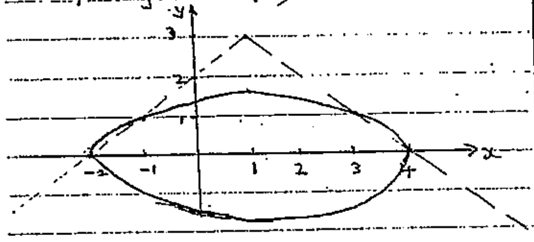
$= |x-1|$



β) $y = \frac{1}{f(x)}$



γ) $y^2 = f(x)$



b) $\lim_{\theta\to 0} \frac{1-\cos\theta}{\theta}$

$= \lim_{\theta\to 0} \frac{1-\cos\theta}{\theta}\times\frac{1+\cos\theta}{1+\cos\theta}$

$= \lim_{\theta\to 0} \frac{1-\cos^2\theta}{\theta(1+\cos\theta)}$

$= \lim_{\theta\to 0} \frac{\sin^2\theta}{\theta(1+\cos\theta)}$

$= \lim_{\theta\to 0}\frac{\sin\theta}{\theta}\times\lim_{\theta\to 0}\frac{\sin\theta}{1+\cos\theta}$

$= 1 \times \frac{0}{1+1}$

$= 0$

c) $z^2 + (1+i)z + k = 0$

$1-2i$ is a root

$\therefore (1-2i)^2 + (1+i)(1-2i) + k = 0$

$1-4i-4+1-2i+i+2+k=0$

$-5i + k = 0$

$k = 5i$

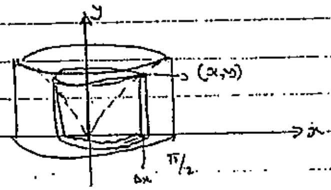
let α be the other root

$\therefore \alpha + 1 - 2i = \frac{-b}{a}$

$= -1-i$

$\therefore \alpha = -2+i$

d)



Volume of shell $= 2\pi rh$

$= 2\pi xy$

$V \doteq \sum_{x=0}^{\pi/2} 2\pi xy\,\Delta x$

$V = \lim_{\Delta x\to 0}\sum_{x=0}^{\pi/2} 2\pi xy\,\Delta x$

$= 2\pi\int_0^{\pi/2} xy\,dx$

$= 2\pi\int_0^{\pi/2} x\sin x\,dx$

$= 2\pi\int_0^{\pi/2} x\frac{d}{dx}(-\cos x)\,dx$

$= 2\pi\left(-x\cos x - \int_0^{\pi/2} -\cos x\,dx\right)$

$= 2\pi\left[-x\cos x + \sin x\right]_0^{\pi/2}$

$= 2\pi[(0+1)-(0+0)]$

$= 2\pi$ cubic units

Q4)

a) i) $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$

sub $(a\cos\theta, b\sin\theta)$

$\frac{a^2\cos^2\theta}{a^2} + \frac{b^2\sin^2\theta}{b} = 1$

$\cos^2\theta+\sin^2\theta = 1$

$1 = 1$

$\therefore$ the point lies on the ellipse

ii) for Q: $y = b$ ---(1)

$\frac{x\cos\theta}{a} + \frac{y\sin\theta}{b} = 1$ ---(2)

Sub (1) into (2) $\frac{x\cos\theta}{a} + \sin\theta = 1$

$x = a\frac{(1-\sin\theta)}{\cos\theta}$

$\therefore$ Q $\left(\frac{a(1-\sin\theta)}{\cos\theta}, b\right)$

for Q': $y = -b$ ------(1)

$\frac{x\cos\theta}{a} + \frac{y\sin\theta}{b} = 1$ ------(2)

Sub (1) into (2)

$\frac{x\cos\theta}{a} - \frac{b\sin\theta}{b} = 1$

$x = \frac{a(1+\sin\theta)}{\cos\theta}$

$\therefore Q' \left(\frac{a(1+\sin\theta)}{\cos\theta}, -b \right)$

$\therefore BQ = \frac{a(1-\sin\theta)}{\cos\theta}$

$B'Q' = \frac{a(1+\sin\theta)}{\cos\theta}$

$\therefore BQ.B'Q' = \frac{a(1-\sin\theta)}{\cos\theta} \cdot \frac{a(1+\sin\theta)}{\cos\theta}$

