GOSFORD HIGH SCHOOL

2013
HIGHER SCHOOL CERTIFICATE
TRIAL EXAMINATION

Mathematics Extension 2

General Instructions

- Reading Time - 5 minutes
- Working Time - 3 hours
- Write using a blue or black pen. Black pen is preferred
- Board approved calculators may be used
- A table of standard integrals is provided at the back of this paper
- Show all necessary working in Questions 11-16

Total marks (100)

Section I

Total marks (**10**)

- Attempt Questions 1-10
- Answer on the Multiple Choice answer sheet provided
- Allow about 15 minutes for this section

Section II

Total marks (**90**)

- Attempt questions 11 – 16
- Answer on the blank paper provided, unless otherwise instructed
- Start a new page for each question
- All necessary working should be shown for every question
- Allow about 2 hours 45 minutes for this section

Section I
10 marks
Attempt Questions 1-10
Allow about 15 minutes for this section

Use the multiple choice answer sheet for Questions 1 – 10.

1. If $Z_1 = 5-2i$ and $Z_2 = 3+4i$ then $Z_1\overline{Z_2} =$

(A) $23+14i$

(B) $7+26i$

(C) $7-26i$

(D) $23-26i$

2. The equation of an ellipse is given by $4x^2+9y^2=36$.The foci and the directrices of this ellipse are:

(A) $(\pm\sqrt{5},0)$ and $x=\pm\frac{9\sqrt{5}}{5}$

(B) $(0,\pm\sqrt{5})$ and $x=\pm\frac{9\sqrt{5}}{5}$

(C) $(\pm\sqrt{5},0)$ and $y=\pm\frac{9\sqrt{5}}{5}$

(D) $(0,\pm\sqrt{5})$ and $y=\pm\frac{9\sqrt{5}}{5}$

3. If $\frac{x+1}{x^2-4}=\frac{a}{x+2}+\frac{b}{x-2}$ then:

(A) $a=-1$ $b=3$

(B) $a=-\frac{1}{4}$ $b=\frac{3}{4}$

(C) $a=-\frac{1}{4}$ $b=\frac{1}{4}$

(D) $a=\frac{1}{4}$ $b=\frac{3}{4}$

4. Given the curve $y = f(x)$, where $f(x)$ is defined for all real x, then the curve $y = -f(x)$ is best described by:

(A) A reflection of $y = f(x)$ in the y-axis.

(B) A reflection of $y = f(x)$ in the x-axis.

(C) A reflection of $y = f(x)$ in the y-axis for $0 \le x$.

(D) A reflection of $y = f(x)$ in the x-axis for $y \le 0$.

5. The equation $x^3 + 2x^2 - 4x + 5 = 0$ has roots α, β and γ. The value of $\frac{1}{\alpha} + \frac{1}{\beta} + \frac{1}{\gamma}$ is

(A) $\frac{5}{4}$

(B) $-\frac{5}{4}$

(C) $\frac{4}{5}$

(D) $-\frac{4}{5}$

6. $\int \sin^3 x \, dx =$

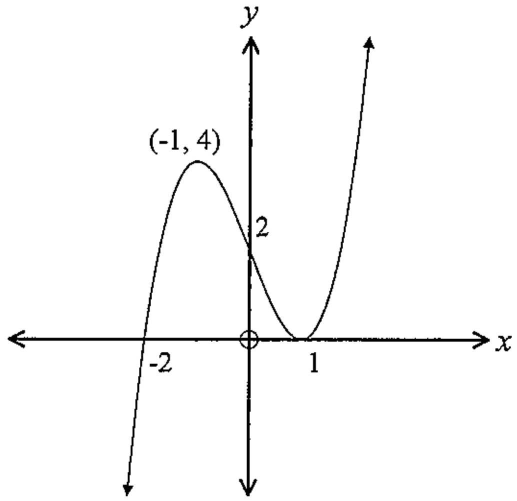
(A) $\cos^3 x - \cos x + c$

(B) $\frac{1}{3}\cos^3 x - \cos x + c$

(C) $\cos^3 x + \cos x + c$

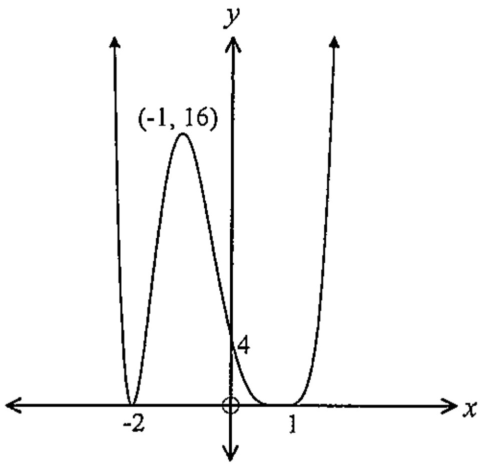
(D) $\frac{1}{3}\cos^3 x + \cos x + c$

7. The graph of the function $y = f(x)$ is drawn below:

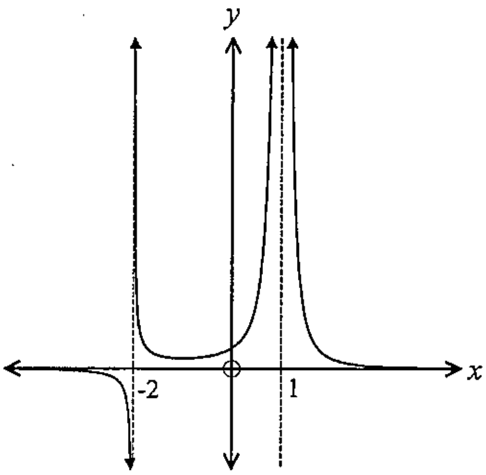


Which of the following graphs best represents the graph $y^2 = f(x)$

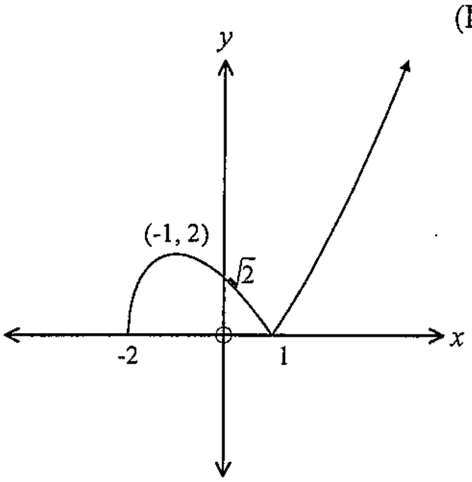
(A)



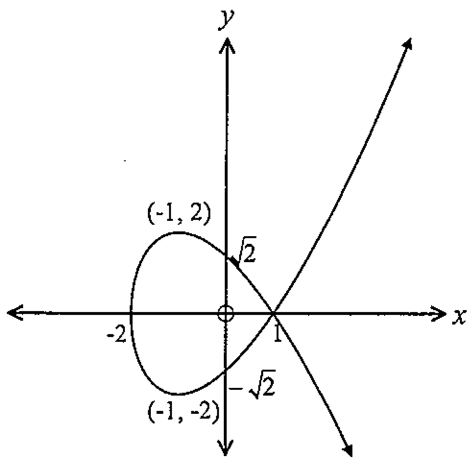
(B)



