GOSFORD HIGH SCHOOL

2012
TRIAL HSC EXAMINATION.

MATHEMATICS EXTENSION 2

General Instructions

- Reading time – 5 minutes
- Working time – 3 hours
- Board approved calculators may be used
- Write using black or blue pen
- A table of standard integrals is provided at the back of this paper
- Show all necessary working in Questions 11 - 16

Total marks – 100

Section I – 10 marks

- Attempt Questions 1 – 10
- Allow about 15 minutes for this section

Section II – 90 marks

- Attempt Questions 11 – 16
- Allow about 2 hours 45 minutes for this section

SECTION I

1. If $Z_1 = 5 - 2i$ and $Z_2 = 3 + 4i$ then $Z_1 - \overline{Z_2} =$

(A) $2 - 6i$

(B) $8 - 6i$

(C) 2+2i

(D) $2 - 2i$

2. If 'e' represents the eccentricity for the conic section $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$, then $e^2 =$

(A) $1 - \frac{b^2}{a^2}$

(B) $1 + \frac{b^2}{a^2}$

(C) $1 - \frac{a^2}{b^2}$

(D) **2**

3. If $\frac{x+1}{x^2 - 4} = \frac{a}{x+2} + \frac{b}{x-2}$ then:

(A) $a = -1 \quad b = 3$

(B) $a = -\frac{1}{4} \quad b = \frac{3}{4}$

(C) $a = -\frac{1}{4} \quad b = \frac{1}{4}$

(D) $a = \frac{1}{4} \quad b = \frac{3}{4}$

4. Given the curve $y = f(x)$, where $f(x)$ is defined for all real x, then the curve $y = f(|x|)$ is best described by:

(A) A reflection of $y = f(x)$ in the y-axis.

(B) A reflection of $y = f(x)$ in the x-axis.

(C) A reflection of $y = f(x)$ in the y-axis for $0 \le x$.

(D) A reflection of $y = f(x)$ in the x-axis for $y \le 0$.

5. Using the recurrence relation $I_n = \int \sec^n x \, dx = \frac{1}{n-1} \tan x \sec^{n-2} x + \frac{n-2}{n-1} I_{n-2}$

then $\int_0^{\frac{\pi}{4}} \sec^4 x \, dx =$

(A) $\frac{4}{3}$

(B) 1

(C) $\frac{5}{6}$

(D) $\frac{6+4\sqrt{2}}{9}$

6. In using the 't' substitution method $\int_0^{\frac{\pi}{2}} \frac{1}{1+\sin x} dx$ could be calculated by:

(A) $\int_0^{\frac{\pi}{4}} \frac{1+t^2}{(t+1)^2} \, dt$

(B) $\int_0^1 \frac{1}{(t+1)^2} \, dt$

(C) $\int_0^1 \frac{2}{(t+1)^2} \, dt$

(D) $\int_0^1 \frac{1+t^2}{(t+1)^2} \, dt$

7. Graphically $|z-1+2i|=16$ represents:

(A) A circle centre $(-1,2)$ radius 4

(B) A circle centre $(-1,2)$ radius 16

(C) A circle centre $(1,-2)$ radius 4

(D) A circle centre $(1,-2)$ radius 16

8. When x^4-kx+1 is divided by x^2+1 the remainder is $3x+2$.
The value of k is:

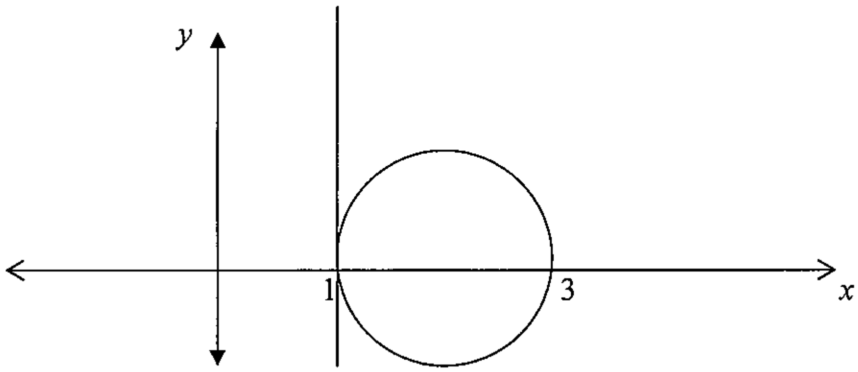
(A) 3

(B) 2

(C) -3

(D) -2

9.



In the diagram above the circle $(x-2)^2 + y^2 = 1$ is drawn. The region bounded by the circle is rotated about the line $x = 1$. Using the method of cylindrical shells the volume of the solid of revolution so formed is given by .

(A) $V = 4\pi \int_{1}^{3} (x-1)\sqrt{1+(x-2)^2}\,dx$

(B) $V = 2\pi \int_{1}^{3} (x-1)\sqrt{1-(x-2)^2}\,dx$

C) $V = 4\pi \int_{-2}^{3} (x-1)\sqrt{1-(x-2)^2}\,dx$

(D) $V = 4\pi \int_{1}^{3} (x-1)\sqrt{1-(x-2)^2}\,dx$

10. A particle of mass, *m*, is projected vertically upwards in a medium where the resistance to the motion has magnitude *mkv*(*k*, a constant: *v*, velocity). If *x* metres is its displacement from its starting point and *g* the force due to gravity then $\frac{dx}{dv} =$

(A) $\frac{v}{g-kv}$

(B) $\frac{-g+kv}{v}$

(C) $\frac{-v}{g+kv}$

(D) $\frac{-g-kv}{v}$

Section II
Total Marks (90)
Attempt Questions 11-16.
Start each question in a new answer booklet.
All necessary working should be shown in every question.

Question 11 (15 marks). Start a new answer booklet.

a) Find $\int \frac{1}{x^2-2x+5}dx$ **2**

b) Find $\int \sin^4 x \cos^3 x dx$. **3**

c) Use the substitution $x = u^2$ to find the exact value of $\int_0^{\frac{1}{2}} \frac{dx}{\sqrt{x}\sqrt{1-x}}$. **3**

d) Evaluate $\int_0^{\frac{\pi}{4}} x \sin x dx$. **3**

e) **i)** For the circle $x^2 + y^2 = 1$ use implicit differentiation to show that $\frac{dy}{dx} = -\frac{x}{y}$ **1**

ii) It is known that the arc length on the continuous curve $y = f(x)$ from $x = a$ to $x = b$ is given by:

$$\text{arc length} = \int_a^b \sqrt{1+\left(\frac{dy}{dx}\right)^2}\, dx.$$ **3**

By using the above formula, show that the circumference of the circle $x^2 + y^2 = 1$ is 2π units.

