Year 12 Trial Examination – Mathematics Extension 1

Question One 12 marks (Start on a new page) Marks

a) If P is the point (-3, 5) and Q is the point (1, -2), find the co-ordinates of the point R which divides the interval PQ externally in the ratio 3:2. **2**

b) When $(x + 3)(x - 2) + 2$ is divided by $x - k$, the remainder is k^2. Find the value of k. **2**

c) Solve $\frac{x}{x-3} > 1$ **3**

d) Find the general solution of $\sin\theta = \cos\theta$ in radians **2**

e) Find the exact value of $\int_0^{\frac{\pi}{2}} 2\sin^2 x\, dx$ **3**

Question Two 12 marks (Start on a new page)

a) Draw a graph of a function $y = f(x)$ for $1 \le x \le 2$ such that $\frac{dy}{dx} > 0$ and $\frac{d^2y}{dx^2} < 0$ for $1 \le x \le 2$ **2**

b) Differentiate:

i) $\frac{1}{1 + 4x^2}$ **2**

ii) $e^{2x} \log_e 2x$ **2**

c) i) Show that $x^2 + 4x + 13 = (x+2)^2 + 9$ **1**

ii) Hence find $\int \frac{1dx}{x^2 + 4x + 13}$ **2**

d) Evaluate $\int_0^{\sqrt{3}} x\sqrt{x^2 + 1}\, dx$ using the substitution that $u = x^2 + 1$ **3**

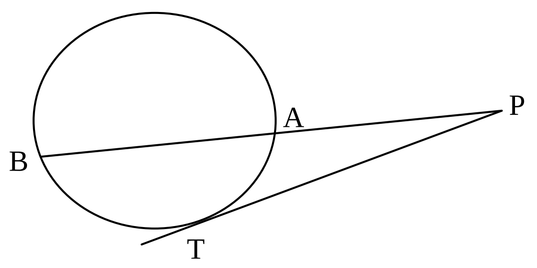
Question Three 12 marks (Start on a new page) **Marks**

a) Given $f(x) = \dfrac{x-1}{x+2}$

i) Write an expression for the inverse function $f^{-1}(x)$ **1**

ii) Write down the domain and range of $f^{-1}(x)$ **2**

b)



PT is a tangent to circle *ABT*. *PAB* is a secant intersecting the circle in *A* and *B*. *PA* = 8 cm and *AB* = 10 cm. Find the length of *PT* giving reasons. **2**

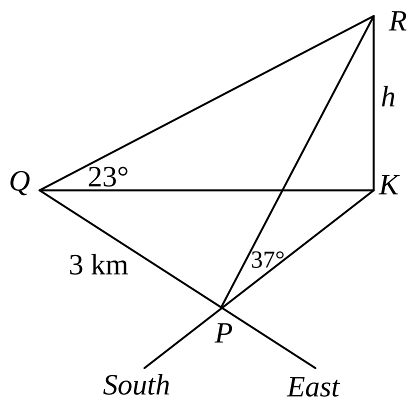
c) Find the gradients of the 2 lines which make angles of 45° with the line whose equation is $2x - 3y + 6 = 0$. **3**

d) i) Express $\cos\theta + \sqrt{3}\sin\theta$ in the form $r\cos(\theta - \alpha)$ where $r > 0$ and $0 < \alpha < \dfrac{\pi}{2}$ **2**

ii) Hence solve $\cos\theta + \sqrt{3}\sin\theta = 1$ for $-2\pi \le \theta \le 2\pi$ **2**

Question Four 12 marks (Start on a new page)

a)



The angle of elevation of a hill top from a place *P* due south of it is 37°. The angle of elevation of this same hill top from a place *Q*, due west of *P*, is 23°. The distance of *Q* from *P* is 3km. If the height of the hill is *h* km.