$= \frac{a^2(1-\sin^2\theta)}{\cos^2\theta}$

$= \frac{a^2\cos^2\theta}{\cos^2\theta}$

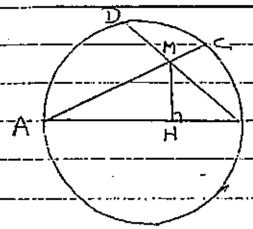
$= a^2$

b) $T = \frac{mv^2}{r}$

$= \frac{12 \times 3.6^2}{1.2}$

$= 129.6$ N.

c)



i) $\Delta ABC \parallel\!| \Delta AMH$

$\angle A$ is common

$\angle AHM = 90°$ given

$\angle ACB = 90°$ AB a diameter angle in a semicircle

$\therefore \Delta ABC \parallel\!| \Delta AMH$ (A.A.A.)

(ii) fro (i) $\Delta ABC \parallel\!| \Delta AMH$

$\therefore \frac{AB}{AM} = \frac{AC}{AH}$

$\therefore AB.AH = AM.AC$ ----(1)

(iii) $\Delta BAD \parallel\!| \Delta BMH$

$\angle B$ common

$\angle BHM = 90°$ given

$\angle BDA = 90°$ angle in a semi circl

$\therefore \frac{AB}{BM} = \frac{BD}{BH}$

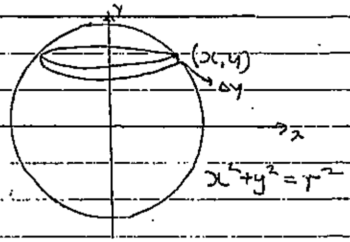
$AB.BH = BM.BD$ -----(2)

Now

(1)+(2) $AB(AH+BH) = AM.AC + BM.BD$.

$AB^2 = AC.AM + BM.BD$.

d)



Volume of a slice $= \pi x^2 \Delta y$

$V \doteq \sum_{y=r-h}^{r} \pi x^2 \Delta y$

$V = \lim_{\Delta y \to 0} \sum_{y=r-h}^{r} \pi x^2 \Delta y$

$= \int_{r-h}^{r} x^2 \, dy$

$\int_{r-h}^{r} r^2 - y^2 \, dy$

$= \pi \left[r^2 y - \frac{y^3}{3} \right]_{r-h}^{r}$

$= \pi \left[\left(r^3 - \frac{r^3}{3}\right) - \left(r^2(r-h) - \frac{(r-h)^3}{3}\right) \right]$

$= \pi \left[\frac{2r^3}{3} - r^3 + r^2h + \frac{r^3 - 3r^2h + 3rh^2 - h^3}{3} \right]$

$= \pi \left[\frac{r^3}{3} + r^2h + \frac{r^3}{3} - r^2h + rh^2 - \frac{h^3}{3} \right]$

$= \frac{\pi}{3}\left(-r^3 + 3r^2h + r^3 - 3r^2h + 3rh^2 + h^3 \right)$

$= \frac{\pi}{3}\left(3rh^2 - h^3 \right)$

$= \frac{\pi h^2}{3}(3r - h)$

Q5)

a) $y = \frac{2+x-x^2}{(x-1)^2}$

$\frac{dy}{dx} = \frac{(x-1)^2(1-2x) - 2(2+x-x^2)(x-1)}{(x-1)^4}$

$= \frac{(x-1)\left((x-1)(1-2x) - 2(2+x-x^2)\right)}{(x-1)^4}$

$= \frac{(x-1)\left(x^2 - 2x^2 - 1 + 2x - 4 - 2x + 2x^2\right)}{(x-1)^4}$

$= \frac{(x-1)(x-5)}{(x-1)^4}$

$= \frac{x-5}{(x-1)^3}$

Turning point when $\frac{dy}{dx} = 0$

$\therefore \frac{(x-5)}{(x-1)^3} = 0$

$\therefore x = 5$.

$f'(4) < 0$

$f'(6) > 0$

$\therefore (5, -\frac{9}{..})$ min

asymptote:

Vertical: $x = 1$

other: $\lim_{x \to \infty} \frac{2+x-x^2}{x^2-2x+1}$

$= \lim_{x \to \infty} \frac{2/x^2 + x/x^2 - 1}{x^2/x^2 - 2x/x^2 + 1/x^2}$

$= \frac{-1}{1}$

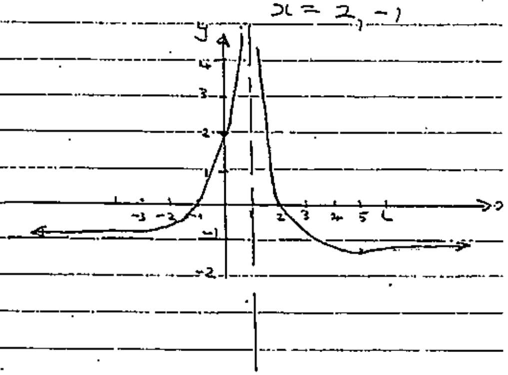
$= -1$

'y' intercept: 2.