Question 12 (15 marks). Start a new answer booklet.

a) Express $\frac{2-5i}{4-3i}$ in the form $x+iy$. Where x and y are real numbers. **1**

b) If z_1 and z_2 represent two complex numbers show that
$\text{Re}(z_1 z_2) = \text{Re}(z_1)\text{Re}(z_2) - \text{Im}(z_1)\text{Im}(z_2)$. **2**

c) Evaluate $\left(\frac{1+i\sqrt{3}}{1-i\sqrt{3}}\right)^{10}$ giving your answer in the form $x+iy$ **2**

d) Represent graphically the set of values of z for which $\left|\frac{z-2}{z+3}\right| = 2$. **3**

e) If z is any point on the circle $|z-1| = 1$ prove that $\arg(z-1) = 2\arg z$ **2**

f) Use De Moivre's Theorem with $n = 3$ to show that
$\cos 3\theta = 4\cos^3\theta - 3\cos\theta$ **2**

g) On an argand diagram the points P,Q,R represent the complex numbers p, q, r respectively. If $p - q + iq - ir = 0$, what type of triangle is PQR? Give reasons for your answer. **3**

Question 13 (15 marks). Start a new answer booklet.

a) Solve over the complex field $z^4 + 81 = 0$ **2**

b) When a polynomial $P(x)$ is divided by $(x-3)$ the remainder is 5. When divided by $(x-4)$ the remainder is 9. Find the remainder when $P(x)$ is divided by $(x-4)(x-3)$ **3**

c) Solve over the complex field $z^4 - z^2 + 16 = 0$ **3**

d) Solve the equation $8x^4 + 44x^3 + 54x^2 + 25x + 4 = 0$ given that it has a root of multiplicity 3 **3**

e) None of the roots α, β, γ of the equation $x^3 - 3px + q = 0$ is zero. Obtain the monic equation whose roots are $\frac{\beta\gamma}{\alpha}, \frac{\alpha\gamma}{\beta}, \frac{\alpha\beta}{\gamma}$ expressing its coefficients in terms of p and q. **4**

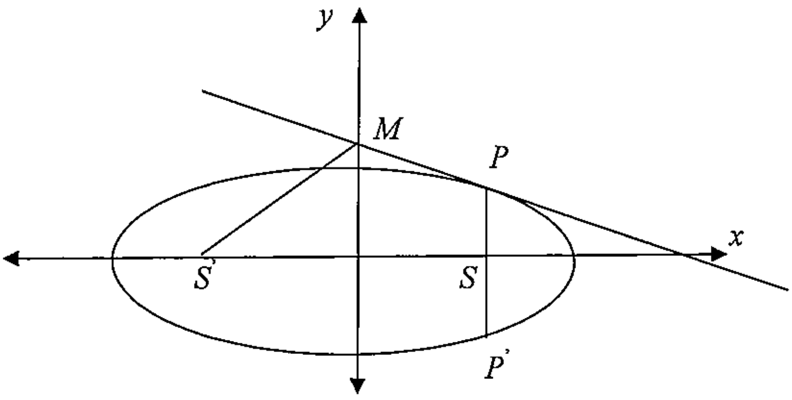
Question 14 (15 marks). Start a new answer booklet.

a) For the hyperbola $\frac{x^2}{16} - \frac{y^2}{9} = 1$

i) Find, in general form, the equation of the chord of contact from the point $(3\frac{1}{5}, 1)$ **2**

ii) Show that the chord in part (i) is a focal chord. **2**

b)



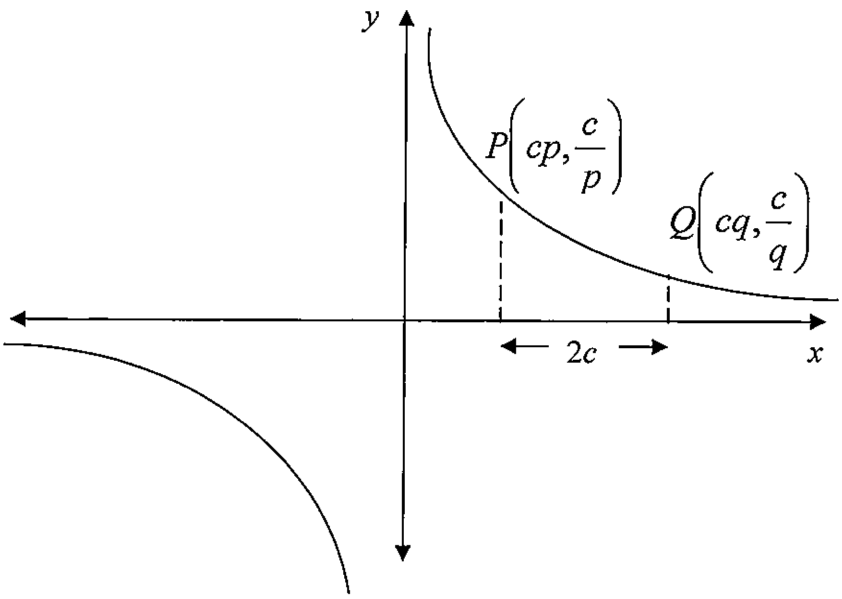
$\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}=1$ is an ellipse.

i) If PSP' is the latus rectum show that P has coordinates $\left(ae, \dfrac{b^2}{a}\right)$ **2**

ii) Show the equation of the tangent to the ellipse at the point P is $aex + ay = a^2$ **3**

iii) If the tangent at P meets the minor axis at M prove that the line joining M to the other focus, S', is parallel to the normal at P. **3**

c)

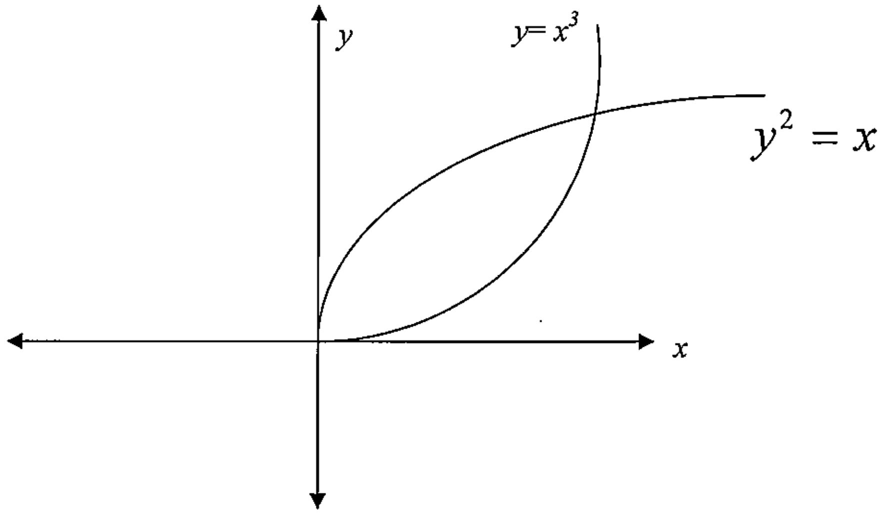


In the diagram a variable chord PQ of the rectangular hyperbola $xy = c^2$ is such that its projection on the x axis has a constant length 2c. **3**
Show that the locus of the midpoint of the chord is $x^2y = c^2(x+y)$.

