**-Total marks – 120**

**Attempt All Questions**

**All questions are of equal value**

**Answer each question in a SEPARATE writing booklet. Extra booklets are available.**

**Question 1** *(15 Marks)* **Marks**

(a) Find $\int x^2 \sin(x^3)\, dx$. **2**

(b) Use integration by parts to evaluate $\int_0^1 \tan^{-1} x\, dx$. **3**

(c) (i) Find the real numbers $a$ and $b$ such that $\frac{x}{(x-1)(x+4)} \equiv \frac{a}{x-1} + \frac{b}{x+4}$. **2**

(ii) Find. $\int \frac{x}{(x-1)(x+4)}\, dx$. **2**

(d) Find $\int \frac{x+4}{x^2-4x+13}\, dx$. **3**

(e) Use the substitution $t = \tan\frac{x}{2}$ to evaluate

$$\int_0^{\frac{\pi}{2}} \frac{dx}{1+\cos x}$$ **3**

**Question 2** *(15 Marks)* **Use a SEPARATE writing booklet.** **Marks**

(a) Let $w = 1 + i$ and $z = 1 - i\sqrt{3}$ **,** simplify the following

(i) $w\overline{z}$ **1**

(ii) $\dfrac{1}{w}$ **1**

(iii) $\dfrac{i(\mathrm{Re}(z) - z)}{\mathrm{Im}(z)}$. **2**

(b) Sketch the region on the Argand diagram where the inequalities $|z| \le 2$ and $\pi \ge \arg z \ge -\frac{\pi}{4}$ hold simultaneously. **3**

(c) Solve the equation $x^2 - 4x + (1 - 4i) = 0$. Answer should be expressed in the form $a + ib$ **4**

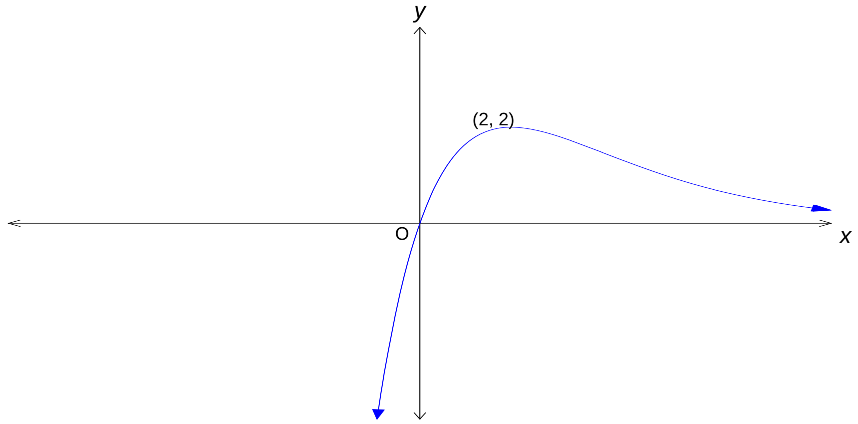
(d) The complex number $z = x + iy$, where $x$ and $y$ are real, satisfies the parametric equation $z = 1 + 2i + t(3 - 4i)$ where $t$ is a real parameter.

(i) Show that the Cartesian equation of the locus of the point $P$ which represents $z$ in an Argand diagram is given by $4x + 3y = 10$. **2**

(ii) Hence find the minimum value of $|\ z\ |$. **2**

**Question 3** *(15 Marks)* **Use a SEPARATE writing booklet.** **Marks**

(a) The curve shown in the diagram is the equation $y = f(x)$. There is a maximum turning point at (2,2) and the curve crosses the x axis at (0,0). The graph has a horizontal asymptote at $y = 0$.




Sketch the following curves on separate diagrams, showing all of the essential features.