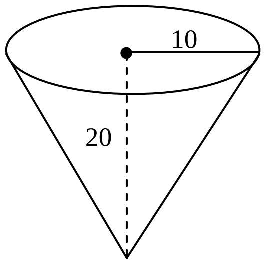
i) Prove that *PK* = *h cot* 37° **2**

ii) Find a similar expression for *QK* **2**

iii) Hence, or otherwise calculate the height of the hill to two decimal places. **2**

Question Four continued **Marks**

b)



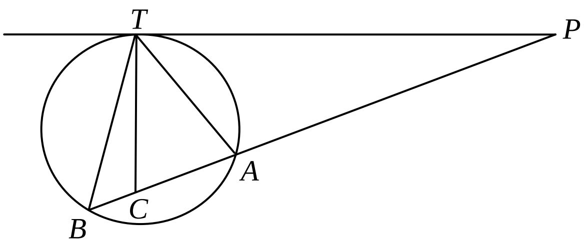
Water is running out of a filled conical funnel at the rate of $5cm^3\,s^{-1}$. The radius of the funnel is 10 cm and the height is 20 cm:

i) How fast is the water level dropping when the water is 10 cm deep? (answer in exact form) **4**

ii) How long does it take for the water to drop to 10 cm deep? (answer to 2 decimal places) **2**

Question Five **12 marks** **(Start on a new page)**

a)



PT is a tangent and PAB is a secant. $TC = TA$.

Prove $\angle BTC = \angle TPA$ **3**

b) Given θ is acute:

i) Write $\sin\frac{\theta}{2}$ in terms of $\cos\theta$ **1**

ii) Prove that $\tan\frac{\theta}{2} = \frac{\sin\theta}{1+\cos\theta}$ **2**

iii) If $\sin\theta = \frac{4}{5}$, find the value of $\tan\frac{\theta}{2}$ **3**

c) Find $\frac{d}{dx}\cos^{-1}(\sin x)$ **3**

Question Six **12 marks** **(Start on a new page)** **Marks**

a) i) Show that the sum of the cubes of three consecutive integers $(n-1),\ n,\ (n+1)$ is $3n^3+6n$. **2**

ii) Using part i), prove by mathematical induction, for all positive integers n, $n \geq 1$ that the sum of the cubes of the three consecutive integers is divisible by 9. **3**

b) Consider the variable point $P(x,\ y)$ on the parabola $x^2 = 2y$. The x value of P is given by $x = t$:

i) write its y value in terms of t **1**

ii) write an expression, in terms of t, for the square of the distance, m from P to the point (6, 0) **1**

iii) hence find the co-ordinates of P such that P is closest to the point (6, 0). **5**

Question Seven **12 marks** **(Start on a new page)**

a) Use the binomial expansion of $(1+x)^{2n}$ to show that:

i) $1-2\binom{2n}{1}+4\binom{2n}{2}-\ \ldots\ldots\ldots\ +4^n\binom{2n}{2n}=1$ **2**

ii) $\binom{2n}{1}-4\binom{2n}{2}+12\binom{2n}{3}-\ \ldots\ldots\ldots\ -n4^n\binom{2n}{2n}=-2n$ **2**

Question Seven continued **Marks**

b) Mr Mac hits a golf ball from a point O towards a flat, elevated green as shown in the diagram below. The hole at the base of the flag is situated in the centre of the green:

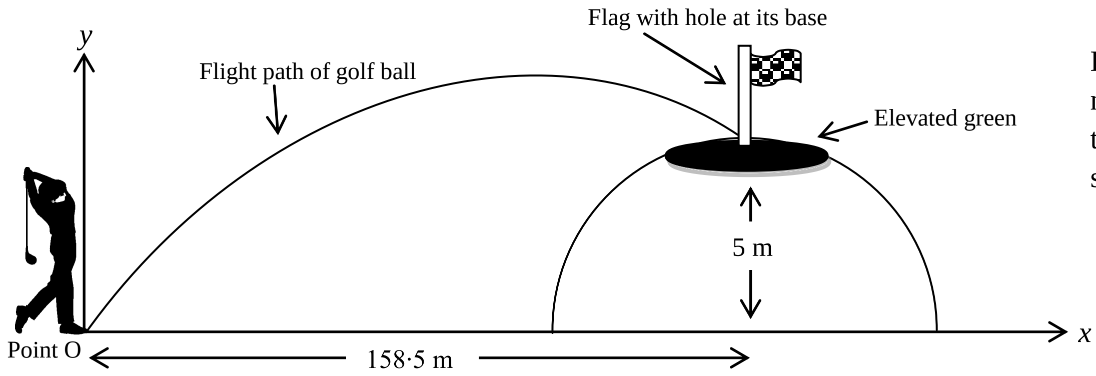


Diagram not to scale

The golf ball is projected from the point O with initial velocity of $v\ ms^{-1}$ at an angle of α to the horizontal. You may assume the only force acting on the golf ball in flight is gravity which is approximately $10ms^{-2}$:

i) Taking the point O as the origin, show the parametric equations of the flight path of the golf ball are given by $x = vt\ \cos\alpha$ and $y = -5t^2 + vt\sin\alpha$ **2**

ii) If the angle of projection is given by $\alpha = \tan^{-1}\left(\frac{1}{\sqrt{3}}\right)$ find v which will enable Mr Mac to hit the ball directly into the hole on the green. The hole is situated 158.5 m horizontally and 5 m vertically from point O. Answer to the nearest integer. **3**

iii) Using your value of v from part (ii), at what angle does the ball strike the green? (answer to the nearest degree) **3**

END OF EXAMINATION

STANDARD INTEGRALS

$$\int x^n \, dx = \frac{1}{n+1} x^{n+1}, \; n \neq -1; \; x \neq 0, \text{ if } n < 0$$

$$\int \frac{1}{x} \, dx = \ln x, \; x > 0$$

$$\int e^{ax} \, dx = \frac{1}{a} e^{ax}, \; a \neq 0$$

$$\int \cos ax \, dx = \frac{1}{a} \sin ax, \; a \neq 0$$

$$\int \sin ax \, dx = -\frac{1}{a} \cos ax, \; a \neq 0$$

$$\int \sec^2 ax \, dx = \frac{1}{a} \tan ax, \; a \neq 0$$

$$\int \sec ax \tan ax \, dx = \frac{1}{a} \sec ax, \; a \neq 0$$

$$\int \frac{1}{a^2 + x^2} \, dx = \frac{1}{a} \tan^{-1} \frac{x}{a}, \; a \neq 0$$

$$\int \frac{1}{\sqrt{a^2 - x^2}} \, dx = \sin^{-1} \frac{x}{a}, \; a \neq 0, \; -a < x < a$$

$$\int \frac{1}{\sqrt{x^2 - a^2}} \, dx = \ln\left(x + \sqrt{x^2 - a^2} \right), \; x > a > 0$$

$$\int \frac{1}{\sqrt{x^2 + a^2}} \, dx = \ln\left(x + \sqrt{x^2 + a^2} \right)$$