You can assume the midpoint of the chord PQ is $\left(\dfrac{c(p+q)}{2}, \dfrac{c(p+q)}{2pq}\right)$

Question 15 (15 marks). Start a new answer booklet.

a)



The sketch shows the region in the first quadrant bounded by the curves $y^2 = x$ and $y = x^3$.

By taking slices perpendicular to the axis of rotation find the volume of the solid formed by revolving this region about the y axis. **3**

b) The base of a solid is the first quadrant area bounded by the line $4x + 5y = 20$ and the coordinate axes. Find the volume of the solid if every plane section perpendicular to the y axis is an equilateral triangle. **3**

c) Consider the functions f and g defined by:

$$f(x) = \frac{x-1}{x+2} \text{ for } x \neq -2$$

$$g(x) = [f(x)]^2$$

i) Sketch $y = f(x)$, clearly labelling the horizontal and vertical asymptotes and the points of intersection with the coordinate axes. **2**

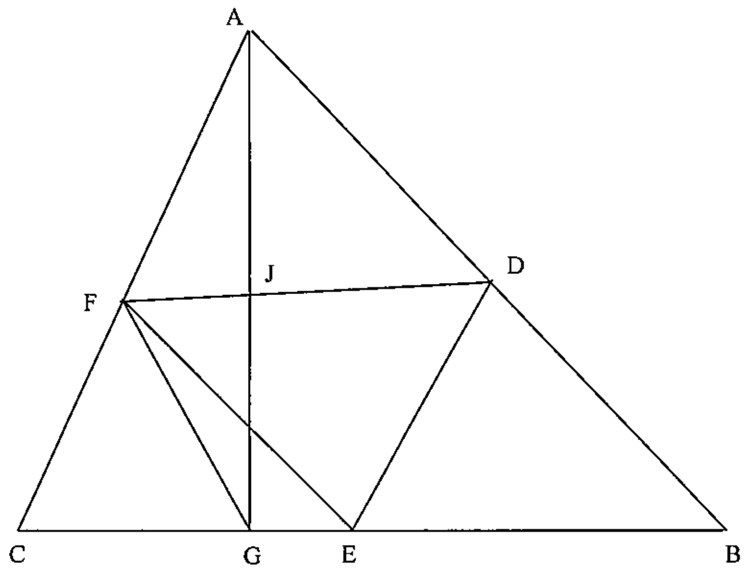
ii) Find all turning points for $y = g(x)$ and determine their nature. **3**

iii) Using the same diagram as used in part (i), sketch the curve $y = g(x)$. **2**

iv) On a separate diagram sketch the curve given by $y = g(-x)$ **2**

Question 16 (15 marks). Start a new answer booklet.

a) ABC is an acute angled triangle. DEF are the midpoints of AB, BC and CA respectively. AG is an altitude of ΔABC. AG and DF intersect at J.



(i) Explain why CB is parallel to FD **1**

(ii) Using the fact that $\Delta AJF \equiv \Delta GJF$,or otherwise, show that $\angle FGC = \angle FCG$ **2**

(iii) Show that DEGF is a cyclic quadrilateral. **2**

b)

A parachutist of mass m **falls freely** until his parachute opens. When it opens he experiences an upward resistance mkv, where v is his speed and k is a positive constant. The parachutist falls freely for a time $\frac{1}{2k}$ seconds and then opens his parachute.

i) Show that when the parachute opens the velocity will be $\frac{g}{2k}$ and

He will have travelled $\frac{g}{8k^2}$ from the starting point (g, gravity). **2**

ii) Show that after the parachute opens $\ddot{x} = g - kv$. **1**

iii) Show that the total distance that he has fallen when his velocity is $\frac{3g}{4k}$ is given by $x = \frac{g}{k^2}\ln 2 - \frac{g}{8k^2}$. **3**

c) Show $\int (a^2 + x^2)^m \, dx = \frac{x(a^2 + x^2)^m}{2m+1} + \frac{2ma^2}{2m+1} \int (a^2 + x^2)^{m-1} dx$ **4**

(hint. Use the fact that $(a^2 + x^2)^m = (a^2 + x^2)(a^2 + x^2)^{m-1}$)

End of the examination.

Name:____________________ Teacher:__________________

Multiple-choice answer sheet

Select the alternative A, B, C or D that best answers the question. Fill in the response oval completely, using a black pen.

Sample: 2 + 4 = (A) 2 (B) 6 (C) 8 (D) 9

A ◯ B ● C ◯ D ◯

If you think you have made a mistake, put a cross through the incorrect answer and fill in the new answer.

A ● B ⊗ C ◯ D ◯

If you change your mind and have crossed out what you consider to be the correct answer, then indicate the correct answer by writing the word correct and drawing an arrow as follows.

correct ↓

A ⊗ B ⊗ C ◯ D ◯

Start here →

1. A ◯ B ◯ C ◯ D ◯
2. A ◯ B ◯ C ◯ D ◯
3. A ◯ B ◯ C ◯ D ◯
4. A ◯ B ◯ C ◯ D ◯
5. A ◯ B ◯ C ◯ D ◯
6. A ◯ B ◯ C ◯ D ◯
7. A ◯ B ◯ C ◯ D ◯
8. A ◯ B ◯ C ◯ D ◯
9. A ◯ B ◯ C ◯ D ◯
10. A ◯ B ◯ C ◯ D ◯

STANDARD INTEGRALS

$$\int x^n \, dx = \frac{1}{n+1} x^{n+1}, \; x \neq 0 \text{ if } n < 0$$

$$\int \frac{1}{x} \, dx = \ln x, \; x > 0$$

$$\int e^{ax} \, dx = \frac{1}{a} e^{ax}, \; a \neq 0$$

$$\int \cos ax \, dx = \frac{1}{a} \sin ax, \; a \neq 0$$

$$\int \sin ax \, dx = -\frac{1}{a} \cos ax, \; a \neq 0$$

$$\int \sec^2 ax \, dx = \frac{1}{a} \tan ax, \; a \neq 0$$

$$\int \sec ax \tan ax \, dx = \frac{1}{a} \sec ax, \; a \neq 0$$

$$\int \frac{1}{a^2 + x^2} \, dx = \frac{1}{a} \tan^{-1} \frac{x}{a}, \; a \neq 0$$

$$\int \frac{1}{\sqrt{a^2 - x^2}} \, dx = \sin^{-1} \frac{x}{a}, \; a > 0, \; -a < x < a$$

$$\int \frac{1}{\sqrt{x^2 - a^2}} \, dx = \ln\left(x + \sqrt{x^2 - a^2}\right), \; x > a > 0$$

$$\int \frac{1}{\sqrt{x^2 + a^2}} \, dx = \ln\left(x + \sqrt{x^2 + a^2}\right)$$

NOTE: $\ln x = \log_e x, \; x > 0$

GHS TRIAL 2012 EXT 2 M.C. SOLUTIONS

1. $5-2i-(3-4i)$

$= 5-2i-3+4i$ (C)