(C)



(D)



8. $\int_{-1}^{1} \frac{1}{x^2+2x+5}\,dx =$

(A) $\frac{\pi}{4}$

(B) $\frac{\pi}{8}$

(C) $\frac{\pi}{16}$

(D) 0

9. Given $3x+2iy-ix+5y=7+5i$ .where x and y are real numbers. Then

(A) $x=-1,\quad y=2$

(B) $x=\frac{39}{11},\quad y=-\frac{8}{11}$

(C) $x=-\frac{3}{5},\quad y=\frac{22}{5}$

(D) $x=-11,\quad y=8$

10. The solutions of the equation $z^3=8$ are $z=$

(A) $2, -\frac{1}{2}+\frac{i\sqrt{3}}{2}, -\frac{1}{2}-\frac{i\sqrt{3}}{2}$

(B) $2, \frac{1}{2}+\frac{i\sqrt{3}}{2}, \frac{1}{2}-\frac{i\sqrt{3}}{2}$

(C) $2, -1+i\sqrt{3}, -1-i\sqrt{3}$

(D) $2, 1+i\sqrt{3}, 1-i\sqrt{3}$

Section II

Total marks (90)
Attempt Questions 11-16
Allow about 2 hours 45 minutes for this section

Answer all questions, starting each question in a new booklet with your name and question number on the front page.Do not write on the back of sheets.

Question 11 (15 marks) Use a separate booklet **Marks**

a) i) Graph $y = f(x)$ where $f(x) = 2x - x^2$ **2**

ii) Hence sketch.

α) $y = \dfrac{1}{f(x)}$ **1**

β) $y = (f(x))^2$ **1**

b) i) Write $\sqrt{3} + i$ in modulus argument form **2**

ii) hence evaluate $(\sqrt{3} + i)^6$. **1**

c) Use the substitution $t = \tan\dfrac{x}{2}$ to evaluate $\displaystyle\int_0^{\frac{\pi}{2}} \frac{2}{5 + 3\cos x}dx$. **3**

d) The area between the co-ordinates axes and the line $2x + 3y = 6$ is rotated about the line $y = 3$ .By taking slices perpendicular to the axis of rotation find the volume formed. **3**

e) Represent on an Argand diagram the region for which the inequalities $|z - 3 - 3i| < 5$ and $\dfrac{\pi}{4} \le \arg z \le \dfrac{3\pi}{4}$ are both satisfied. **2**

End of Question 11

Question 12 (15 marks) Use a separate booklet **Marks**

a) i) Find the Cartesian equation of the curve whose parametric equations are $x = 2\cos\theta, y = \sin\theta$ **2**

ii) Find the equation of the normal to this curve at the point P where $\theta = \frac{\pi}{4}$ **3**

b) i) Show that the tangent from the origin to the curve $y = \ln x$ has a gradient of $\frac{1}{e}$ **2**

ii) Hence find the set of values of the real number k for which the equation $\ln x = kx$ has two distinct real roots. **1**

c) i) Use the substitution $x = u^2 (u > 0)$ to show that **3**

$$\int_4^9 \frac{\sqrt{x}}{x-1}\,dx = 2 + \log_e\left(\frac{3}{2}\right).$$

ii) Hence use integration by parts to find the value of **2**

$$\int_4^9 \frac{1}{\sqrt{x}} \log_e(x-1)\,dx$$

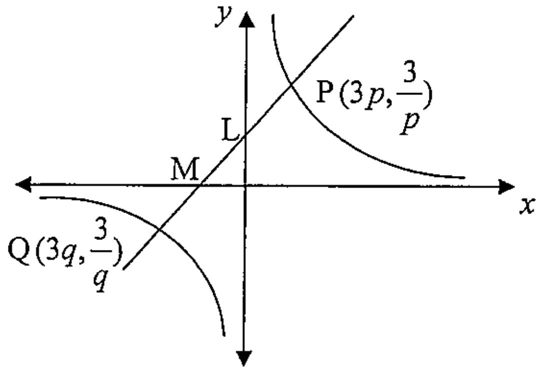
d) Describe the set of points in the complex plane that satisfies **2**

$|Z+1| = |Z-i|$

End of Question 12

Question 13 (15 marks) Use a separate booklet **Marks**

a)



A chord PQ of the rectangular hyperbola $xy = 9$ meets the asymptotes at L and M as shown.

i) Show that the equation of chord PQ is: $pqy + x = 3(p+q)$ **2**

ii) Find the coordinates of N, the mid-point of PQ **2**

iii) Show that PL= MQ. **2**

iv) If the chord PQ is a tangent to the parabola $y^2 = 3x$, find the locus of N **3**

b) If $1-2i$ is a root of the equation $z^2 - (3+i)z + c = 0$.

i) Explain why the conjugate $1+2i$ cannot be a root of the equation. **1**

ii) Show that the other root is $2+3i$. **1**

iii) Find the value of c. **1**

c) For a real number, r, the polynomial $P(x) = 8x^3 - 4x^2 - 42x + 45$ is divisible by $(x-r)^2$. Find r. **3**

End of Question 13

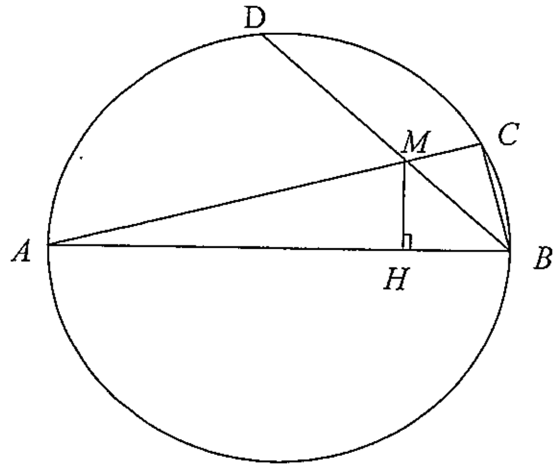
Question 14 (15 marks) Use a separate booklet **Marks**

a)

i) Express $z = 2i$ and $w = -1 + \sqrt{3}i$ in modulus argument form. On an argand diagram plot the points P and Q which represent z and w . **2**

ii) On the same diagram construct vectors which represent $z + w$ and $z - w$. Hence deduce the exact values of $\arg(z + w)$ and $\arg(z - w)$. **2**

b)



AB is the diameter of a circle. Chords AC and BD intersect at M. H is a point on AB such that MH is perpendicular to AB.

i) Prove that triangle ABC is similar to triangle AMH. **2**

ii) Show that $AB.AH = AC.AM$. **1**

iii) Prove that $AB^2 = AC.AM + BD.BM$ **2**

c) The area between the curve $y = 8x - x^2$, the x axis and the line $x = 4$ is rotated about the line $x = 4$. Find the volume generated by using:

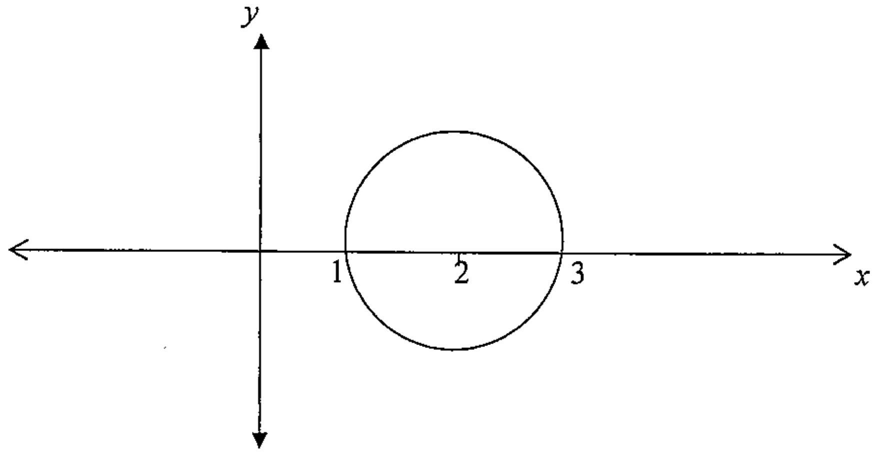
i) cylindrical shells. **3**

ii) slicing . **3**

End of Question 14

Question 15 (15 marks) Use a separate booklet **Marks**

a) i)



In the diagram above the circle $(x-2)^2 + y^2 = 1$ is drawn. The region bounded by the circle is rotated about the line $x = 1$. Use the method of cylindrical shells to show that the volume of the solid of revolution so formed is given by.

$$V = 4\pi\int_1^3 (x-1)\sqrt{1-(x-2)^2}\,dx$$ **2**

ii) By using the substitution $x - 2 = \sin u$, or otherwise, calculate the volume of the solid of revolution. **3**

b) i) Find the three cube roots of unity. **2**

ii) If ω is one of the complex roots of unity prove the other is ω^2 and show that $1+\omega+\omega^2 = 0$. **2**

iii) Prove that if n is a positive integer, then $1+\omega^n+\omega^{2n} = 3$ or 0 depending on whether n is or is not a multiple of 3. **3**

c) The equation $x^3 + x^2 - 2x - 3 = 0$ has roots α, β and γ. Find the equation with roots α^2, β^2 and γ^2 **3**

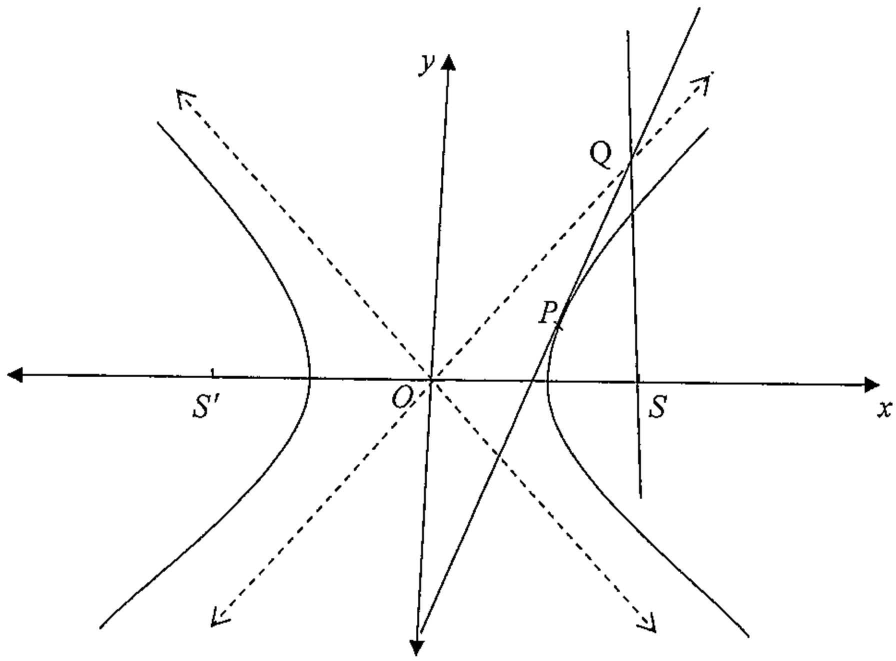
End of Question 15

Question 16 (15 marks) Use a separate booklet

a) Show that the derivative of $y = x^{x+1}$ is $\left(1+\frac{1}{x}+\ln x\right)x^{x+1}$. **2**

b) Sketch the graph of $y = \frac{2+x-x^2}{(x-1)^2}$ clearly showing any turning points and any asymptotes. **4**

c)



d) The point P $(a\sec\theta, b\tan\theta)$ on the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ with focus, S, is such that the tangent at P, the latus rectum through S, and one asymptote are concurrent. Prove that SP is parallel to the other asymptote. **4**

(you may assume the equation of the tangent at P is $\frac{x\sec\theta}{a}-\frac{y\tan\theta}{b}=1$)

e) i) Show that $(1-\sqrt{x})^{n-1}.\sqrt{x}=(1-\sqrt{x})^{n-1}-(1-\sqrt{x})^n$ **2**

ii) If $I_n=\int_0^1(1-\sqrt{x})^n\,dx$ for $n\geq 0$ show that $I_n=\frac{n}{n+2}I_{n-1}$ for $n\geq 1$ **3**

STANDARD INTEGRALS

$$\int x^n\,dx = \frac{1}{n+1}x^{n+1}, \quad n \neq -1; \quad x \neq 0, \text{ if } n < 0$$

$$\int \frac{1}{x}\,dx = \ln x, \quad x > 0$$

$$\int e^{ax}\,dx = \frac{1}{a}e^{ax}, \quad a \neq 0$$

$$\int \cos ax\,dx = \frac{1}{a}\sin ax, \quad a \neq 0$$

$$\int \sin ax\,dx = -\frac{1}{a}\cos ax, \quad a \neq 0$$

$$\int \sec^2 ax\,dx = \frac{1}{a}\tan ax, \quad a \neq 0$$

$$\int \sec ax \tan ax\,dx = \frac{1}{a}\sec ax, \quad a \neq 0$$

$$\int \frac{1}{a^2+x^2}\,dx = \frac{1}{a}\tan^{-1}\frac{x}{a}, \quad a \neq 0$$

$$\int \frac{1}{\sqrt{a^2-x^2}}\,dx = \sin^{-1}\frac{x}{a}, \quad a > 0, \quad -a < x < a$$

$$\int \frac{1}{\sqrt{x^2-a^2}}\,dx = \ln\left(x+\sqrt{x^2-a^2}\right), \quad x > a > 0$$

$$\int \frac{1}{\sqrt{x^2+a^2}}\,dx = \ln\left(x+\sqrt{x^2+a^2}\right)$$

NOTE : $\ln x = \log_e x, \quad x > 0$

EXT2 TRIAL SOLUTIONS 2013

1. C
2. A
3. D
4. B
5. C
6. B
7. D
8. B
9. A
10. C

1) $(5-2i)(3-4i)$

$= 15 - 20i - 6i - 8$

$= 7 - 26i$

(C)

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2) $4x^2 + 9y^2 = 36$

$\frac{x^2}{9} + \frac{y^2}{4} = 1.$

$b^2 = a^2(1-e^2)$

$4 = 9(1-e^2)$

$\frac{4}{9} = 1 - e^2$

$e^2 = \frac{5}{9}$

$e = \frac{\sqrt{5}}{3}.$

focus $(\pm ae, 0)$

$= (\pm\sqrt{5}, 0)$

directrix $x = \pm\frac{a}{e}$

$= \pm\frac{9}{\sqrt{5}}$

$\therefore$ $= \pm\frac{9\sqrt{5}}{5}$

$\therefore$ (A)