NOTE: $\ln x = \log_e x, \; x > 0$

QUESTION 1

$a)\quad P(-3,5)\ G(1,-2)$

$$x=\frac{-6-3}{-3+2}=9$$

$$y=\frac{10+6}{-3+2}=-16$$

$$\therefore R(9,-16)$$

$b)\ \mathrm{P}(k)=(k+3)(k+2)+2=k^2$

$$k-4=0$$

$$k=4$$

$c)\quad \frac{x}{x-3}>1 \qquad ,x\neq 3$

$$x(x-3)>(x-3)^2$$

$$x(x-3)-(x-3)^2>0$$

$$(x-3)(3)>0$$

$$x>3$$

$d)\quad \sin\theta=\cos\theta$

$$\tan\theta=1$$

$$\theta=\tan^{-1}1+n\pi$$

$$\theta=\frac{\pi}{4}+n\pi$$

$e)\quad \int_0^{\frac{\pi}{2}} 2\sin^2 x\, dx$

$$\int_0^{\frac{\pi}{2}} (1-\cos 2x)\ dx$$

$$=\left[x-\frac{1}{2}\sin 2x\right]_0^{\frac{\pi}{2}}$$

$$=\frac{\pi}{2}-\frac{1}{2}\sin\pi-0$$

$$=\pi$$

QUESTION 2

$2bi)\quad y=\frac{1}{1+4x^2}$

$$y'=\frac{-8}{\left(1+4x^2\right)}$$

$bii)\quad \text{y} = e^{2x}\log_e 2x$

$$y=e^{2x}\left(2\ln x+\frac{1}{x}\right)$$

$ci)\quad \text{RHS} = (x+2)^2+9$

$= x^2+4x+4+9$
$= x^2+4x+13$
$= \text{LHS}$

cii) $\int\frac{1dx}{x^2+4x+13}$

$$\int\frac{1dx}{(x+2)^2+9}$$

$$=\frac{1}{3}\tan^{-1}\left(\frac{x+2}{3}\right)+c$$

$2d)\quad \int_0^{\sqrt{3}} x\sqrt{x^2+1}\,dx \qquad u=x^2+1,\quad du=2xdx$

$=\frac{1}{2}\int_1^4\sqrt{u}\,du \qquad x=\sqrt{3}\ ,\ u=4$

$=\frac{1}{3}\left[u^{\frac{3}{2}}\right]_1^4 \qquad x=0,\quad u=1$

$=\frac{7}{3}$

Question 3

$ai)\ y=\frac{x-1}{x+2}$

Question 3

ai) $y=\dfrac{x-1}{x+2}$

$x=\dfrac{y-1}{y+2}$

$xy+2x=y-1$

$y(1-x)=2x+1$

$y=\dfrac{2x+1}{1-x}$

$f^{-1}=\dfrac{2x+1}{1-x}$

$ii)$ Domain all real, $x\neq 1$

range all real, $y\neq -2$

$b)$ $PT^2=BP\times PA$ $\left(\text{prod of secant and tangent}\right)$

$=18\times 8$

$PT=12$

$c)$ $\tan\theta=\dfrac{m_1-m_2}{1+m_1m_2}$

$$1=\frac{\frac{2}{3}-m_2}{1+\frac{2}{3}m_2}$$

$$1+\frac{2}{3}m_2=\frac{2}{3}-m_2$$

$$\frac{1}{3}=\frac{5}{3}m_2$$

$$\therefore m_2=\frac{-1}{5}$$

since $45^\circ+45^\circ=90^\circ$,

$m_1=5$

$di)\ \cos\theta + \sqrt{3}\sin\theta = r\cos(\theta - \alpha)$

$$= r\cos\theta\cos\alpha - r\sin\theta\sin\alpha$$

$$r\cos\alpha = 1$$
$$r\sin\alpha = \sqrt{3}$$
$$\tan\alpha = \sqrt{3}$$
$$\alpha = \frac{\pi}{3}$$
$$\therefore \cos\theta + \sqrt{3}\sin\theta = 2\cos\left(\theta - \frac{\pi}{3}\right)$$

$ii)\ \cos\theta + \sqrt{3}\sin\theta = 1$

$$2\cos\left(\theta - \frac{\pi}{3}\right) = 1$$
$$\cos\left(\theta - \frac{\pi}{3}\right) = \frac{1}{2}$$
$$\theta - \frac{\pi}{3} = \frac{\pi}{3}, \frac{5\pi}{3}, \frac{-\pi}{3}, \frac{-5\pi}{3}, \frac{-7\pi}{3}$$
$$\theta = \frac{2\pi}{3}, \frac{-4\pi}{3}, 0, 2\pi, -2\pi$$

QUESTION 4

$ai)\ \tan 37 = \frac{PK}{h}$

$$\text{PK} = h\cot 37$$

$aii)\ \tan 23 = \frac{h}{QK}$

$$QK = h\cot 23$$

$4aiii)\ 3^2 + h^2\cot^2 37 = h^2\cot^2 23$

$$h\left(\cot^2 23 - \cot^2 37\right) = 9$$
$$h = \frac{9}{\left(\cot^2 23 - \cot^2 37\right)}$$
$$h = 1.54km$$

$$4bi)\ v=\frac{1}{3}\pi r^2 h$$

$$=\frac{1}{3}\pi\left(\frac{h}{2}\right)^2 h, \qquad r=\frac{h}{2}$$

$$=\frac{1}{12}\pi h^3$$

$$\frac{dv}{dt}=\frac{1}{4}\pi h^2$$

$$\frac{dh}{dt}=\frac{dh}{dv}\times\frac{dv}{dt}$$

$$=\frac{4}{\pi h^2}\times -5$$

$$=\frac{-20}{\pi h^2}$$

$$when\ h=10, \qquad \frac{dh}{dt}=\frac{-1}{5\pi}cm/s$$

$$ii)\ \frac{dv}{dt}=-5$$

$$v=-5t+c$$

$$when\, t=0, v=\frac{1}{12}\pi\left(20\right)^3=\frac{250\pi}{3}$$

$$\frac{250\pi}{3}=-5t+\frac{2000\pi}{3}$$

$$5t=\frac{1750\pi}{3}$$

$$t=\frac{350\pi}{3}$$

$$t=367s$$

Question 5

5a) $\angle PTA = \angle BTA$ (angle between tan and chord)