$= 2+2i$

2. (B)

3. $x+1 = a(x-2)+b(x+2)$

$x=2)$ $3 = 4b$

$\therefore \frac{3}{4} = b$

$x=-2)$ $-1 = -4a$ (D)

$\frac{1}{4} = a$

$\therefore$ RHS $= \frac{1}{4(x+2)} + \frac{3}{4(x-2)}$

4. (C)

5. $\frac{1}{3}\tan\frac{\pi}{4}\sec^2\frac{\pi}{4} + \frac{2}{3}I_2$

$= \frac{2}{3} + \frac{2}{3}I_2$

$I_2 = \int_0^{\frac{\pi}{4}} \sec^2 x\, dx$ (A)

$= \left[\tan x\right]_0^{\frac{\pi}{4}}$

$= 1.$

$\therefore I_4 = \frac{2}{3}+\frac{2}{3} = \frac{4}{3}$

6. $\int_0^1 \frac{\frac{2\,dt}{1+t^2}}{1+\frac{2t}{1+t^2}}$ $\quad t=\tan\frac{x}{2}$, $dx = \frac{2}{1+t^2}dt$

$x=\frac{\pi}{2}$ $t=1$, $x=0$ $t=0$

$= \int_0^1 \frac{\frac{2}{1+t^2}}{\frac{1+t^2+2t}{1+t^2}}\,dt$

$= \int_0^1 \frac{2\,dt}{(1+t)^2}$ (C)

7. (D)

8. $x^4-kx+1 = (x^2+1)Q(x)+3x+2$

(when $x=i$)

$i^4-ik+1 = 3i+2$

$1-ik+1 = 3i+2$ (C)

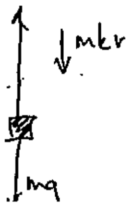
$-ik = 3i$

$-3i = ik$

$\therefore k=-3$

9. $V = 2\pi\int_1^3 (2y)(x-1)\,dx$ $\quad y^2=1-(x-2)^2$, $y=\sqrt{1-(x-2)^2}$

$= 4\pi\int_1^3 (x-1)\left(\sqrt{1-(x-2)^2}\right)dx$ (D)

10. 

$-mkv-mg = m\ddot{x}$

$-kv-g = \ddot{x}$ (C)

$-kv-g = \frac{v\,dv}{dx}$

$-k-g$ ...

GHS TRIAL 2012 MATHEMATICS SOLUTIONS

Q11 a) $\int \frac{dx}{x^2-2x+5}$

$= \int\frac{dx}{(x^2-2x+1)+4}$

Let $u=x-1$, $\frac{du}{dx}=1$, $du=dx$

$= \int\frac{dx}{(x-1)^2+4}$

$= \int\frac{du}{u^2+4}$

$= \frac{1}{2}\tan^{-1}\frac{u}{2}+c$

$= \frac{1}{2}\tan^{-1}\frac{x-1}{2}+c$

b) $\int \sin^4 x\cos^3 x\,dx$

$= \int \sin^4 x.\cos^2 x.\cos x\,dx$

$= \int \sin^4 x(1-\sin^2 x).\cos x\,dx$

$= \int(\sin^4 x-\sin^6 x)\cos x\,dx$

$= \int\sin^4 x.\cos x\,dx - \int\sin^6 x.\cos x\,dx$

Let $u=\sin x$, $\frac{du}{dx}=\cos x$, $du=\cos x\,dx$

$= \int u^4\,du - \int u^6\,du$

$= \frac{u^5}{5}-\frac{u^7}{7}+c.$

$= \frac{\sin^5 x}{5}-\frac{\sin^7 x}{7}+c.$

c) $\int_0^{\frac{1}{2}}\frac{dx}{\sqrt{x(1-x)}}$ $\quad u^2=x$, $2u = \frac{dx}{du}$, $2u\,du=dx$

$x=\frac{1}{2}$ $u=\frac{1}{\sqrt{2}}$, $x=0$ $u=0$

$= \int_0^{\frac{1}{\sqrt{2}}}\frac{2u\,du}{\sqrt{u^2}.\sqrt{1-u^2}}$

$= 2\int_0^{\frac{1}{\sqrt{2}}}\frac{du}{\sqrt{1-u^2}}$

$= 2\left[\sin^{-1}u\right]_0^{\frac{1}{\sqrt{2}}}$

$= 2\left(\frac{\pi}{4}-0\right)$

$= \frac{\pi}{2}$

d) $\int_0^{\frac{\pi}{4}} x\sin x\,dx$ (by parts)

$= -\left[x\cos x\right]_0^{\frac{\pi}{4}} + \int_0^{\frac{\pi}{4}} 1.\cos x\,dx$

$= -\left[\frac{\pi}{4}.\frac{1}{\sqrt{2}}-0\right]+\left[\sin x\right]_0^{\frac{\pi}{4}}$

$= -\frac{\pi}{4\sqrt{2}}+\left(\frac{1}{\sqrt{2}}-0\right)$

$= \frac{4-\pi}{4\sqrt{2}}$

e) (i) $x^2+y^2=1$

$2x+2y\frac{dy}{dx}=0$

$\frac{dy}{dx}=\frac{-2x}{2y}$

$=-\frac{x}{y}$

(ii) Semi circle arc length

$=\int_{-1}^{1}\sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx$

$=\int_{-1}^{1}\sqrt{1+\left(\frac{-x}{y}\right)^2}\,dx$

$=\int_{-1}^{1}\sqrt{1+\frac{x^2}{y^2}}\,dx$

$=\int_{-1}^{1}\sqrt{\frac{y^2+x^2}{y^2}}\,dx$

$=\int_{-1}^{1}\frac{1}{\sqrt{1-x^2}}\,dx$

$=\left[\sin^{-1}x\right]_{-1}^{1}$

$=\sin^{-1}1-\sin^{-1}(-1)$

$=\frac{\pi}{2}+\frac{\pi}{2}$

$=\pi$ units

$\therefore$ Circumference of full circle $=2\pi$ units as req.

