New South Wales

Higher School Certificate

4 Unit Mathematics

Examinations 1975-1980

Typeset with $\mathcal{A}\mathcal{M}\mathcal{S}$-TEX

NSW HSC 4 Unit Mathematics Examination 1975

Question 1.

Sketch the graphs (showing the main features - do NOT use squared paper) of:

(i) $y = \cos^2(x/2) \quad (0 \leqslant x \leqslant 4\pi)$;

(ii) $y = e^{-x^2}$;

(iii) $y = \log|x|$;

(iv) $y = \frac{\sin x}{x}$;

(v) $y = \max(x, 1-x)$ wher $\max(a,b)$ denote the greater of the two numbers a and b, i.e.,

$$\begin{aligned} \max(a,b) &= a \text{ if } a \geqslant b \\ &= b \text{ if } a < b. \end{aligned}$$

Question 2.

(i) Define "the greatest common divisor" of two integers a, b.

(ii) Use Euclid's algorithm to compute the greatest common divisor of 1081 and 943.

(iii) Given two positive integers a, b with their greatest common divisor d expressed in the form

$$d = pa + qb$$

where p and q are integers, determine what value or values the greatest common divisor of p and q can have.

Question 3.

(i) Evaluate:

(a) $\int_0^1 te^{-2t}\,dt$; (b) $\int_{-1}^1 \frac{dx}{(x+2)(x+5)}$;

(c) $\int_0^1 x^2\sqrt{1-x^2}\,dx$.

(ii) Evaluate:

$$\lim_{h\to 0} \frac{2^h - 1}{h}$$

Question 4.

In the Cartesian plane, R denotes the region consisting of those points whose co-ordinates (x, y) simultaneously satisfy the relations

$$y \leqslant \sin x, \quad 0 \leqslant x \leqslant \pi, \quad y \geqslant 0.$$

Prove that the area of R is 2(units of area) and calculate the volume of the solid formed when R is rotated (through one complete revolution) about

(i) the x-axis; (ii) the y-axis.

Question 5.

(i) A particle moves in the x-axis with acceleration at position x given by

$$\frac{d^2x}{dt^2} = -p^2 x$$

where p is a positive constant. If at zero time the particle is released from rest at the point $x = a$ derive formulae for

(a) its velocity v at position x;

(b) its position x at time t.

(ii) The rise and fall of the tide at a certain harbour may be taken as simple harmonic, the interval between successive high tides being 12.5 hours. The harbour entrance has a depth of 15 metres at high tide and 7 metres at low tide. If low tide occurs at 11 a.m. on a certain day find the earliest time thereafter that a ship requiring a minimum depth of 13 metres of water can pass through the entrance.

Question 6.

(i) Explain, with proof and with the aid of a diagram, why

$$\begin{bmatrix} \cos\alpha & \sin\alpha \\ \sin\alpha & -\cos\alpha \end{bmatrix}$$

is called a reflection matrix.

(ii) Prove that the product of two reflection matrices represents a rotation.

(iii) Given $A = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$ compute the elements of the matrix A^{8n+4} where n is an integer.

Question 7.

(i) For the matrix $\begin{bmatrix} 5 & 3 \\ 3 & 5 \end{bmatrix}$ find the eigenvalues and corresponding eigenvectors.

(ii) Prove that, in the Cartesian plane, the equation

$$5x^2 + 6xy + 5y^2 + 4x - 4y = 0$$

represents an ellipse and determine

(a) the co-ordinate of its centre;

(b) the equations of its principal axes.

Illustrate by a sketch.

Question 8.

(i) For the sphere (in three-dimensional Cartesian space)

$$x^2 + y^2 + z^2 - 2x - 6y + 2z + 2 = 0$$

find

(a) the co-ordinates of the centre;

(b) the equation of the tangent plane at the point $(3, 4, 1)$.

(ii) S is a given sphere and β a given plane (not intersecting S). With any point P on β as centre, a sphere is drawn with radius equal to the length of the tangent from P to S. Prove that all such spheres have two points in common.

Question 9.

(i) Prove that the series $\sum_{n=1}^{\infty} \frac{1}{n^s}$ is convergent for $s > 1$.

(ii) Determine whether the series

$$\sum_{n=1}^{\infty} \frac{2n+3}{n^3+1}$$

is convergent or divergent.

(iii) It follows from part (i) that the series

$$\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \cdots + \frac{1}{n^2} + \cdots$$

is convergent and, in fact, it is known that its sum is $\pi^2/6$. It is also given that the sum of the first thousand terms of this series is 1,644 (with four-figure accuracy). Hence we obtain an approximation to the value of π from the following calculations:

$$6(1.644) = 9.864$$

$$\sqrt{9.864} = 3.141 \text{ (to four-figure accuracy)}$$

and therefore $\pi = 3.141$ approximately.