'x' intercept: $x^2 - x - 2 = 0$

$(x-2)(x+1) = 0$

$x = 2, -1$



b) Let $P(x) = (x^2-2x-3)Q(x) + ax$

$P(x) = (x+1)(x-3)Q(x) + ax+b$

$P(-1) = 6$

$-a + b = 6$ ---(1)

$P(3) = -2$

$3a + b = -2$ ----(2)

(1)-(2) $-4a = 8$

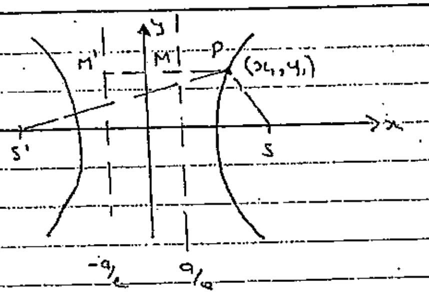
$a = -2$

Sub $a = -2$ into (1)

$b = 4$

$\therefore$ remainder $-2x + 4$.

c)



i) $x^2 - y^2 = a^2$

$b^2 = a^2(e^2 - 1)$

but $b = a$

$b^2 = b^2(e^2 - 1)$

$1 = e^2 - 1$

$2 = e^2$

$e = \sqrt{2}.$

ii) by definition

$SP = ePM$

$= e(x_1 - a/e)$

$= \sqrt{2}(x_1 - \frac{a}{\sqrt{2}})$

$= \sqrt{2}x_1 - a.$

also $S'P = ePM'$

$= \sqrt{2}(x_1 + \frac{a}{\sqrt{2}})$

$= \sqrt{2}x_1 + a.$

iii) $SP\,S'P = (\sqrt{2}x_1 + a)(\sqrt{2}x_1 - a)$

$= 2x_1^2 - a^2$

Now

$OP^2 = x_1^2 + y_1^2$

$= x_1^2 + x_1^2 - a^2$

$= 2x_1^2 - a^2$

$\left\{ \begin{array}{l} x_1^2 - y_1^2 = a^2 \\ y_1^2 = x_1^2 - a^2 \end{array} \right.$

$\therefore SP\,S'P = OP^2$

d) Let $M_{235} = M_0 e^{-kt}$, $M_{238} = M_0 e^{-ct}$

$\therefore \frac{1}{2} = e^{-0.71k}$, $\frac{1}{2} = e^{-4.51c}$

$\ln\frac{1}{2} = -0.71k$ $\ln\frac{1}{2} = -4.51c$

$\therefore k = 0.9736$, $c = 0.1537$

$\therefore M_{235} = M_0 e^{-0.9736t}$, $M_{238} = M_0 e^{-0.1537t}$

Now $M_{238} = 137.7\, M_{23}$

$\frac{M_0 e^{-0.1537t}}{M_0 e^{-0.9736t}} = 137.7$

$e^{0.8226t} = 137.7$

$t = \frac{\ln 137.7}{0.8226}$

$= 5.99$ billion years.

(Q6)

a) $z^3 + mz^2 + nz + 6 = 0$

(i) m, n real $\Rightarrow 1+i$ is a root.

let 3 root be α

$\therefore \alpha(1+i)(1-i) = -\frac{d}{a}$

$2\alpha = -6$

$\alpha = -3$

$\therefore$ other roots $1+i, -3$

ii) $1+i+1-i-3 = -b/a$

$-1 = -m$

$1 = m$

If -3 is a root

$(-3)^3 + (-3)^2 - 3n + 6 = 0$

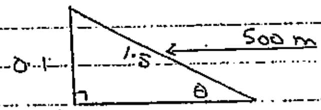
$-12 - 3n = 0$

$3n = -12$

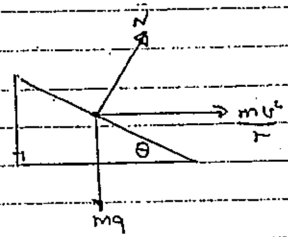
$n = -4.$

$\therefore m = 1,\ n = -4$

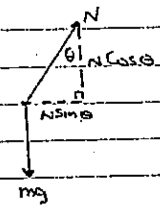
b)



i)



ii)