12 a) $\frac{2-5i}{4-3i}\times\frac{4+3i}{4+3i}$

$=\frac{8+6i-20i+15}{16+9}$

$=\frac{23-14i}{25}$

$=\frac{23}{25}-\frac{14i}{25}$

b) let $z_1=a+ib$ $z_2=c+id$

$\therefore$ LHS $=\text{Re}\left[(a+ib)(c+id)\right]$

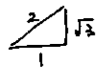
$=\text{Re}\left[ac+iad+ibc-bd\right]$

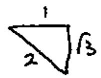
$=ac-bd$

RHS $=a\times c-b\times d$

$=ac-bd$

$\therefore$ LHS = RHS as req.

c) $1+i\sqrt{3}\Rightarrow$  $2\text{cis}\frac{\pi}{3}$

$1-i\sqrt{3}\Rightarrow$  $2\text{cis}\left(-\frac{\pi}{3}\right)$

$\therefore\frac{1+i\sqrt{3}}{1-i\sqrt{3}}=\text{cis}\frac{2\pi}{3}$

$\therefore z^{10}=\text{cis}\frac{20\pi}{3}$

$=\text{cis}\frac{2\pi}{3}$

$=1\left(-\frac{1}{2}+\frac{\sqrt{3}i}{2}\right)$

$=-\frac{1}{2}+\frac{\sqrt{3}i}{2}$

d) $\left|\frac{z-2}{z+3}\right|=2$

$\therefore|z-2|=2|z+3|$

$\sqrt{(x-2)^2+y^2}=2\sqrt{(x+3)^2+y^2}$

$x^2-4x+4+y^2=4\left[x^2+6x+9+y^2\right]$

$x^2-4x+4+y^2=4x^2+24x+36+4y^2$

$-32=3x^2+28x+3y^2$

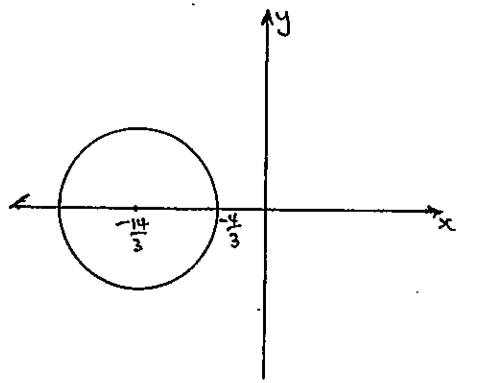
$-\frac{32}{3}=x^2+\frac{28}{3}x+y^2$

$-\frac{32}{3}+\frac{196}{9}=x^2+\frac{28}{3}x+\frac{196}{9}+y^2$

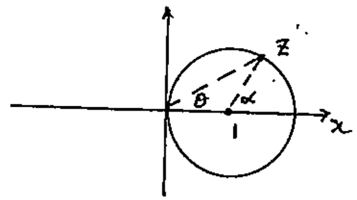
$\frac{100}{9}=\left(x+\frac{14}{3}\right)^2+y^2$

$\therefore$ Circle centre $\left(-\frac{14}{3},0\right)$

Radius $=\frac{10}{3}$.



e)



let $\alpha=\arg(z-1)$

$\theta=\arg z$

$\therefore\alpha=2\theta$ (since ∠ at the centre of a circle is twice the ∠ at the circumference standing on the same arc)

$\therefore\arg(z-1)=2\arg z$

as req.

f) let $z=\cos\theta+i\sin\theta$

$\therefore z^3=(\cos\theta+i\sin\theta)^3$

$=\cos3\theta+i\sin3\theta$ (De Moivre's Theorem) —(1)

Expanding:

$z^3=\cos^3\theta+3\cos^2\theta.i\sin\theta+3\cos\theta.i^2\sin^2\theta+i^3\sin^3\theta$

$=\cos^3\theta+3\cos^2\theta.i\sin\theta-3\cos\theta\sin^2\theta-i\sin^3\theta$ —(2)

Equating real parts in (1) and (2)

$\cos3\theta=\cos^3\theta-3\cos\theta\sin^2\theta$

$=\cos^3\theta-3\cos\theta(1-\cos^2\theta)$

$=\cos^3\theta-3\cos\theta+3\cos^3\theta$

$=4\cos^3\theta-3\cos\theta$

as req.

g) $p-q+iq-ir=0$

$\therefore p-q=i(r-q)$

$\therefore$ RQ rotated 90° gives PQ

$\therefore\triangle PQR$ is right angled

also since $|p-q|=|i||r-q|$

$\therefore|p-q|=|r-q|$ and PQ = RQ.

~~right angled isosceles~~

Q13 a) $z^4 + 81 = 0$

$z^4 = -81$

If $z = r \operatorname{cis}\theta$

$\therefore (r\operatorname{cis}\theta)^4 = 81 \operatorname{cis}\pi$ (since $\operatorname{cis}\pi = -1$)

$\therefore r = 3 \quad \theta = \frac{\pi}{4}$

$\therefore z_1 = 3\operatorname{cis}\frac{\pi}{4} = \frac{3}{\sqrt{2}}(1+i)$

$z_2 = 3\operatorname{cis}\frac{3\pi}{4} = \frac{3}{\sqrt{2}}(-1+i)$

$z_3 = 3\operatorname{cis}\left(-\frac{3\pi}{4}\right) = \frac{3}{\sqrt{2}}(-1-i)$

$z_4 = 3\operatorname{cis}\left(-\frac{\pi}{4}\right) = \frac{3}{\sqrt{2}}(1-i)$

b) $P(x) = (x-4)(x-3)Q(x) + ax + b$

$P(3) = 5 \quad \therefore \quad 5 = 3a + b$ — ①

$P(4) = 9 \quad \therefore \quad 9 = 4a + b$ — ②

① − ② : $-4 = -a$

$\therefore a = 4$

$b = -7$

$\therefore$ Remainder $= 4x - 7$

c) $z^4 - z^2 + 16 = 0$

Using the quadratic formula:

$z^2 = \frac{1 \pm \sqrt{1-64}}{2}$

$= \frac{1 \pm \sqrt{63}\,i}{2}$

$= \frac{1 \pm 3\sqrt{7}\,i}{2}$

Let $z = x + iy$

$(x+iy)^2 = \frac{1}{2} \pm \frac{3\sqrt{7}}{2}i$

Equating real/imaginary parts

$x^2 - y^2 = \frac{1}{2}$ —(1)

$2xy = \pm\frac{3\sqrt{7}}{2}$ —(2)

(2) $\Rightarrow xy = \pm\frac{3\sqrt{7}}{4}$

From (2) $y = \pm\frac{3\sqrt{7}}{4x}$

Sub in (1) $x^2 - \left(\pm\frac{3\sqrt{7}}{4x}\right)^2 = \frac{1}{2}$

$x^2 - \frac{63}{16x^2} = \frac{1}{2}$

$16x^4 - 63 = 8x^2$

$16x^4 - 8x^2 - 63 = 0$

$\therefore x^2 = \frac{8 \pm \sqrt{64 + 4(16)(63)}}{32}$

$= \frac{8 \pm 64}{32}$

$= \frac{9}{4}$, ~~$-\frac{7}{4}$~~ since x is real

$x = \pm\frac{3}{2}$

If $x = 3/2$: $y = \pm\frac{3\sqrt{7}}{6}$

$= \pm\frac{\sqrt{7}}{2}$

If $x = -3/2$: $y = \pm\frac{3\sqrt{7}}{-6}$

$= \mp\frac{\sqrt{7}}{2}$

$\therefore z = \frac{3}{2} \pm \frac{\sqrt{7}}{2}, \; -\frac{3}{2} \pm \frac{\sqrt{7}}{2}$.