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3) $\frac{x+1}{x^2-4} = \frac{a}{x+2} + \frac{b}{x-2}$

$x+1 = a(x-2) + b(x+2)$

let $x = 2$: $3 = 4b$

$b = \frac{3}{4}$

let $x = -2$: $-1 = -4a$

$a = \frac{1}{4}$

$\therefore$ (D)

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(4) (B)

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5) $x^3 + 2x^2 - 4x + 5 = 0$

$\alpha\beta + \alpha\gamma + \beta\gamma = \frac{c}{a}$

$= -4$

$\alpha\beta\gamma = -\frac{d}{a}$

$= -5$

$\frac{1}{\alpha} + \frac{1}{\beta} + \frac{1}{\gamma}$

$= \frac{\beta\gamma + \alpha\gamma + \alpha\beta}{\alpha\beta\gamma}$

$= \frac{-4}{-5}$

$= \frac{4}{5}$

$\therefore$ (C)

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6) $\int \sin^3 x \, dx$

$= \int \sin x . \sin^2 x \, dx$

$= \int \sin x (1 - \cos^2 x) \, dx$

$= \int \sin x - \sin x \cos^2 x \, dx$

$= -\cos x + \frac{\cos^3 x}{3} + c$

(B)

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7) (D)

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8) $\int_{-1}^{1} \frac{1}{x^2+2x+5} \, dx$

$= \int_{-1}^{1} \frac{1}{x^2+2x+1+4} \, dx$

$= \int_{-1}^{1} \frac{1}{(x+1)^2 + 4}$

$= \left[\frac{1}{2} \tan^{-1}\left(\frac{x+1}{2}\right)\right]_{-1}^{1}$

$= \frac{1}{2}(\tan^{-1} 1 - \tan^{-1} 0)$

$= \frac{1}{2}\left(\frac{\pi}{4} - 0\right)$

$= \frac{\pi}{8}$ (C)

9) $3x + 2iy - ix + 5y = 7 + 5i$

$\therefore \quad 3x + 5y = 7 \quad \cdots (1)$

$-x + 2y = 5 \quad \cdots (2)$

$(2)\times 3 \quad -3x + 6y = 15 \quad \cdots (3)$

$(1)+(3) \quad 11y = 22$

$y = 2$

Sub into (1) $\quad 3x + 10 = 7$

$3x = -3$

$x = -1$

$\therefore$ (A)

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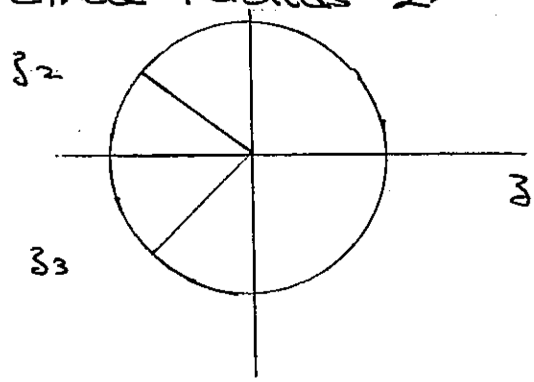
10) $z^3 = 8$

zeros equally spaced around a circle radius 2.



$z = 2$

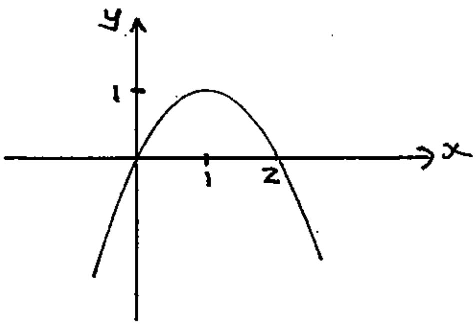
$z_2 = 2\,\text{Cis}\,\frac{2\pi}{3} = 2\left(-\frac{1}{2} + \frac{\sqrt{3}}{2}\right) = -1 + \sqrt{3}i$

$z_3 = 2\,\text{Cis}\left(-\frac{2\pi}{3}\right) = 2\left(-\frac{1}{2} - \frac{\sqrt{3}}{2}\right) = -1 - \sqrt{3}i$

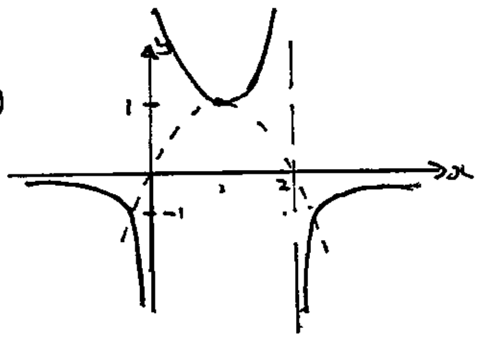
(C)

SECTION II

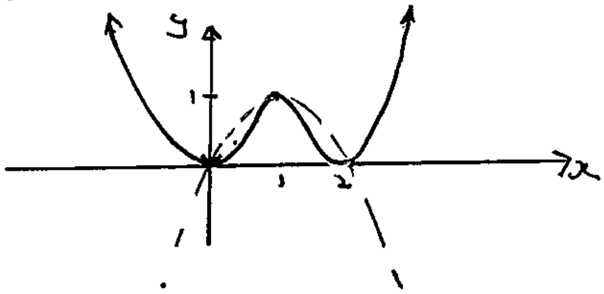
Q11) a)(i) $y = 2x - x^2$



α)



β)



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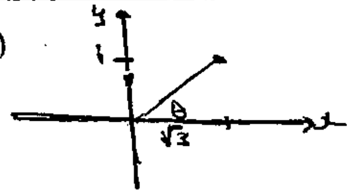
b) i)



$\sqrt{3} + i = 2\left(\cos\frac{\pi}{6} + i\sin\frac{\pi}{6}\right)$

(ii) $\left(\sqrt{3}+i\right)^6 = \left(2\left(\cos\frac{\pi}{6} + i\sin\frac{\pi}{6}\right)\right)^6$

$= 64\left(\cos\pi + i\sin\pi\right)$

$= 64(-1 + 0)$

$= -64$

---

c) $\displaystyle\int_0^{\frac{\pi}{2}} \frac{2}{5 + 3\cos x}\,dx.$

$t = \tan\frac{x}{2}$: $x = 0, \; t = 0$

$x = \frac{\pi}{2} \quad t = 1.$

$= \displaystyle\int_0^1 \frac{2}{5 + 3\left(\frac{1-t^2}{1+t^2}\right)} \times \frac{2}{1+t^2}\,dt$

$= 4\displaystyle\int_0^1 \frac{1}{5(1+t^2) + 3 - 3t^2}\,dt$

$= 4\displaystyle\int_0^1 \frac{1}{2t^2 + 8}\,dt$

$= 2\int_0^1 \frac{1}{t^2+4}\,dt$

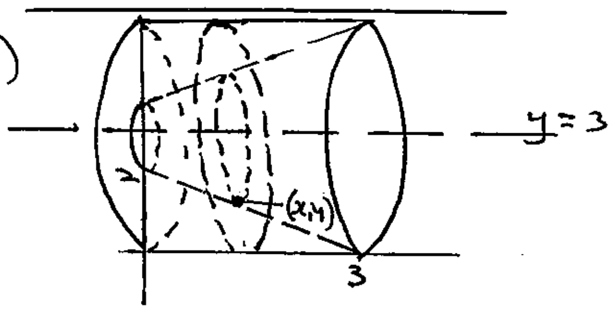
$= 2\left[\frac{1}{2}\tan^{-1}\frac{t}{2}\right]$