(i) $y = f(x+2)$ **1**

(ii) $y = \dfrac{1}{f(x)}$ **2**

(iii) $y = \left(f(x)\right)^2$ **2**

(iv) $y = -x \times f(x)$. **2**

(b) (i) Show that $P(x) = x^5 - 3x^4 + 4x^3 - 4x^2 + 3x - 1$ has $x = 1$ as a root of multiplicity 3. **2**

(ii) Verify that $x = i$ is also a root of $P(x)$. **1**

(iii) Hence solve the equation $P(x) = 0$. **2**

(c) Let $\alpha,\ \beta,\ \gamma$ be the roots of the equation $x^3 - 2x^2 - 5x - 1 = 0$. Form an equation whose roots are $\dfrac{1}{\sqrt{\alpha}}, \dfrac{1}{\sqrt{\beta}}$ and $\dfrac{1}{\sqrt{\gamma}}$. **3**

**Question 4** *(15 Marks)* **Use a SEPARATE writing booklet.** **Marks**

(a) For what values of $k$ does the equation $\dfrac{x^2}{5-k}+\dfrac{y^2}{k-3}=1$ represent:

(i) a circle? **2**

(ii) a hyperbola? **2**

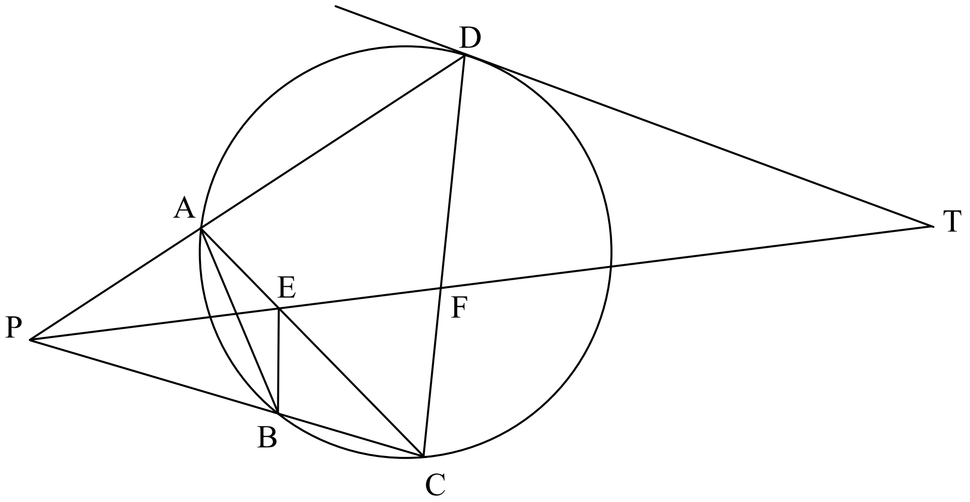
(b) The points $P\left(cp,\dfrac{c}{p}\right)$ and $Q\left(cq,\dfrac{c}{q}\right)$ are points on the rectangular hyperbola $xy = c^2$.
Tangents to the rectangular hyperbola at $P$ and $Q$ intersect at the point $R(X,Y)$.

(i) Show that the tangent to the rectangular hyperbola at any point $T\left(ct,\dfrac{c}{t}\right)$ has equation $x + t^2y - 2ct = 0$. **1**

(ii) Find the coordinates $R$. **2**

(iii) If $P$ and $Q$ are variable points on the rectangular hyperbola which move so that $p^2 + q^2 = 2$, show that the equation of the locus of $R$ is given by $xy + y^2 = 2c^2$. **3**

(c) *ABCD* is a cyclic quadrilateral. *DA* is produced and *CB* produced meet at *P*. *T* is a point on the tangent at *D* to the circle through *A*, *B*, *C* and *D*. *PT* cuts *CA* and *CD* at *E* and *F* respectively. *TF* = *TD*.