OR a better way:

$z^4 - z^2 + 16 = 0$

can be written as

$z^4 + 8z^2 + 16 - 9z^2 = 0$

$(z^2+4)^2 - (3z)^2 = 0$

$(z^2+4-3z)(z^2+4+3z) = 0$

$(z^2-3z+4)(z^2+3z+4) = 0$

$\therefore z = \frac{3 \pm \sqrt{9-16}}{2}, \; \frac{-3 \pm \sqrt{9-16}}{2}$

$= \frac{3 \pm \sqrt{7}\,i}{2}, \; \frac{-3 \pm \sqrt{7}\,i}{2}$

as before.

d) Let $P(x) = 8x^4 + 44x^3 + 54x^2 + 25x + 4 = 0$

$P'(x) = 32x^3 + 132x^2 + 108x + 25$

$P''(x) = 96x^2 + 264x + 108$

If $P(x)$ has a root of multiplicity 3 then $P''(x) = 0$ has a common root

$P''(x) \Rightarrow 12(8x^2 + 22x + 9) = 0$

$\Rightarrow 12(4x+9)(2x+1) = 0$

$\therefore x = -\frac{9}{4}, -\frac{1}{2}$

Test in $P(x) = 0$

Since $P(-\frac{1}{2}) = 0$

$\therefore$ triple root is $x = -\frac{1}{2}$

$\therefore P(x) = (2x+1)^3(x+4)$

$\therefore x = -\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, -4$.

e) Given $x^3 - 3px + q = 0$

$\alpha + \beta + \gamma = 0$

$\Sigma\alpha\beta = -3p$

$\alpha\beta\gamma = -q$

$\therefore$ if roots are $\frac{\beta\gamma}{\alpha}, \frac{\alpha\gamma}{\beta}, \frac{\alpha\beta}{\gamma}$

$\Sigma\alpha \Rightarrow \frac{\beta\gamma}{\alpha} + \frac{\alpha\gamma}{\beta} + \frac{\alpha\beta}{\gamma}$

$= \frac{(\beta\gamma)^2 + (\alpha\gamma)^2 + (\alpha\beta)^2}{\alpha\beta\gamma}$

$= \frac{(\beta\gamma + \alpha\gamma + \alpha\beta)^2 - 2(\alpha\beta\gamma^2 + \alpha\beta^2\gamma + \alpha^2\beta\gamma)}{\alpha\beta\gamma}$

$= \frac{(-3p)^2 - 2(\alpha\beta\gamma(\alpha+\beta+\gamma))}{-q}$

$= \frac{9p^2 - 0}{-q}$

$= \frac{9p^2}{-q}$

$\Sigma\alpha\beta \Rightarrow \frac{\beta\gamma}{\alpha}\cdot\frac{\alpha\gamma}{\beta} + \frac{\beta\gamma}{\alpha}\cdot\frac{\alpha\beta}{\gamma} + \frac{\alpha\gamma}{\beta}\cdot\frac{\alpha\beta}{\gamma}$

$= \gamma^2 + \beta^2 + \alpha^2$

$= (\alpha+\beta+\gamma)^2 - 2(\Sigma\alpha\beta)$

$= 0 - 2(-3p)$

$= 6p$.

$\Sigma\alpha\beta\gamma \Rightarrow \frac{\beta\gamma}{\alpha}\cdot\frac{\alpha\gamma}{\beta}\cdot\frac{\alpha\beta}{\gamma} = \alpha\beta\gamma = -q$

$\therefore$ with $\Sigma\alpha = \frac{9p^2}{-q}$

$\Sigma\alpha\beta = 6p$

and $\Sigma\alpha\beta\gamma = -q$

and the required equation being of the form:

$x^3 - \Sigma\alpha . x^2 + \Sigma\alpha\beta . x - \Sigma\alpha\beta\gamma = 0$

req. equation is:

$x^3 + \frac{9p^2}{q}x^2 + 6px + q = 0$

An alternative solution:

Since $\alpha\beta\gamma = -q$

$\beta\gamma = \frac{-q}{\alpha}$

imilarly $\alpha\gamma = \frac{-q}{\beta}$ and $\alpha\beta = \frac{-q}{\gamma}$)

ince $\frac{\beta\gamma}{\alpha}$ is a root

then so is $\frac{-q}{\alpha^2}$

$\therefore x = \frac{-q}{\alpha^2}$

$\alpha^2 = \frac{-q}{x}$

$\alpha = \sqrt{\frac{-q}{x}}$

Sub this in $x^3 - 3px + q = 0$

$\left(\sqrt{\frac{-q}{x}}\right)^3 - 3p\left(\sqrt{\frac{-q}{x}}\right) + q = 0$

$\sqrt{\frac{-q}{x}}\left(\frac{-q}{x}\right) - 3p\sqrt{\frac{-q}{x}} + q = 0$

$\sqrt{\frac{-q}{x}}\left[\frac{-q}{x} - 3p\right] = -q$

Square both sides:

$\frac{-q}{x}\left(\frac{q^2}{x^2} + \frac{6pq}{x} + 9p^2\right) = q^2$

$\frac{-q^3}{x^3} - \frac{6pq^2}{x^2} - \frac{9p^2q}{x} = q^2$

$-q^3 - 6pq^2x - 9p^2qx^2 = q^2x^3$

$(\div q^2)$ — since monic!

$-q - 6px - \frac{9p^2}{q}x^2 = x^3$

$\therefore$ eqn is

$x^3 + \frac{9p^2}{q}x^2 + 6px + q = 0$

14 a) (i) Chord of contact:

$\frac{xx_0}{a^2} - \frac{yy_0}{b^2} = 1$

$(x_0 y_0) = \left(\frac{16}{5}, 1\right)$

$a^2 = 16$, $b^2 = 9$

$\therefore \frac{\frac{16}{5}x}{16} - \frac{y}{9} = 1$

$\frac{x}{5} - \frac{y}{9} = 1$

$9x - 5y = 45$

$\therefore 9x - 5y - 45 = 0$

(ii) $e^2 = 1 + \frac{b^2}{a^2}$

$e^2 = 1 + \frac{9}{16}$

$e^2 = \frac{25}{16}$

$\therefore e = \frac{5}{4}$

$\therefore$ Focus $= (ae, 0)$

$= (4 . \frac{5}{4}, 0)$

$= (5, 0)$

Sub $(5,0)$ into $9x - 5y - 45 = 0$

LHS $= 45 - 0 - 45$

$= 0$

$=$ RHS

$\therefore$ chord passes through the focus $\therefore$ focal chord.

b)(i) $S = (ae, 0)$ focus

$\therefore$ equation of PSP' is $x = ae$

to find P sub $x = ae$ into

$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

$\frac{a^2e^2}{a^2} + \frac{y^2}{b^2} = 1$

$e^2 + \frac{y^2}{b^2} = 1$

$b^2e^2 + y^2 = b^2$

$y^2 = b^2(1 - e^2)$

$y = \pm b\sqrt{1 - e^2}$

P is in 1st Quad.