$= \tan^{-1}(\tfrac{1}{2}) - \tan^{-1}(0)$

$= \tan^{-1}(\tfrac{1}{2})$

$= 0.46$ (2 dec. places)

d)



Volume of a Slice $= \pi\left(3^2 - (3-y)^2\right)$

$V \doteq \lim_{\Delta x \to 0} \sum_{x=0}^{3} \pi\left(3^2 - (3-y)^2\right)\Delta x$

$V = \pi\int_0^3 3^2 - (3-y)^2\,dx$

$= \pi\int_0^3 9 - (9 - 6y + y^2)\,dx$

$= \pi\int_0^3 6y - y^2\,dx$

$= \pi\int_0^3 6\left(\frac{6-2x}{3}\right) - \left(\frac{6-2x}{3}\right)^2 dx$

$= \pi\int_0^3 12 - 4x - \frac{1}{9}(6-2x)^2\,dx$

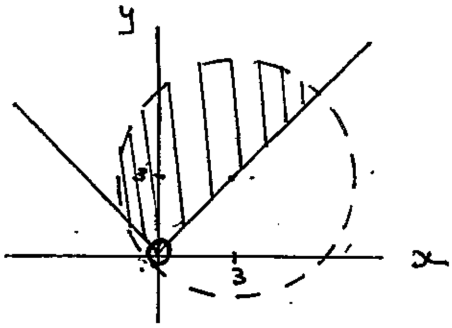
$= \pi\left[12x - 2x^2 - \frac{1}{9}\frac{(6-2x)^3}{-6}\right]_0^3$

$= \pi\left[12x - 2x^2 + \frac{(6-2x)^3}{54}\right]_0^3$

$= \pi\left[36 - 18 + 0 - (0 - 0 + 4)\right]$

$= 14\pi$ cubic units.

e)



Q 12)

a) i) $x = 2\cos\theta$ ---(1)

$y = \sin\theta$ ---(2)

(1) $\Rightarrow$ $\frac{x}{2} = \cos\theta$ ---(3)

$(3)^2 + (2)^2$: $\left(\frac{x}{2}\right)^2 + y^2 = \sin^2\theta + \cos^2\theta$

$\frac{x^2}{4} + y^2 = 1.$

ii) $\frac{x^2}{4} + y^2 = 1$

$\frac{x}{2} + 2y\frac{dy}{dx} = 0$

$2y\frac{dy}{dx} = -\frac{x}{2}.$

$\frac{dy}{dx} = -\frac{x}{4y}.$

$\theta = \frac{\pi}{4}$: $x = 2\cos\frac{\pi}{4} = \sqrt{2}$

$y = \sin\frac{\pi}{4} = \frac{\sqrt{2}}{2}$

$\therefore \frac{dy}{dx} = \frac{-\sqrt{2}}{2\sqrt{2}}$

$= -\frac{1}{2}$

$\therefore$ gradient normal $= 2.$

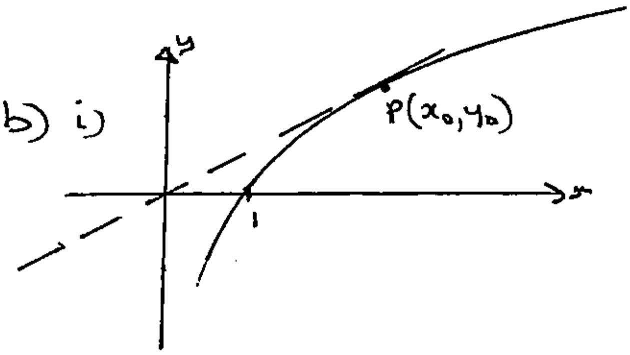
$\therefore$ eqn normal

$y - \frac{\sqrt{2}}{2} = 2(x - \sqrt{2})$

$y - \frac{\sqrt{2}}{2} = 2x - 2\sqrt{2}$

$2x - y - 2\sqrt{2} + \frac{\sqrt{2}}{2} = 0$

b) i)



let $P(x_0, y_0)$ be the point where the tangent hits the curve:

$$y = \ln x$$

$$\frac{dy}{dx} = \frac{1}{x}$$

at $x = x_0$ $\quad \frac{dy}{dx} = \frac{1}{x_0}$

$\therefore$ eqn tangent: $y - y_0 = \frac{1}{x_0}(x - x_0)$

now this tangent passes through the origin:

$$\therefore \quad 0 - y_0 = \frac{1}{x_0}(x - x_0)$$

$$-y_0 = -1$$

$$y_0 = 1$$

as (x_0, y_0) is on the curve

$$\therefore \quad 1 = \log_e x_0$$

$$\therefore \quad x_0 = e$$

Now gradient $= \frac{1}{x_0}$

$= \frac{1}{e}$

---

ii) two distinct real roots

$\Rightarrow$ $y = kx$ hits the curve $y = \ln x$ twice:

$$\therefore \quad 0 < k < \frac{1}{e}$$

---

c i) $\displaystyle\int_4^9 \frac{\sqrt{x}}{} \, dx$

$x = u^2$ $\quad$ at $x = 4$ $\quad u = 2$

$\frac{dx}{du} = 2u$ $\quad$ $x = 9$ $\quad u = 3.$

$dx = 2u\,du$

$$\int_2^3 \frac{u}{u^2 - 1} \times 2u \, du.$$

$$= 2\int_2^3 \frac{u^2}{u^2 - 1} \, du$$

$$\frac{u^2}{u^2 - 1} = a + \frac{b}{u+1} + \frac{c}{u-1}$$

$$u^2 = a(u^2 - 1) + b(u - 1) + c(u + 1)$$

$u = 1$: $\quad 1 = 2c$

$\frac{1}{2} = c$

$u = -1$: $\quad 1 = -2b$

$-\frac{1}{2} = b$

$u = 0$: $\quad 0 = -a - b + c$

$0 = -a + \frac{1}{2} + \frac{1}{2}$

$a = 1.$

$$\therefore \frac{u^2}{u^2 - 1} = 1 - \frac{1}{2(u+1)} + \frac{1}{2(u-1)}$$

$$2\int_2^3 \frac{u^2}{u^2 - 1}$$

$$\frac{2}{2}\int_2^3 2 - \frac{1}{u+1} + \frac{1}{u-1} \, du.$$

$$= \Big[2u - \ln(u+1) + \ln(u-1)\Big]_2^3$$

$$= \left[2u + \ln\left(\frac{u-1}{u+1}\right)\right]_2^3$$

$$= \left(6 + \ln\left(\frac{2}{4}\right)\right) - \left(4 + \ln\left(\frac{1}{3}\right)\right)$$