Vertically:

$N\cos\theta = mg$ --- (1)

horizontally:

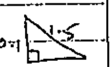
$N\sin\theta = \frac{mv^2}{r}$ --- (2)

(2) ÷ (1) $\tan\theta = \frac{v^2}{rg}$

$v_0^2 = rg\tan\theta$

$= 500g\tan\theta$

iii) $v_0^2 = 500 \times 9.8 \times \frac{0.1}{1.5}$



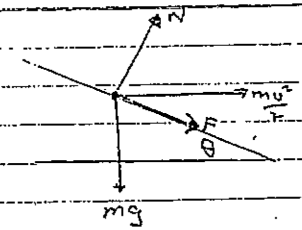
(for small θ $\tan\theta \doteqdot \sin\theta$)

$= 326\frac{2}{3}$

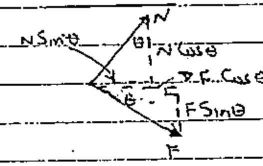
$v_0^2 = 18$ m/s.

iv) outer

v)



vi)



Vertically:

$N\cos\theta = F\sin\theta + mg$

$N\cos\theta - F\sin\theta = mg$ --- (1)

horizontally

$N\sin\theta + F\cos\theta = \frac{mv^2}{r}$ --- (2)

(1) × Sinθ : $N\sin\theta\cos\theta - F\sin^2\theta = mg\sin\theta$ --- (3)

(2) × Cosθ : $N\sin\theta\cos\theta + F\cos^2\theta = \frac{mv^2}{r}\cos\theta$ --- (4)

(4) − (3) $F(\sin^2\theta + \cos^2\theta) = \frac{mv^2\cos\theta}{r} - mg\sin\theta$

$F = \frac{mv^2\cos\theta}{500} - mg\sin\theta$

vii) $F = \frac{mv^2\cos\theta}{500} - mg\sin\theta$

$= \frac{m(2v_0)^2\cos\theta}{500} - mg\sin\theta$ $(v = 2v_0)$

$= \frac{4mv_0^2\cos\theta}{500} - mg\sin\theta$

but $v_0^2 = 500g\tan\theta$

$\therefore F = \frac{4m\,500g\tan\theta\cos\theta}{500} - mg\sin\theta$

$= 4mg\sin\theta - mg\sin\theta$

$= 3mg\sin\theta$

$= 3mg \times \frac{0.1}{1.5}$

$= 3mg \times \frac{1}{15}$

$= \frac{1}{5}mg$

$\therefore F = \frac{1}{5}$ the weight of the train

Q7)

a) $P(x) = x^4 - 6x^3 + 13x^2 - ax - b$

Let roots be $\alpha, \alpha, \beta, \beta$

$2\alpha + 2\beta = \frac{-b}{a}$

$2\alpha + 2\beta = 6$

$\therefore \alpha + \beta = 3$ ---(1)

$\alpha^2 + 4\alpha\beta + \beta^2 = \frac{c}{a}$

$(\alpha+\beta)^2 + 2\alpha\beta = 13$ ---(2)

Sub (1) into (2)

$9 + 2\alpha\beta = 13$

$\alpha\beta = 2$

Now $2\alpha^2\beta + 2\alpha\beta^2 = \frac{-d}{a}$

$2\alpha\beta(\alpha+\beta) = a$

$4 \times 3 = a$

$12 = a$

also $\alpha^2\beta^2 = -b$

$(\alpha\beta)^2 = -b$

$2^2 = -b$

$4 = -b$

$-4 = b$

$\therefore a = 12 \quad b = -4$

(b) i) $\uparrow$ $v = u$, $\downarrow mg$, $\downarrow R = mkv^2$

i) $ma = -mg - mkv^2$

$a = -g - kv^2$

$\frac{dv}{dt} = -g - kv^2$

$\frac{dt}{dv} = \frac{-1}{g + kv^2}$

$= -\frac{1}{k}\left(\frac{1}{\frac{g}{k} + v^2}\right)$

$-kt\,\frac{dt}{dv} = \frac{1}{\frac{g}{k} + v^2}$

$-kt = \sqrt{\frac{k}{g}}\tan^{-1}\sqrt{\frac{k}{g}}\,v + c$

$t = 0, \quad v = u$

$0 = \sqrt{\frac{k}{g}}\tan^{-1}\sqrt{\frac{k}{g}}\,u + c$

$\therefore c = -\sqrt{\frac{k}{g}}\tan^{-1}\sqrt{\frac{k}{g}}\,u$

$\therefore -kt = \sqrt{\frac{k}{g}}\tan^{-1}\sqrt{\frac{k}{g}}\,v - \sqrt{\frac{k}{g}}\tan^{-1}\sqrt{\frac{k}{g}}\,u$