***Copy this diagram into your writing booklet.***

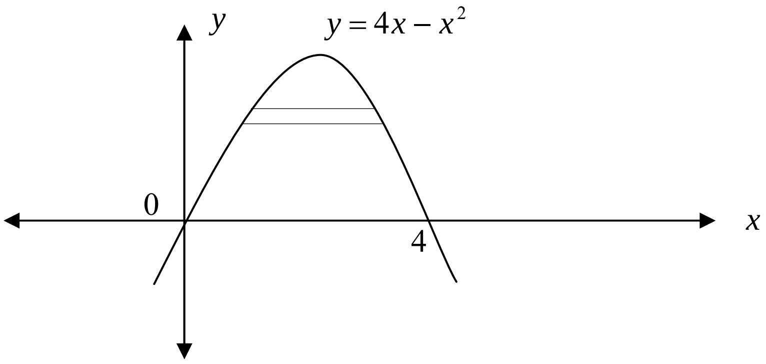
(i) Show that *AEFD* is a cyclic quadrilateral. **2**

(ii) Show that *PBEA* is a cyclic quadrilateral. **3**

**Question 5** *(15 Marks)* **Use a SEPARATE writing booklet.** **Marks**

(a) Find the general solution for the equation $\cos 3x = -\sin 2x$ **3**

(b)



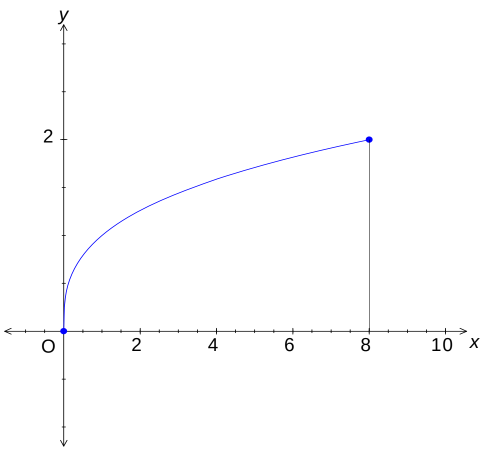

The area bounded by the curve $y = 4x - x^2$ and the *x*-axis is rotated about the y-axis.

(i) Use strips perpendicular to the axis of rotation and show the *x*-coordinates of the end-points of these strips are $2 - \sqrt{4-y}$ and $2 + \sqrt{4-y}$. **2**

(ii) Find the maximum value *y*. **1**

(iii) Hence find the volume of the solid of revolution, in terms of $\pi$. **5**

(c) The sketch below shows the region enclosed by the curve $y = x^{\frac{1}{3}}$, the *x* axis and the ordinate $x = 8$.




Find the volume generated when this region is rotated about the line $x = 8$, using the method of cylindrical shells. **4**

**Question 6** ***(15 Marks)*** **Use a SEPARATE writing booklet.** **Marks**

(a) Given that $z^n - \dfrac{1}{z^n} = 2i\sin n\theta$ and $\left(z - \dfrac{1}{z}\right)^5 = z^5 - 5z^3 + 10z - \dfrac{10}{z} + \dfrac{5}{z^3} - \dfrac{1}{z^5}$

(i) prove that: $\sin^5\theta = \dfrac{1}{16}\left[\sin 5\theta - 5\sin 3\theta + 10\sin\theta\right]$ **3**

(ii) find the general solutions of the equation $16\sin^5\theta = \sin 5\theta$. **4**

(b) A particle, of mass *m*, is projected vertically upwards in a resisted medium where the resistance is proportional to its velocity and *mk* is the constant of variation. The velocity of projection is given by $u\ ms^{-1}$.

(i) Show that after a time *t* seconds, the height above the ground is:

$$x_1 = \frac{g + ku}{k^2}\left(1 - e^{-kt}\right) - \frac{gt}{k}.$$ **5**

(ii) At the same time another particle is dropped from a height *h* metres vertically above the first particle. Given that at the time *t* seconds, its distance from the ground is:

$$x_2 = h + \frac{g}{k^2}\left(1 - e^{-kt}\right) - \frac{gt}{k},$$

show that the two particles will meet at a time *T* where

$$T = \frac{1}{k}\log\left(\frac{u}{u - kh}\right).$$ **3**

**Question 7** *(15 Marks)* **Use a SEPARATE writing booklet.** **Marks**

(a) i) Find the greatest and least values of $e^{x^2-x}$ in the domain $0 \le x \le 2$. **2**

ii) Hence show that $2e^{-\frac{1}{4}} < \int_0^2 e^{x^2-x}dx < 2e^2$ **1**

(b) (i) Using the substitution $u = a - x$, prove that $\int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx$. **2**