$$= 6 - 4 + \ln\left(\frac{\frac{1}{2}}{\frac{1}{3}}\right)$$

$$= 2 + \ln\left(\frac{3}{2}\right)$$

---

ii) $\displaystyle\int_4^9 \frac{1}{\sqrt{x}} \ln(x - 1) \, dx$

$$= \int_4^9 \frac{d}{dx}\left(2\sqrt{x}\right) \ln(x - 1) \, dx$$

$= \left[2\sqrt{x}\ln(x-1)\right]_4^9 - \int_4^9 \frac{2\sqrt{x}}{x-1}\,dx$

$= 6\ln 8 - 4\ln 3 - 2\left(2+\ln\left(\frac{3}{2}\right)\right)$

$= 6\ln 8 - 4\ln 3 - 4 - 2\ln\left(\frac{3}{2}\right)$

$= -4 + 2\ln\left(\frac{1024}{27}\right)$

---

d) $|z+1| = |z-i|$

$|x+iy+1| = |x+iy-i|$

$\sqrt{(x+1)^2+y^2} = \sqrt{x^2+(y-1)^2}$

$x^2+2x+1+y^2 = x^2+y^2-2y+1$

$2x = -2y$

$y = -x$

Q13) i) $m_{PQ} = \frac{\frac{3}{p}-\frac{3}{q}}{3p-3q}$

$= \frac{\frac{3(q-p)}{pq}}{3(p-q)}$

$= -\frac{1}{pq}$

$\therefore$ eqn $y - \frac{3}{p} = -\frac{1}{pq}(x-3p)$

$pqy - 3q = -x + 3p$

$x + pqy = 3(p+q)$

---

ii) $N = \left(\frac{3p+3q}{2}, \frac{\frac{3}{p}+\frac{3}{q}}{2}\right)$

$= \left(\frac{3(p+q)}{2}, \frac{3}{2}\left(\frac{1}{p}+\frac{1}{q}\right)\right)$

---

iii) $L\left(0, \frac{3}{pq}(p+q)\right)$ $M\left(3(p+q), 0\right)$

mid pt $LM = \left(\frac{3(p+q)}{2}, \frac{3}{2pq}(p+q)\right)$

$= \left(\frac{3(p+q)}{2}, \frac{3}{2}\left(\frac{1}{p}+\frac{1}{q}\right)\right)$

As this is the same mid pt as the mid pt of PQ then $PL = MQ$

---

iv) $pqy + x = 3(p+q)$ --- (1)

$y^2 = 3x$ --- (2)

(2) $\Rightarrow$ $x = \frac{y^2}{3}$

Sub into (1)

$pqy + \frac{y^2}{3} = 3(p+q)$

$y^2 + 3pqy - 9(p+q) = 0$

as PQ is a tangent there must be only one solution

ie $\Delta = 0$

$9p^2q^2 + 36(p+q) = 0$

$p^2q^2 = -4(p+q)$

$\therefore$ $p+q = -\frac{p^2q^2}{4}$

Now locus of N

$x = \frac{3}{2}(p+q)$

$y = \frac{3}{2pq}(p+q)$

ie $x = \frac{3}{2} \times -\frac{p^2q^2}{4}$

$x = -\frac{3p^2q^2}{8}$ --- (1)

$y = \frac{3}{2pq} \times \frac{-p^2q^2}{4}$

$y = \frac{-3pq}{8}$ --- (2)

from (2) $pq = -\frac{8y}{3}$

Sub into (1)

$x = -\frac{3}{8} \times \frac{64y^2}{9}$

$x = -\frac{8y^2}{3}$

---

b) i) Conjugate is only a root if the coefficients are real. As coefficient of z is not real the conjugate cannot be a root.

ii) if α is the other root then $\alpha + 1 - 2i = -\frac{b}{a}$

i.e. $\alpha + 1 - 2i = 3 + i$

$\alpha = 2 + 3i$

iii) $\overline{(1-2i)(2+3i)} = \frac{c}{a}$

$(1-2i)(2+3i) = c$

$c = 2 + 3i - 4i + 6$

$= 8 - i$

c) $8x^3 - 4x^2 - 42x + 45$

if divisible by $(x-r)^2$ the r is a double root.

$P(x) = 8x^3 - 4x^2 - 42x + 45$

$P'(x) = 24x^2 - 8x - 42$

$P'(r) = 0$

$24r^2 - 8r - 42 = 0$

$12r^2 - 4r - 21 = 0$

$(6r+7)(2r-3) = 0$

$\therefore r = -\frac{7}{6}$, or $\frac{3}{2}$

Now $P\left(\frac{3}{2}\right) = 8\left(\frac{27}{8}\right) - 4\left(\frac{9}{4}\right) - 63 + 45$

$= 27 - 9 - 63 + 45$

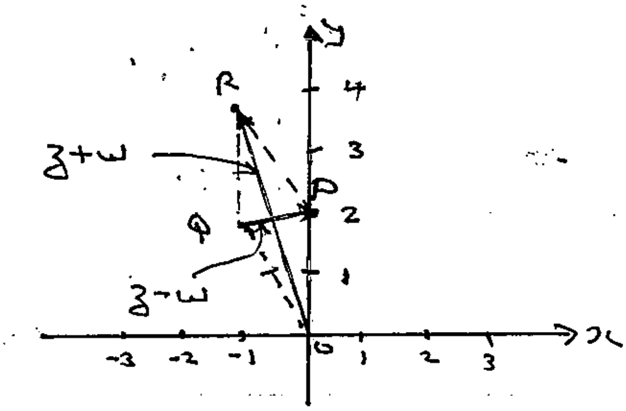
$= 0$

$\therefore r = \frac{3}{2}$.

Q14) a) i)

$z = 2i = 2\left(\cos\frac{\pi}{2} + i\sin\frac{\pi}{2}\right)$

$\omega = -1 + \sqrt{3}i = 2\left(\cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3}\right)$



As OPRQ is a rhombus (OP = OQ)

$\arg(z+\omega) = \frac{\pi}{2} + \left(\frac{1}{2}\left(\frac{2\pi}{3} - \frac{\pi}{2}\right)\right)$

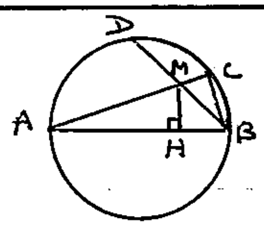
$= \frac{\pi}{2} + \frac{\pi}{12}$

$= \frac{7\pi}{12}$

$\arg(z-\omega) = \frac{5\pi}{12} - \frac{\pi}{3}$

$= \frac{\pi}{12}$

b)



i) $\Delta ABC \;|||\; \Delta AMH$

$\angle A$ is common

$\angle ACB = 90°$ angle in a semi-circle - given AB diam

$\angle AHM = 90°$ given

$\therefore \angle ACB = \angle AHM$

$\therefore \Delta ABC \;|||\; \Delta AMH$ equiangular

ii) $\frac{AB}{AM} = \frac{AC}{AH}$

$AB.AH = AM.AC$.

iii) $\Delta BAD \;|||\; \Delta BMH$ (equiangular)
$\angle B$ common

$\angle BHM = 90°$ given
$\angle BDA = 90°$ angle in a semicircle

$\therefore \frac{AB}{BM} = \frac{BD}{BH}$

$\therefore AB.BH = BM.BD$.

