THE HILLS GRAMMAR SCHOOL

TRIAL HIGHER SCHOOL CERTIFICATE EXAMINATION

2003

MATHEMATICS

EXTENSION 2

Time Allowed: Three hours (plus 5 minutes reading time)

Teacher Responsible: Mr D Price

SPECIAL INSTRUCTIONS:

- This paper contains 8 questions. ALL questions to be attempted.
- ALL questions are of equal value.
- ALL necessary working should be shown in every question in the booklets provided.
- Start each question in a new booklet.
- A table of standard integrals is supplied at the back of this paper.
- Board approved calculators may be used.
- Hand up your paper in ONE bundle, together with this question paper.
- ALL HSC course outcomes are being assessed in this task. The Course Outcomes are listed on the back of this sheet.

Students are advised that this is a Trial Examination only and cannot in any way guarantee the content or the format of the Higher School Certificate Examination.

Question 1 **Marks**

(a) Evaluate **8**

(i) $\int_0^{\frac{\pi}{2}} \sin^2\theta\cos\theta\,d\theta$

(ii) $\int_0^1 \frac{x}{\sqrt{x+1}}\,dx$

(iii) $\int_0^1 \tan^{-1}x\,dx$

(b) (i) Express $\frac{3x+1}{(x+1)(x^2+1)}$ in the form $\frac{A}{x+1} + \frac{Bx+C}{x^2+1}$. **4**

(ii) Hence, or otherwise, find $\int \frac{3x+1}{(x+1)(x^2+1)}\,dx$.

(c) By using the substitution $t = \tan\frac{x}{2}$, evaluate $\int_0^{\frac{\pi}{2}} \frac{dx}{\sin x + \cos x + 1}$. **3**

Question 2 (Start a NEW booklet) **Marks**

(a) Let $z = 2(\cos\frac{2\pi}{9} + i\sin\frac{2\pi}{9})$ **8**

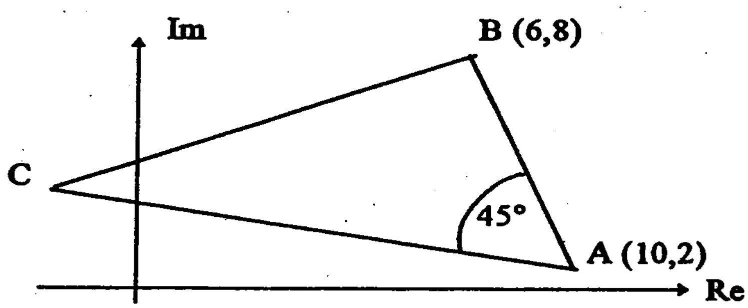
(i) Write down the modulus and argument of the complex numbers

$z, \bar{z}, z^2$, and $\frac{1}{z}$

(ii) Hence, or otherwise, clearly plot the points on an Argand diagram corresponding to $z, \bar{z}, z^2$, and $\frac{1}{z}$ labelling them A, B, C and D respectively.

(b) Find the two square roots of $-3 + 4i$ expressing each root in the form $a + ib$ where a, b are real. **4**

(c) **3**



(NOT TO SCALE)

ΔABC is drawn on the Argand plane where angle BAC = 45°, A represents the complex number $10 + 2i$ and B represents $6 + 8i$.

If the length of side AC is twice the length of AB then find the complex number that point C represents.