$\therefore t = \sqrt{\frac{1}{kg}}\tan^{-1}u\sqrt{\frac{k}{g}} - \sqrt{\frac{1}{kg}}\tan^{-1}v\sqrt{\frac{k}{g}}$

max height $v = 0$

$\therefore t = \frac{1}{\sqrt{kg}}\tan^{-1}u\sqrt{\frac{k}{g}}$

ii) $ma = -mg - mkv^2$

$a = -g - kv^2$

$v\frac{dv}{dx} = -g - kv^2$

$\frac{dv}{dx} = \frac{-g - kv^2}{v}$

$\frac{dx}{dv} = \frac{v}{-g - kv^2}$

$-\frac{dx}{dv} = \frac{v}{g + kv^2}$

$-x = \frac{1}{2k}\ln(g + kv^2) + c$

$x = 0, \quad v = u$

$0 = \frac{1}{2k}\ln(g + ku^2) + c$

$\therefore c = -\frac{1}{2k}\ln(g + ku^2)$

$\therefore -x = \frac{1}{2k}\ln(g + kv^2) - \frac{1}{2k}\ln(g + ku^2)$

$x = \frac{1}{2k}\ln(g + ku^2) - \frac{1}{2k}\ln(g + kv^2)$

$= \frac{1}{2k}\ln\left(\frac{g + ku^2}{g + kv^2}\right)$

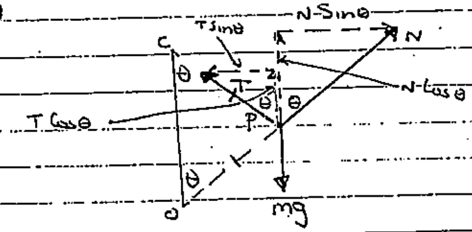
greatest height $v = 0$

$x = \frac{1}{2k}\ln\left(\frac{g + ku^2}{g}\right)$

$= \frac{1}{2k}\ln\left(1 + \frac{ku^2}{g}\right)$

c) Forces at P

(i)



(N.B. $CP = OP = 1.3$ $\therefore \triangle CPO$ isosceles

$\therefore \angle COP = \angle OCP = \theta$)

Vertically

$T\cos\theta + N\cos\theta = mg$

$(T + N)\cos\theta = mg$

horizontally

$T\sin\theta - N\sin\theta = \frac{mv^2}{r}$

$(T - N)\sin\theta = \frac{mv^2}{r}$

ii) no force on the Sphere

$\Rightarrow N = 0$

$\therefore T\sin\theta = \frac{mv^2}{r}$ ---(1)

$T\cos\theta = mg$ ---(2)

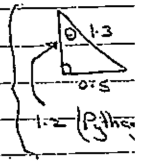
(1) ÷ (2) $\tan\theta = \frac{v^2}{rg}$

$v^2 = rg\tan\theta$

$= 0.5 \times 9.8 \times \frac{0.5}{1.2}$

$= 2.04$

$v = 1.43$ m/s



Q8)

a) $y = x^{x+1}$

$\ln y = \ln x^{x+1}$

$\ln y = (x+1)\ln x$

$\ln y = x\ln x + \ln x$

$\frac{1}{y}\frac{dy}{dx} = x \cdot \frac{1}{x} + \ln x + \frac{1}{x}$

$\frac{1}{y}\frac{dy}{dx} = 1 + \frac{1}{x} + \ln x$

$\frac{dy}{dx} = \left(1 + \frac{1}{x} + \ln x\right)y$

$= \left(1 + \frac{1}{x} + \ln x\right)x^{x+1}$

b) $\arg\left(\frac{z+i}{z-i}\right) = 0$

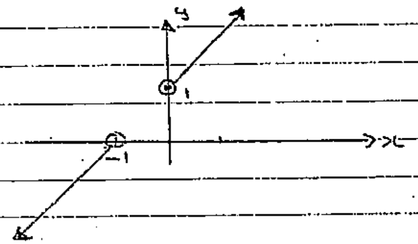
$\arg(z+i) - \arg(z-i) = 0$

$\arg(z+i) = \arg(z-i)$

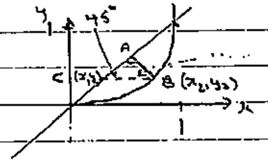
$\Rightarrow$ the angle from the positive OX direction from $-i$ to z equals the angle from the positive direction from i to Z

$\therefore -1$, i and Z must be collinear.