$\therefore y = b\sqrt{1 - e^2}$

For an ellipse $e^2 = 1 - \frac{b^2}{a^2}$

$\frac{b^2}{a^2} = 1 - e^2$

$\therefore y = b\sqrt{\frac{b^2}{a^2}}$

$= b \times \frac{b}{a}$

$= \frac{b^2}{a}$

$\therefore P = \left(ae, \frac{b^2}{a}\right)$

(ii) Gradient of tangent $= \frac{dy}{dx}$

for $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

$\frac{2x}{a^2} + \frac{2y}{b^2}\frac{dy}{dx} = 0$

$$\frac{dy}{dx} = \frac{-\frac{2x}{a^2}}{\frac{2y}{b^2}} = \frac{-b^2x}{a^2y}$$

at P) $\frac{dy}{dx} = \frac{-b^2 \cdot ae}{a^2 \cdot \frac{b^2}{a}} = -e$

equation of tangent $\Rightarrow$

$$y - \frac{b^2}{a} = -e(x - ae)$$

$$ay - b^2 = -aex + a^2e^2$$

$$aex + ay = a^2e^2 + b^2$$

$$aex + ay = a^2\left(1 - \frac{b^2}{a^2}\right) + b^2$$

$$aex + ay = a^2 - b^2 + b^2$$

$aex + ay = a^2$ as req.

ii) to find M sub $x = 0$

$\therefore ay = a^2$

$y = a$

$\therefore M = (0, a)$

gradient of $MS' = \frac{0 - a}{-ae - 0}$

$= \frac{1}{e}$

From (1) gradient of tangent at P $= -e$

$\therefore$ gradient of normal at P $= \frac{1}{e}$

$\therefore$ normal is parallel to MS'

c) Midpoint $PQ = \left(\frac{c(p+q)}{2}, \frac{c(p+q)}{2pq}\right)$

$\therefore x = \frac{c(p+q)}{2}$ — (1)

$y = \frac{c(p+q)}{2pq}$ — (2)

Also from data

$cq - cp = 2c$

$q - p = 2$

$q = 2 + p$

$\therefore x = \frac{c(p+2+p)}{2}$

$= \frac{c(2p+2)}{2}$

$= c(p+1)$

$\therefore \frac{x}{c} - 1 = p$ — (3)

Also $y = \frac{c(p+2+p)}{2p(p+2)}$

$= \frac{2c(p+1)}{2p(p+2)}$

$= \frac{c(p+1)}{p(p+2)}$

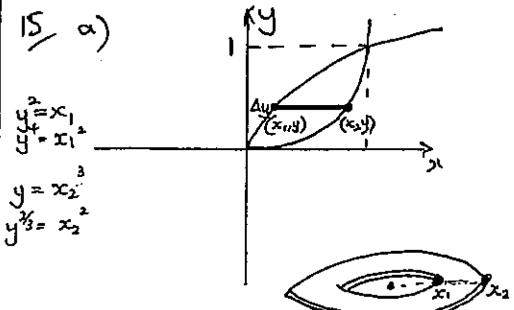
Sub (3) in here:

$$y = \frac{c\left(\frac{x}{c} - 1 + 1\right)}{\left(\frac{x}{c} - 1\right)\left(\frac{x}{c} - 1 + 2\right)}$$

$$= \frac{x}{\left(\frac{x}{c} - 1\right)\left(\frac{x}{c} + 1\right)}$$

On simplifying $x^2y - c^2(\ldots$

15 a)



$A(y) = \pi x_2^2 - \pi x_1^2$

$= \pi\left(y^{2/3} - y^4\right)$

$\Delta V = \pi\left(y^{2/3} - y^4\right) \cdot \Delta y$

$$V = \pi\int_0^1 y^{2/3} - y^4 \, dy$$

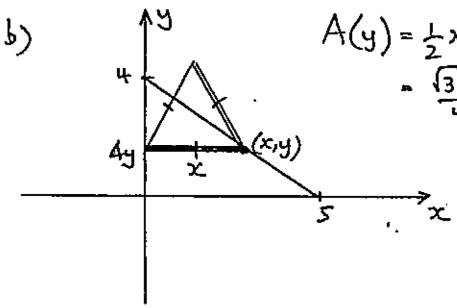
$$= \pi\left[\frac{y^{5/3}}{5/3} - \frac{y^5}{5}\right]_0^1$$

$$= \frac{\pi}{5}\left[3y^{5/3} - y^5\right]_0^1$$

$$= \frac{\pi}{5}\left[(3 - 1) - 0\right]$$

$= \frac{2\pi}{5}$ units^3

b)



$A(y) = \frac{1}{2}x^2 \sin 60°$

$= \frac{\sqrt{3}x^2}{4}$

$\Delta V = \frac{\sqrt{3}x^2}{4} \cdot \Delta y$

$$V = \int_0^4 \frac{\sqrt{3}x^2}{4} \, dy$$

Now $4x + 5y = 20$

$4x = 20 - 5y$

$x = 5 - \frac{5}{4}y$

$x^2 = 25 - \frac{25}{2}y + \frac{25y^2}{16}$

$$\therefore V = \frac{\sqrt{3}}{4}\int_0^4 25 - \frac{25}{2}y + \frac{25y^2}{16} \, dy$$

$$= \frac{\sqrt{3}}{4}\left[25y - \frac{25}{4}y^2 + \frac{25y^3}{48}\right]_0^4$$

$$= \frac{25\sqrt{3}}{4}\left[\left(4 - 4 + \frac{64}{48}\right) - 0\right]$$

$= \frac{25\sqrt{3}}{3}$ units^3

c) (i) and (iii)

A: $x = -2$ is a vertical asymptote

as $x \to \infty$ $y \to 1^-$

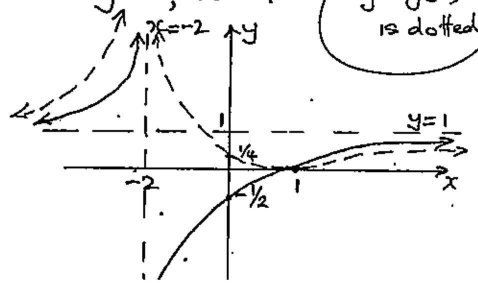
as $x \to -\infty$ $y \to 1^+$

$\therefore y = 1$ is a horizontal asymptote

I: $x = 0$, $y = -\frac{1}{2}$

$y = 0$, $x = 1$

$y = g(x)$ is dotted



i) $y = \left(\frac{x-1}{x+2}\right)^2$

$= \frac{(x-1)^2}{(x+2)^2}$

Stat pts occur when $y' = 0$

$y' = \frac{(x+2)^2 . 2(x-1) - (x-1)^2 . 2(x+2)}{(x+2)^4}$

$= \frac{2(x+2)(x-1)\left[x+2 - (x-1)\right]}{(x+2)^4}$

$= \frac{2(x-1)(3)}{(x+2)^3}$

$= \frac{6(x-1)}{(x+2)^3}$

∴ Stat pt at $(1, 0)$

to determine nature:

x	1^-	1	1^+
y	-ve	0	+ve

\ __ /

∴ $(1,0)$ is a min T.P.