Question 3 (Start a NEW booklet) **Marks**

(a) For the hyperbola $9x^2 - 16y^2 = 144$ find, **7**

(i) eccentricity

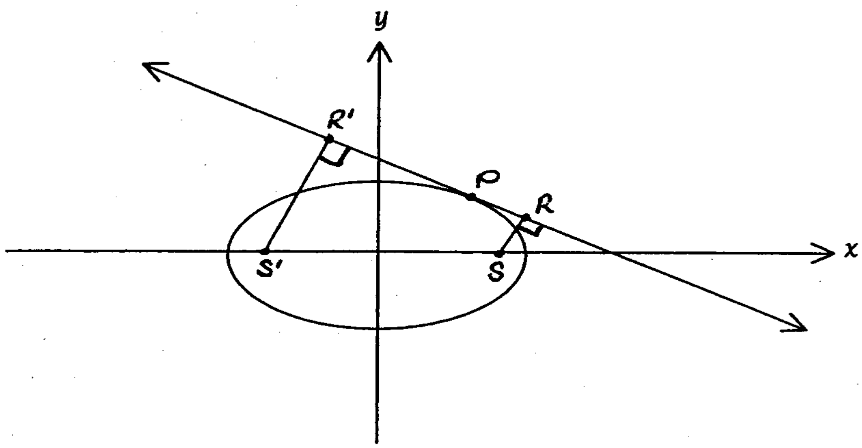
(ii) co-ordinates of the foci

(iii) equations of the directrices

(iv) equation of the asymptotes

Hence, sketch this hyperbola showing the foci, directrices and asymptotes.

(b) **8**



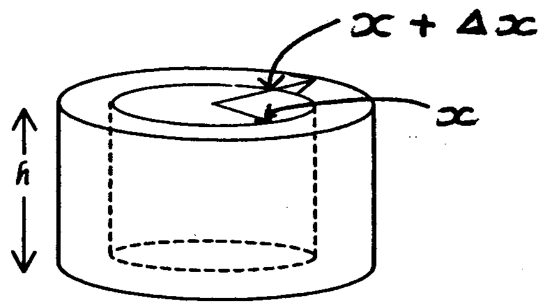
Consider the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ where $a > b > 0$.

(i) Show that the tangent to the ellipse at the point $P\,(a\cos\theta,\, b\sin\theta)$ has equation $bx\cos\theta + ay\sin\theta = ab$.

(ii) R and R' are the feet of the perpendiculars from the foci S and S' on to the tangent at P. Show that $SR \,.\, S'R' = b^2$.

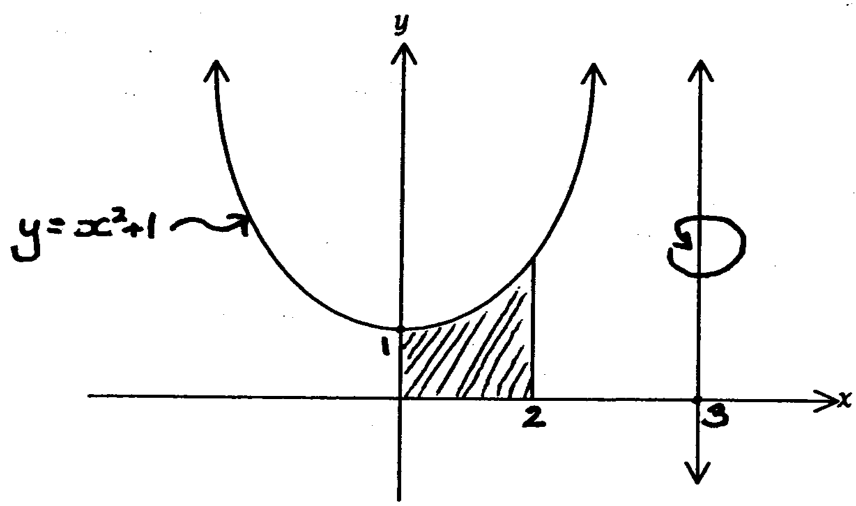
Question 4 (Start a NEW booklet) **Marks**

(a) (i) 8



Show that the volume of a cylindrical shell of height h cm with inner and outer radii x cm and $(x + \Delta x)$ cm respectively is $2\pi x h \Delta x$ cm^3 when terms involving $(\Delta x)^2$ can be neglected.