(ii) Hence show that $\int_0^\pi x\cos^2 x\,dx = \frac{\pi^2}{4}$. **3**

(c) Given that $\cos(n\theta) + i\sin(n\theta) = (\cos\theta + i\sin\theta)^n$ where n is a positive integer,

(i) prove that

$$\cos(n\theta) = \cos^n\theta - \binom{n}{2}\cos^{n-2}\theta\sin^2\theta + \binom{n}{4}\cos^{n-4}\theta\sin^4\theta - \dots\dots\dots \text{ and}$$

$$\sin(n\theta) = \binom{n}{1}\cos^{n-1}\theta\sin\theta - \binom{n}{3}\cos^{n-3}\theta\sin^3\theta + \dots\dots\dots\dots$$ **4**

(ii) hence deduce that

$$\tan(n\theta) = \frac{\binom{n}{1}\tan\theta - \binom{n}{3}\tan^3\theta + \dots\dots\dots}{1 - \binom{n}{2}\tan^2\theta + \binom{n}{4}\tan^4\theta - \dots\dots\dots}$$ **3**

Question 8 *(15 Marks)* Use a SEPARATE writing booklet. Marks

(a) Use the compound angle formulae for $\cos(x+y)$ and $\cos(x-y)$ to prove the result
$\cos S - \cos T = -2\sin\left(\frac{S+T}{2}\right)\sin\left(\frac{S-T}{2}\right)$. **2**

(b) For $n = 0,1,2,3,.....$,define $I_n = \int_0^{\frac{\pi}{4}} \frac{1-\cos 2nx}{\sin 2x}\,dx$.

(i) Evaluate $I_1$ **2**

(ii) Using the result proven in part (a), show that for $r \geq 1$:
$$I_{2r+1} - I_{2r-1} = \frac{1-(-1)^r}{2r}.$$ **3**

(iii) Hence evaluate $I_9$ **3**

(c) $P(a\cos\theta, b\sin\theta)$ and $Q(a\cos\phi, b\sin\phi)$ are points on the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ whose eccentricity is *e*.

(i) Find the equation of the chord *PQ*. **2**

(ii) If *PQ* is a focal chord of this ellipse show that $e = \frac{\sin(\phi-\theta)}{\sin\phi - \sin\theta}$. **3**

**END OF PAPER**

## STANDARD INTEGRALS

$$\int x^n \, dx = \frac{1}{n+1} x^{n+1}, \; n \neq -1; \; x \neq 0, \text{if } n < 0$$

$$\int \frac{1}{x} \, dx = \ln x, \quad x > 0$$

$$\int e^{ax} \, dx = \frac{1}{a} e^{ax}, \; a \neq 0$$

$$\int \cos ax \, dx = \frac{1}{a} \sin ax, \; a \neq 0$$

$$\int \sin ax \, dx = -\frac{1}{a} \cos ax, \; a \neq 0$$

$$\int \sec^2 ax \, dx = \frac{1}{a} \tan ax, \; a \neq 0$$

$$\int \sec ax \tan ax \, dx = \frac{1}{a} \sec ax, \; a \neq 0$$

$$\int \frac{1}{a^2 + x^2} \, dx = \frac{1}{a} \tan^{-1} \frac{x}{a}, \; a \neq 0$$

$$\int \frac{1}{\sqrt{a^2 - x^2}} \, dx = \sin^{-1} \frac{x}{a}, \; a > 0, \; -a < x < a$$

$$\int \frac{1}{\sqrt{x^2 - a^2}} \, dx = \ln\left(x + \sqrt{x^2 - a^2}\right), \quad x > a > 0$$

$$\int \frac{1}{\sqrt{x^2 + a^2}} \, dx = \ln\left(x + \sqrt{x^2 + a^2}\right)$$

NOTE: $\ln x = \log_e x, \quad x > 0$