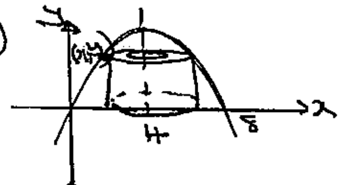
adding this to the result in (ii) gives

$AB.AH + AB.BH = AM.AC + BM.BD$

$AB(AH + BH) = AM.AC + BM.BD$

$AB^2 = AC.AM + BM.BD$

c) i)



Volume of a shell $= 2\pi(4-x)y.\Delta x$

$V \doteq \lim_{\Delta x \to 0} \sum_{x=0}^{4} 2\pi(4-x)(8x-x^2)\Delta x$

$V = 2\pi\int_0^4 (4-x)(8x-x^2)\,dx$

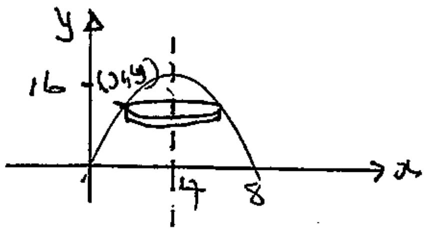
$= 2\pi\int_0^4 32x - 12x^2 + x^3\,dx$

$= 2\pi\left[16x^2 - 4x^3 + \frac{x^4}{4}\right]_0^4$

$= 2\pi\left[\left(256 - 256 + \frac{256}{4}\right) - 0\right]$

$= 128\pi$ cubic units

ii)



Volume of slice $= \pi(4-x)^2\Delta y$

$V \doteq \lim_{\Delta y \to 0} \sum_{y=0}^{16} \pi(4-x)^2\,\Delta y$

$V = \pi\int_0^{16} (4-x)^2\,dy$.

$V = \pi\int_0^{16}\left(4 - (4 + \sqrt{16-y})\right)^2$

$= \pi\int_0^{16} 16 - y\,dy$

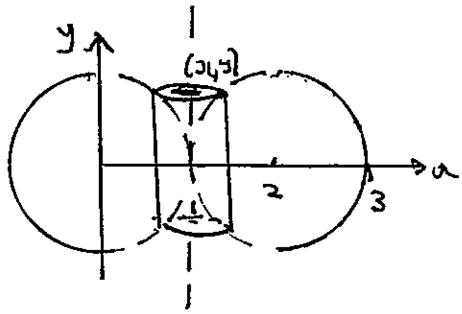
$= \pi\left[16y - \frac{y^2}{2}\right]_0^{16}$

$= \pi\left((256 - 128) - 0\right)$

$= 128\pi$ cubic units.

Q15)

a) i)



Volume of a shell $= 2\pi rh$

$= 2\pi(x-1)2y\,\Delta x$

$V \doteq \lim_{\Delta x \to 0} 2\pi(x-1)2y\,\Delta x$

$V = 4\pi\int_1^3 (x-1)2y\,dx$

$= 4\pi\int_1^3 (x-1)\sqrt{1-(x-2)^2}\,dx$

$$\left\{\begin{aligned} (x-2)^2 + y^2 &= 1 \\ y^2 &= 1-(x-2)^2 \\ y &= \sqrt{1-(x-2)^2} \end{aligned}\right\}$$

ii) $V = 4\pi\int_1^3 (x-1)\sqrt{1-(x-2)^2}\,dx$

$x - 2 = \sin u$ $x = 1,\ u = -\frac{\pi}{2}$

$x = \sin u + 2$ $x = 3,\ u = \frac{\pi}{2}.$

$\frac{dx}{du} = \cos u$

$dx = \cos u\,du.$

$V = 4\pi\int_{-\pi/2}^{\pi/2} (\sin u+1)\sqrt{1-\sin^2 u}\,.\cos u\,du$

$= 4\pi\int_{-\pi/2}^{\pi/2} (\sin u+1)\cos^2 u\,du.$

$= 4\pi\int_{-\pi/2}^{\pi/2} \sin u\cos^2 u\,du + 4\pi\int_{-\pi/2}^{\pi/2}\cos^2 u\,du$

$= 4\pi\left[\frac{\cos^3 u}{3}\right]_{-\pi/2}^{\pi/2} + 2\pi\int_{-\pi/2}^{\pi/2}\cos 2u + 1\,du$

$= 4\pi(0) + 2\pi\left[\frac{1}{2}\sin u + u\right]_{-\pi/2}^{\pi/2}$

$= 2\pi\left[\left(\frac{1}{2}\sin\pi + \frac{\pi}{2}\right) - \left(\frac{1}{2}\sin(-\pi) - \frac{\pi}{2}\right)\right]$

$= 2\pi\left[\left(0 + \frac{\pi}{2}\right) - \left(0 - \frac{\pi}{2}\right)\right]$

$= 2\pi(\pi)$ cubic units.

$= 2\pi^2$ units^3

b) i) $z^3 = 1$

$z^3 - 1 = 0$

$(z-1)(z^2+z+1) = 0$

$z = 1$ $z = \frac{-1 \pm \sqrt{1-4}}{2}.$

$z = 1, \frac{-1+i\sqrt{3}}{2}, \frac{-1-i\sqrt{3}}{2}.$

let $\omega = \frac{-1+i\sqrt{3}}{2}$

$\therefore \omega^2 = \frac{1-2i\sqrt{3}-3}{4}$

$= \frac{-2-2i\sqrt{3}}{4}$

$= \frac{-1-i\sqrt{3}}{2}.$

= the other complex root.

$\therefore$ roots are $1, \omega, \omega^2$

Now $1+\omega+\omega^2 = -\frac{b}{a}$ (sum of the roots)

$= 0.$

iii) if n is a multiple of 3

let $n = 3k.$

$1 + \omega^{3k} + \omega^{6k}.$

$= 1 + (\omega^3)^k + (\omega^3)^{2k}$

$= 1 + 1^k + 1^{2k}$

$= 1 + 1 + 1$

$= 3.$

if 'n' is not a multiple of 3

let $n = 3k+1$ or $3k+2.$

$n = 3k+1$: $1 + \omega^{3k+1} + \omega^{6k+2}$

$= 1 + \omega.(\omega^3)^k + \omega^2(\omega^3)^{2k}$

$= 1 + \omega + \omega^2$

$= 0$

$n = 3k+2$: $1 + \omega^{3k+2} + \omega^{6k+4}$

$= 1 + \omega^2(\omega^3)^k + \omega^4(\omega^3)^{2k}$

$= 1 + \omega^2 + \omega^4$

$= 1 + \omega^2 + \omega^3.\omega$

$= 1 + \omega^2 + \omega$

$= 0.$

c) $x^3 + x^2 - 2x - 3 = 0$

Equation will be of the form.