(ii)

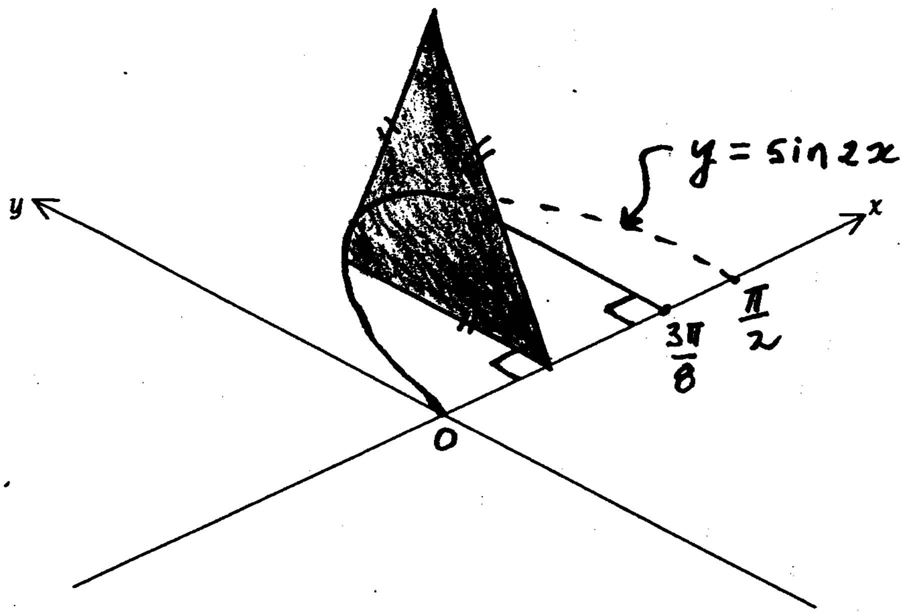


The region bounded by the curve $y = x^2 + 1$, the line $x = 2$ and the co-ordinate axes is rotated about the line $x = 3$ to form a solid of revolution.

Show that the volume V of the solid is given by $V = 2\pi \int_0^2 (3 - x)(x^2 + 1)\, dx$.

Hence find V.

(b) **7**



The base of a certain solid is the region between the x-axis and the curve $y = \sin 2x$ between $x = 0$ and $x = \dfrac{3\pi}{8}$.

Each plane section of the solid perpendicular to the x-axis is an equilateral triangle with one side in the base of the solid. An example of such a plane section is shown in the diagram above.

Show that the volume V of this solid is given by $V = \dfrac{\sqrt{3}}{4}\displaystyle\int_0^{\frac{3\pi}{8}} \sin^2 2x \, dx$.

Hence, evaluate V correct to two significant figures.

Question 5 (Start a NEW booklet) **Marks**

(a) (i) On the **same** diagram, sketch the graphs (**without** the use of calculus) of: **7**

$$y = \frac{1}{x^2+1} \quad \text{and} \quad y = \frac{x^2}{x^2+1}.$$

Mark on your diagram the co-ordinates of the intersection points of these two graphs.

(ii) On the diagram in (a)(i) above, shade the region R where

$\frac{1}{x^2+1} \geq \frac{x^2}{x^2+1}$. Find the area of R.

(b) Let the functions $g(x)$ and $f(x)$ be defined as follows: **6**

$$g(x) = \frac{1}{4}(4+x)(2-x)$$

$$f(x) = \begin{cases} g(x) & \text{for } x < 0 \\ g(-x) & \text{for } x \geq 0 \end{cases}$$

(i) By first sketching $y = g(x)$, make a neat sketch of $y = f(x)$ showing the co-ordinates of all critical points (i.e. where $f'(x) = 0$ or $f'(x)$ is undefined).

(ii) Hence, or otherwise, sketch the graph of $y = f'(x)$.