$\angle TPA + \angle PTA = \angle TAC$ (ext angle of Δ)

$\therefore \angle TCA = \angle TPA + \angle PTA$ (equal angles of isos Δ)

$\angle BTC = \angle TCB - \angle BTA$ (equal angles of isos Δ)

$\therefore \angle BTC = \angle TPA$

$$bi)\ \cos\theta = 1 - 2\sin^2\frac{\theta}{2}$$

$$2\sin^2\theta = 1 - \cos\theta$$

$$\sin\theta = \sqrt{\frac{1-\cos\theta}{2}}$$

$$bii)\ RHS = \frac{\sin\theta}{1+\cos\theta}$$

$$= \frac{2\sin\frac{\theta}{2}\cos\frac{\theta}{2}}{1+\left(2\cos^2\frac{\theta}{2}-1\right)}$$

$$= \frac{2\sin\frac{\theta}{2}\cos\frac{\theta}{2}}{\left(2\cos^2\frac{\theta}{2}\right)}$$

$$= \frac{\sin\frac{\theta}{2}}{\cos\frac{\theta}{2}}$$

$$= \tan\frac{\theta}{2} = RHS$$

$biii)\ \tan\frac{\theta}{2} = \frac{\sin\theta}{1+\cos\theta}$

$= \frac{\frac{4}{5}}{\left(1+\frac{3}{5}\right)}$

$= \frac{1}{2}$

$c)\ \frac{d}{dx}\cos^{-1}(\sin x)$

$= \frac{-\frac{d}{dx}\sin x}{\sqrt{1-(\sin x)^2}}$

$= \frac{-\cos x}{\sqrt{\cos^2 x}}$

$= -1$ for ist and 4th quad

$= 1$ for 2nd and 3rd quad, $\cos x \neq 0$

Question 6

$ai)\ S_{(n)} = (n-1)^2 + n^3 + (n+1)^3$

$= n^3 - 3n^2 + 3n - 1 + n^3 + n^3 + 3n^2 + 3n + 1$

$= 3n^3 + 6n$

$= RHS$

$ii)\ S_{(1)} = 9, \therefore\ S_{(1)}$ is divisible by 9

Assume that $S_{(n)}$ is divisible by 9 when $n = k$

$ie\ S_{(k)} = 9m,$ then

$S_{(k+1)} = 3(k+1)^3 + 6(k+1)$

$= 3k^3 + 6k + 9(k^2 + k + 1)$

$= 9m + 9(k^2 + k + 1)$

$\therefore S_{(k+1)}$ is divisible by 9. Hence by the pple of MI, the statement is true for all $n \geq 1$

$b)\quad x^2 = 2y$

$i)\quad x = t,\ t^2 = 2y$

$\therefore y = \frac{t^2}{2}$

$ii)\ m^2 = (t-6)^2 + \left(\frac{t^2}{2}\right)^2$

$= t^2 - 12t + 36 + \frac{t^4}{4}$

$iii)\ \frac{dm^2}{dt} = 2t - 12 + t^3 = 0$

$(t-2)(t^2 + 2t + 6) = 0$

$t = 2,\quad t^2 + 2t + 6 = 0$

$P(2,2)\qquad \Delta = 4 - 24 = -20$

$\therefore$ no soln

QUESTION 7

$ai)\quad (1+x)^{2n} = {}^{2n}C_0x^0 + {}^{2n}C_1x^1 + {}^{2n}C_2x^2 + \ldots + {}^{2n}C_{2n}x^{2n}$

$let\ \ x = -2$

$\therefore (-1)^{2n} = 1 + {}^{2n}C_1(-2) + {}^{2n}C_2(-2)^2 + \ldots.. + {}^{2n}C_{2n}(-2)^{2n}$