Now if Z lies between -1 and i then angle would be opposite in sign. $\therefore Z$ cannot be between -1 and i



c)



Area of cross-section

$= AB^2$

Now from triangle ABC

$\frac{AB}{BC} = \sin 45°$

$AB = BC \sin 45$

$= \frac{x_2 - x_1}{\sqrt{2}}$

$\therefore$ Volume of slice

$= \left(\frac{x_2-x_1}{\sqrt{2}}\right)^2 \Delta y$

Volume $\doteq \sum_{y=0}^{1} \left(\frac{x_2-x_1}{\sqrt{2}}\right)^2 \Delta y$

$V = \lim_{\Delta y \to 0} \sum_{y=0}^{1} \left(\frac{x_2-x_1}{\sqrt{2}}\right)^2 \Delta y$

$V = \int_0^1 \left(\frac{x_2-x_1}{\sqrt{2}}\right)^2 dy$.

Now $y = x_2^2$ and $y = x_1$

$\therefore x_2 = \sqrt{y}$

$\therefore V = \int_0^1 \left(\frac{\sqrt{y}-y}{\sqrt{2}}\right)^2 dy$

$= \int_0^1 \frac{y - 2y^{3/2} + y^2}{2} dy$

$= \frac{1}{2}\left[\frac{y^2}{2} - \frac{4y^{5/2}}{5} + \frac{y^3}{3}\right]_0^1$

$= \frac{1}{2}\left[\frac{1}{2} - \frac{4}{5} + \frac{1}{3}\right]$

$= \frac{1}{60}$ cubic units

d)

i) $\cos[(n-1)\theta + \theta] - \cos[(n-1)\theta - \theta] = -2\sin(n-1)\theta\sin\theta$

L.H.S

$= \cos(n-1)\theta\cos\theta - \sin(n-1)\theta\sin\theta$

$- (\cos(n-1)\theta\cos\theta + \sin(n-1)\theta\sin\theta)$

$= -2\sin(n-1)\theta\sin\theta$

$=$ R.H.S

ii) $U_n - U_{n-2} = \int \cos n\theta \operatorname{cosec}\theta \, d\theta - \int \cos(n-2)\theta \operatorname{cosec}\theta \, d\theta$

$= \int \operatorname{cosec}\theta \left(\cos n\theta - \cos(n-2)\theta\right) d\theta$

$= \int \operatorname{cosec}\theta \left(-2\sin(n-1)\theta\sin\theta\right) d\theta$ from (i)

$= -2\int \sin(n-1)\theta \, d\theta$

$= -2\left(\frac{-\cos(n-1)\theta}{n-1}\right)$

$= \frac{2\cos(n-1)\theta}{n-1}$

iii) $\int_0^{\pi/2} \frac{\cos 2\theta - \cos 8\theta}{\sin\theta} d\theta$

$= \int_0^{\pi/2} \cos 2\theta \operatorname{cosec}\theta - \cos 8\theta \operatorname{cosec}\theta \, d\theta$

Now $U_n - U_{n-2} = \frac{2\cos(n-1)\theta}{n-1}$

$\therefore U_n = \frac{2\cos(n-1)\theta}{n-1} + U_{n-2}$.

$\therefore U_8 = \frac{2\cos 7\theta}{7} + U_6$

$U_6 = \frac{2\cos 5\theta}{5} + U_4$

$U_4 = \frac{2\cos 3\theta}{3} + U_2$

$\therefore U_8 = \frac{2\cos 7\theta}{7} + \frac{2\cos 5\theta}{5} + \frac{2\cos 3\theta}{3} + U_2$

$U_8 - U_2 = \left[\frac{2\cos 7\theta}{7} + \frac{2\cos 5\theta}{5} + \frac{2\cos 3\theta}{3}\right]_0^{\pi/2}$

$= \left[\left(\frac{2\cos\frac{7\pi}{2}}{7} + \frac{2\cos\frac{5\pi}{2}}{5} + \frac{2\cos\frac{3\pi}{2}}{3}\right) - \left(\frac{2}{7} + \frac{2}{5} + \frac{2}{3}\right)\right]$

$= 0 - \frac{142}{105}$

$\therefore U_2 - U_8 = \frac{142}{105}$.