(c) By considering symmetry, draw a neat sketch of the graph $|x| - |y| = 1$. **2**

Question 6 (Start a NEW booklet) **Marks**

(a) A particle moving along the x-axis has position $x(t)$ at time t seconds. If $v(x)$ is the velocity of this particle at position x then show that **2**

$$\frac{d^2x}{dt^2} = v\frac{dv}{dx}$$

(b) A particle of mass 1 kg is moving along the x-axis and experiences a **resistive** force of magnitude kv^2 where $k > 0$ is a constant and v is the speed of the particle. **13**

(i) Find v as a function of time, t, given that the initial speed is 1 ms^{-1}.

(ii) Find v as a function of position, x, given that at $x = 0$ the speed is 1 ms^{-1}.

(iii) Suppose now that a variable **pushing** force of magnitude $\frac{c}{v}$ (where $c > 0$ is a constant) as well as the **resistive** force of magnitude kv^2 is acting on this particle.

(α) Show that the terminal velocity V is given by $V^3 = \frac{c}{k}$.

(β) Show that $\frac{dv}{dx} = c\left(\frac{1}{v^2} - \frac{v}{V^3}\right)$.

(γ) Hence, find v as a function of x, given that $v = \frac{V}{2}$ when $x = 0$.

Question 7 (Start a NEW booklet) **Marks**

(a) (i) Let the polynomial $P(x) = (x-\alpha)^3\, Q(x)$ where $Q(x)$ is also a polynomial and α is a real zero of $P(x)$. **5**

Show that $P'(\alpha) = P''(\alpha) = 0$.

(ii) Hence, or otherwise, solve the equation

$$2x^4 + 9x^3 + 6x^2 - 20x - 24 = 0$$

given that it has a triple root (i.e. a root of multiplicity 3).

(b) Let α, β, γ be the roots of the equation $x^3 - 5x + 7 = 0$. **5**

(i) Find $\dfrac{1}{\alpha} + \dfrac{1}{\beta} + \dfrac{1}{\gamma}$.

(ii) Find the equation whose roots are $2\alpha - 1$, $2\beta - 1$, $2\gamma - 1$. Hence, or otherwise, find the value of

$$(2\alpha - 1)(2\beta - 1) + (2\beta - 1)(2\gamma - 1) + (2\alpha - 1)(2\gamma - 1).$$

(c) Let $P(x)$ be a polynomial with $P(-\frac{1}{2}) = 3$. When $P(x)$ is divided by $(x-4)$ the remainder is -1. **5**

Find the polynomial $R(x)$ of minimum degree such that $P(x) = (x-4)(2x+1)Q(x) + R(x)$ where $Q(x)$ is a polynomial.

Question 8 **(Start a NEW booklet)** **Marks**

(a) Let $I_n = \int_0^1 (1-t^2)^{\frac{n-1}{2}}\,dt$ where $n = 0, 1, 2, \ldots$ **11**

(i) Show that $t^2(1-t^2)^{\frac{n-3}{2}} = (1-t^2)^{\frac{n-3}{2}} - (1-t^2)^{\frac{n-1}{2}}$

(ii) Using integration by parts, show that $nI_n = (n-1)I_{n-2}$ for $n = 2, 3, 4, \ldots$

(iii) Let $J_n = nI_nI_{n-1}$ for $n = 1, 2, 3, \ldots$

By using the principle of mathematical induction, prove that

$$J_n = \frac{\pi}{2} \text{ for } n = 1, 2, 3, \ldots$$

(iv) Briefly explain why $0 < I_n < I_{n-1}$ for $n = 1, 2, 3, \ldots$

(v) Deduce that $\sqrt{\dfrac{\pi}{2(n+1)}} < I_n < \sqrt{\dfrac{\pi}{2n}}$ for $n = 1, 2, 3, \ldots$

(b) Show that $\displaystyle\sum_{r=1}^{r=n} \frac{r\binom{n}{r}}{\binom{n}{r-1}} = \frac{n}{2}(n+1)$ **4**

END OF EXAMINATION