ii) A: $x = -2$ still a V.A

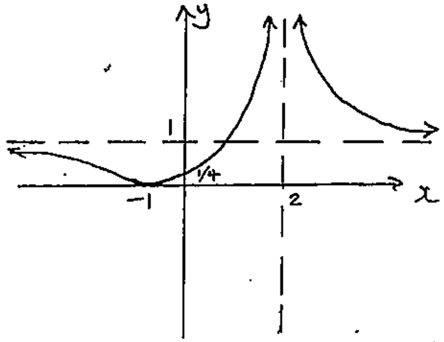
$y = 1$ still a H.A.

I. $(1,0)$ is a Min T.P.

$x = 0$ $y = \frac{1}{4}$.

(iv) $y = g(-x)$

(a reflection about the y axis)



16(b)(i) Since $\Delta AFD \parallel\!| \Delta ACB$

(2 sides in proportion about an equal ∠)

∴ ∠AFD = ∠ACB (corresponding equal ∠'s in similar Δ's)

∴ CB ∥ FD

(ii) AF = FG (corresponding sides in given congruent Δ's)

and since AF = FC (F is given midpoint)

∴ FG = FC

∴ ∠FGC = ∠FCG (equal base ∠'s in Isosceles Δ)

(iii) Using a similar approach to (i) AC ∥ DE

∴ ∠FCG = ∠DEB

let ∠FCG = θ

∴ ∠FGC = θ (part (ii))

∴ $F\hat{D}E = D\hat{E}B = \theta$

(alternate ∠'s in parallel lines)

∴ ∠FGC = $F\hat{D}E$

∴ DEGF is a cyclic quadrilateral because the exterior ∠ is equal to the interior opposite ∠.

b) (i) $m\ddot{x} = mg$ (↓mg)

$\ddot{x} = g$

$\frac{dv}{dt} = g$

$v = gt + c_1$

$(t=0, v=0)$ $0 = 0 + c_1$

$\therefore v = gt$

(when $t = \frac{1}{2k}$) $v = \frac{g}{2k}$ as req.

and $\frac{dx}{dt} = gt$

$x = \frac{gt^2}{2} + c_2$

$(x=0, t=0)$ $c_2 = 0$.

$\therefore x = \frac{gt^2}{2}$

$(t = \frac{1}{2k})$ $x = \frac{g\left(\frac{1}{4k^2}\right)}{2}$

$= \frac{g}{8k^2}$

(ii) ↑mkv ↓mg $m\ddot{x} = mg - mkv$

$\ddot{x} = g - kv$

(iii) $v\frac{dv}{dx} = g - kv$

$\frac{dv}{dx} = \frac{g-kv}{v}$

$\frac{dx}{dv} = \frac{v}{g-kv}$

$= \frac{-\frac{1}{k}(g-kv)}{g-kv} + \frac{\frac{g}{k}}{g-kv}$

$$\therefore x = -\frac{1}{k}\int 1\,dv + \frac{g}{k}\int \frac{dv}{g-kv}$$

$$x = -\frac{v}{k} - \frac{g}{k^2}\ln|g-kv| + c$$

when $v = \frac{g}{2k}$ $x = \frac{g}{8k^2}$

$$\frac{g}{8k^2} = \frac{-g}{2k^2} - \frac{g}{k^2}\ln\left|g - \frac{kg}{2k}\right| + c$$

$$\therefore c = \frac{5g}{8k^2} + \frac{g}{k^2}\ln\left|g - \frac{kg}{2k}\right|$$

$$= \frac{5g}{8k^2} + \frac{g}{k^2}\ln\left|\frac{2kg-kg}{2k}\right|$$

$$= \frac{5g}{8k^2} + \frac{g}{k^2}\ln\left|\frac{g}{2}\right|$$

$$\therefore x = -\frac{v}{k} - \frac{g}{k^2}\left[\ln|g-kv| - \frac{5}{8} - \ln\left|\frac{g}{2}\right|\right]$$

(when $v = \frac{3g}{4k}$)

$$x = \frac{-3g}{4k^2} - \frac{g}{k^2}\left[\ln\left(g - \frac{3g}{4}\right) - \frac{5}{8} - \ln\frac{g}{2}\right]$$

$$= \frac{-3g}{4k^2} - \frac{g}{k^2}\left[\ln\left(\frac{\frac{4g-3g}{4}}{\frac{g}{2}}\right) - \frac{5}{8}\right]$$

$$= \frac{-3g}{4k^2} + \frac{5g}{8k^2} - \frac{g}{k^2}\ln\left(\frac{1}{2}\right)$$

$$= \frac{g}{\ }\ln 2 - \frac{g}{\ }$$ as req.

c) $\int (a^2+x^2)^m dx$

$$= \int (a^2+x^2)(a^2+x^2)^{m-1}dx$$

$$= \int a^2(a^2+x^2)^{m-1} + x^2(a^2+x^2)^{m-1}dx$$

$$= a^2\int (a^2+x^2)^{m-1}dx + \int x^2(a^2+x^2)^{m-1}dx$$

$$= a^2\int (a^2+x^2)^{m-1}dx + \int x\left[x(a^2+x^2)^{m-1}\right]dx$$

$$= a^2\int (a^2+x^2)^{m-1}dx + \frac{x}{2m}(a^2+x^2)^m - \int \frac{1}{2m}(a^2+x^2)^m dx$$

$$= a^2\int (a^2+x^2)^{m-1}dx + \frac{x}{2m}(a^2+x^2)^m - \frac{1}{2m}\int (a^2+x^2)^m dx$$

$$\therefore \int (a^2+x^2)^m dx + \frac{1}{2m}\int (a^2+x^2)^m dx$$

$$= a^2\int (a^2+x^2)^{m-1}dx + \frac{x}{2m}(a^2+x^2)^m$$

$$\therefore \int (a^2+x^2)^m dx\left[1 + \frac{1}{2m}\right] = RHS$$

$$\therefore \int (a^2+x^2)^m dx = \frac{RHS}{\frac{2m+1}{2m}}$$

$$\therefore \int (a^2+x^2)^m dx = \frac{2a^2m}{2m+1}\int (a^2+x^2)^{m-1}dx + \frac{x}{2m+1}(a^2+x^2)^m$$

as req.