$(x^{1/2})^3 + (x^{1/2})^2 - 2(x^{1/2}) - 3 = 0$

$x^{3/2} + x - 2x^{1/2} - 3 = 0$

$x^{3/2} - 2x^{1/2} = 3 - x$

$x^{1/2}(x-2) = 3 - x$

$x(x-2)^2 = (3-x)^2$

$x(x^2 - 4x + 4) = 9 - 6x + x^2$

$x^3 - 4x^2 + 4x = 9 - 6x + x^2$

$x^3 - 5x^2 + 10x - 9 = 0.$

Q16) a) $y = x^{x+1}$

$\ln y = \ln x^{x+1}$

$\ln y = (x+1)\ln x$

$\ln y = x\ln x + \ln x$

$\frac{1}{y}\frac{dy}{dx} = x \cdot \frac{1}{x} + \ln x + \frac{1}{x}$

$\frac{1}{y}\frac{dy}{dx} = 1 + \ln x + \frac{1}{x}$

$\frac{dy}{dx} = (1 + \ln x + \frac{1}{x})y$

$= (1 + \frac{1}{x} + \ln x)x^{x+1}$

---

b) $y = \frac{2 + x - x^2}{(x-1)^2}$

asymptotes: $x = 1$

horizontal: $\lim_{x \to \infty} \frac{\frac{2}{x^2} + \frac{x}{x^2} - \frac{x^2}{x^2}}{\frac{x^2}{x^2} - \frac{2x}{x^2} - \frac{1}{x^2}}$

$= \frac{0 + 0 - 1}{1 - 0 - 0}$

ie $y = -1$.

'y' intercept: 2.

'x' intercept: $2 + x - x^2 = 0$

$(2-x)(1+x) = 0$

$x = 2, -1$

$\frac{dy}{dx} = \frac{(x-1)^2(1-2x) - 2(2+x-x^2)(x-1)}{(x-1)^4}$

$= \frac{(x-1)[(x-1)(1-2x) - 2(2+x-x^2)]}{(x-1)^4}$

$= \frac{(x-1)(x-5)}{(x-1)^4}$

$= \frac{x-5}{(x-1)^3}$.

turning pts. $\frac{dy}{dx} = 0$.

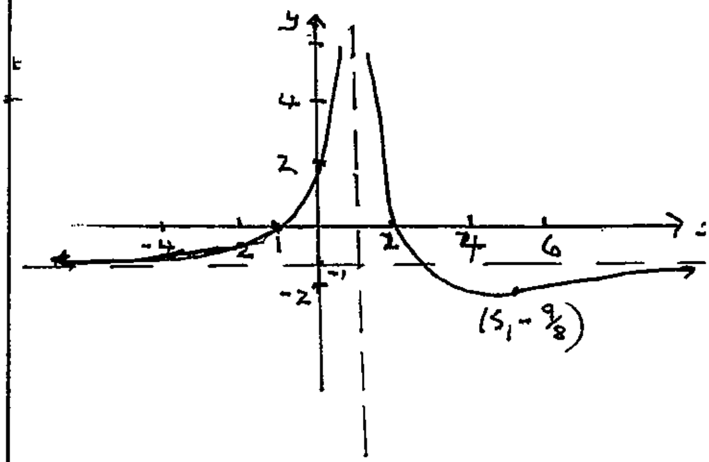
$x - 5 = 0$

$x = 5$.

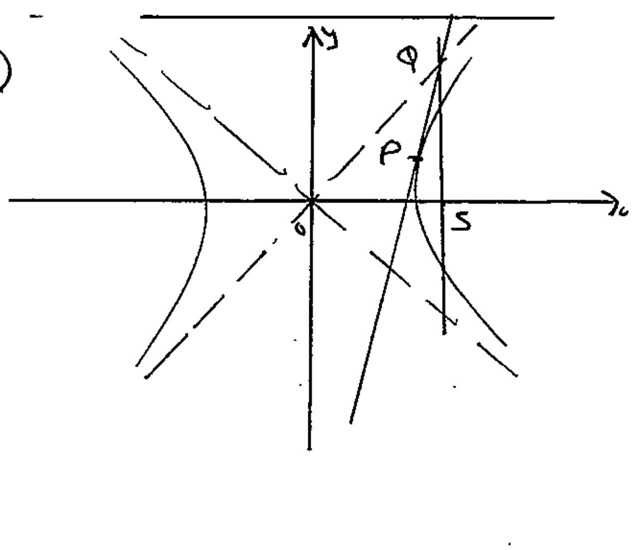
$y = -\frac{9}{8}$

$f(x)$	4	5	6
$f'(x)$	−	0	+

$\therefore (5, -\frac{9}{8})$ min.



c)



eqn tangent at P: $\frac{x\sec\theta}{a} - \frac{y\tan\theta}{b} = 1$ ...(1)

eqn asymptote: $y = \frac{bx}{a}$ ...(2)

eqn latus rectum: $x = ae$ ...(3)

Solving (2) and (3)

Q (ae, be)

Sub into the equation of the tangent.

$$\frac{ae\sec\theta}{a} - \frac{be\tan\theta}{b} = 1$$

$$e\sec\theta - e\tan\theta = 1.$$

$$e = \frac{1}{\sec\theta - \tan\theta} \quad \text{--- (A)}$$

Now gradient SP

$$= \frac{b\tan\theta}{a\sec\theta - ae}.$$

$$= \frac{b\tan\theta}{a\sec\theta - a\left(\frac{1}{\sec\theta - \tan\theta}\right)} \quad \text{from (A)}$$

$$= \frac{b\tan\theta(\sec\theta - \tan\theta)}{a\sec\theta(\sec\theta - \tan\theta) - a}$$

$$= \frac{b\tan\theta\sec\theta - b\tan^2\theta}{a\sec^2\theta - a\sec\theta\tan\theta - a}$$

$$= \frac{b\tan\theta\sec\theta - b(\sec^2\theta - 1)}{a(\sec^2\theta - \sec\theta\tan\theta - 1)}$$

$$= \frac{b(\tan\theta\sec\theta - \sec^2\theta + 1)}{a(\sec^2\theta - \sec\theta\tan\theta - 1)}$$

$$= \frac{-b}{a}$$

Which is the gradient of the other asymptote.

∴ SP || to the other asymptote.

e) i) $(1-\sqrt{x})^{n-1}\sqrt{x} = (1-\sqrt{x})^{n-1} - (1-\sqrt{x})^n$

$$\text{RHS} = (1-\sqrt{x})^{n-1} - (1-\sqrt{x})^n$$
$$= (1-\sqrt{x})^{n-1}(1-(1-\sqrt{x}))$$
$$= (1-\sqrt{x})^{n-1}(1-1+\sqrt{x})$$
$$= (1-\sqrt{x})^{n-1}.\sqrt{x}$$
$$= \text{L.HS}$$

ii) $\int_0^1 (1-\sqrt{x})^n$

$$I_n = \int_0^1 \frac{d}{dx}(x)(1-\sqrt{x})^n$$

$$= \left[x(1-\sqrt{x})^n\right]_0^1 - n\int_0^1 x(1-\sqrt{x})^{n-1} \cdot -\frac{1}{2\sqrt{x}}\, d$$

$$= 0 + \frac{n}{2}\int_0^1 \sqrt{x}(1-\sqrt{x})^{n-1}\,dx$$

$$= \frac{n}{2}\int_0^1 (1-\sqrt{x})^{n-1}dx - \frac{n}{2}\int_0^1 (1-\sqrt{x})^n$$

$$I_n = \frac{n}{2}I_{n-1} - \frac{n}{2}I_n$$

$$2I_n = nI_{n-1} - nI_n$$

$$nI_n + 2I_n = nI_{n-1}$$

$$I_n(n+2) = nI_{n-1}$$

$$I_n = \frac{n}{n+2}I_{n-1}$$