$\therefore \left[(-1)^2\right]^n = 1 - 2\,{}^{2n}C_1 + 4\,{}^{2n}C_2 + \ldots.. + \left[(-2)^2\right]^n\ {}^{2n}C_{2n}$

$ie.1 = 1 - 2\,{}^{2n}C_1 + 4\,{}^{2n}C_2 + \ldots.. + 4^n\ {}^{2n}C_{2n}\quad as\ \ reqd$

$aii)\quad Differentiate\ \ wrt\ \ x$

$2n(1+x)^{2n-1} = 0 + {}^{2n}C_1 + 2x\ {}^{2n}C_2 + 3x^2\ {}^{2n}C_3 + \ldots.. + 2nx^{2n-1}\ {}^{2n}C_{2n}$

$now\ \ let\ \ x = -2$

$2n(-1)^{2n-1} = {}^{2n}C_1 - 4\ {}^{2n}C_2 + 12\,{}^{2n}C_3 + \ldots.. + 2n\,{}^{2n}C_{2n}\left[(-2)^2\right]^n.(-2)^{-1}$

$2n(-1) = {}^{2n}C_1 - 4\ {}^{2n}C_2 + 12\,{}^{2n}C_3 + \ldots.. + n\,{}^{2n}C_{2n}\left[4\right]^n.(-1)$

$\therefore -2n = {}^{2n}C_1 - 4\ {}^{2n}C_2 + 12\,{}^{2n}C_3 + \ldots.. - n\,{}^{2n}C_{2n}\left[4\right]^n\quad as\ reqd$

Question 7b continued

7bi) *horizontal* $\ddot{x}=0$

$\dot{x}=c_1$

$\dot{x}=v\cos\alpha$

$x=\int v\cos\alpha\,dt$

$=vt\cos\alpha+c_2$

$t=0, x=0 \therefore c_2=0$

$\therefore x=vt\cos\alpha$

vertical $\ddot{y}=-10$

$\dot{y}=-10t+c_3$

$\dot{y}=v\sin\alpha$

$t=0,$

$\therefore c_3=v\sin\alpha$

$\dot{y}=-10t+v\sin\alpha$

$y=\int -10t+v\sin\alpha\,dt$

$\therefore\ y=-5t^2+vt\sin\alpha$

ii) $\alpha=\tan^{-1}\left(\frac{1}{\sqrt{3}}\right)$

$\cos\alpha=\frac{\sqrt{3}}{2}\qquad \sin\alpha=\frac{1}{2}$

$vt\cos\alpha=x$

$t=\frac{x}{v\cos\alpha}$

sub into

$$y=-5t^2+vt\sin\alpha$$

$$=-5\left(\frac{x^2}{v^2\cos^2\alpha}\right)+\frac{vx}{v\cos\alpha}\times\sin\alpha$$

$$=\frac{-5x^2\sec^2\alpha}{v^2}+x\tan\alpha$$

$$=\frac{-5x^2}{v^2}\left(1+\tan^2\alpha\right)+x\tan\alpha$$

using $x=158.5, y=5, \tan\alpha=\frac{1}{\sqrt{3}}$

then $$5=\frac{-5\times158.5^2\left(1+\frac{1}{3}\right)}{v^2}+158.5\times\frac{1}{\sqrt{3}}$$

$$v^2=\frac{5\times158.5^2\times4\times\sqrt{3}}{\left(158.5\times3-15\sqrt{3}\right)}$$

$$v^2=1935.98$$

$$v=43.999m/s$$

$$v=44m/s$$

$$7biii)\ y = \frac{-5x^2}{v^2\cos^2\alpha} + x\tan\alpha$$

$$= \frac{-5x^2}{44^2\left(\frac{\sqrt{3}}{2}\right)^2} + x\left(\frac{1}{\sqrt{3}}\right)$$

$$= \frac{-5}{1452}x^2 + \frac{\sqrt{3}}{3}x$$

$$\dot{y} = \frac{-5}{726}x + \frac{\sqrt{3}}{3}$$

$$m_{\tan} = \frac{-5\times\frac{317}{2}}{726} + \frac{\sqrt{3}}{3}$$

$$acute\tan\theta = 27^\circ$$

$\therefore$ ball strikes at 153°

End of assessment