Without using any other knowledge of the value of π, investigate the accuracy of this approximation so obtained for π.

Question 10.

(i) If w is a complex number $i - 1$ indicate on the Argand diagram the points

$$w, \overline{w}, iw, \tfrac{1}{w}.$$

(ii) On the Argand diagram, P represents the complex number z and R the number $\frac{1}{z}$; and a square $PQRS$ is drawn in the plane with PR as a diagonal. If P lies on the circle $|z| = 2$ prove that Q will lie on an ellipse whose equation has the form

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

and specify numerical values for a and b.

NSW HSC 4 Unit Mathematics Examination 1976.

1. (i) Use Euclid's algorithm to find the highest common factor (i.e., the greatest common divisor) of 7313 and 6319 and express the result in the form

$$7313p + 6319q$$

where p and q are integers.

(ii) a, b, c are three given integers. If d is the highest common factor of a, b and D is the highest common factor of c, d prove that D is the highest common factor of the three integers a, b, c.

2. (i) It is given that the constants p, q are such that the equation

$$x^3 + px + q = 0$$

has three real non-zero roots α, β, γ. Evaluate (in terms of p, q)

(a) $\alpha^2 + \beta^2 + \gamma^2$;

(b) $\alpha^4 + \beta^4 + \gamma^4$;

(c) $\alpha^{-1} + \beta^{-1} + \gamma^{-1}$;

and prove that p must be strictly negative.

(ii) Use Euclid's algorithm to show that the polynomials (over the field of real numbers)

$$1 + x + x^2 \text{ and } 1 + x^2$$

are relatively prime and hence obtain a decomposition of

$$\frac{1}{(1 + x + x^2)(1 + x^2)}.$$

into the sum of two partial fractions.

3. A particle of unit mass is projected vertically upwards from the ground with initial speed u. If the air resistance at any instant is proportional to the velocity at that instant (say, resistance $= -kv$) prove that the particle reaches its highest point in time T given by

$$kT = \log(1 + ku/g)$$

where g is the acceleration due to gravity (assumed constant).

If the highest point reached is at a height h above the ground prove that

$$hk = u - gT.$$

4. (i) Using graphical considerations, or otherwise, prove that, for $x > 0$,

$$\log_e x \leqslant x - 1.$$

(ii) If $\{p_1, p_2, p_3, \ldots, p_n\}$ is a set of n positive numbers adding to unity, i.e.,

$$p_1 + p_2 + p_3 + \cdots + p_n = 1 \text{ and each } p_r > 0,$$

prove that

$$\sum_{r=1}^{n} \log(np_r) \leqslant 0.$$

(iii) Given n positive numbers $x_1, x_2, x_3, \ldots, x_n$ prove that

$$x_1 x_2 x_3 \ldots x_n \leqslant \left[\tfrac{1}{n}(x_1 + x_2 + x_3 + \cdots + x_n)\right]^n.$$

Also investigate the case of equality in this relation.

5. (i) If z is the complex number $x + iy$ (where x and y are real numbers) define

(a) $|z|$ (the modulus of z);

(b) $\overline{z}$ (the conjugate of z);

(c) $\Re(z)$ (the real part of z).

Also prove that, for any two complex numbers z, w,

$$|z + w| \leqslant |z| + |w|.$$

(ii) Given

$$w = \frac{1 + 2i}{3 + 4i}$$

determine

(a) $|w|$;

(b) $\overline{w}$;

(c) $\Re(w)$;

(d) $\Re(2w + \overline{w})$.

(iii) Describe, in geometric terms, the loci (in the Argand plane) represented by

(a) $|z - 2i| = 1$;

(b) $z\overline{z} = z + \overline{z}$.

6. Prove that the series

$$1 - \frac{x^2}{(1!)^2} + \frac{x^4}{(2!)^2} - \frac{x^6}{(3!)^2} + \cdots + \frac{(-1)^n x^{2n}}{(n!)^2} + \cdots$$

is convergent for all values of x. (Any theorem or test used should be quoted carefully.)

Denoting the sum of this series by $f(x)$ prove that $f(1) > 0$ and $f(2) < 0$.

It appears that the equation $f(x) = 0$ has a root lying between 1 and 2; describe how you would make a guess at the value of this root.

7. (i) Describe briefly the content and significance of the Fundamental Theorem of the Calculus.

(ii) Evaluate

(a) $\int_0^1 xe^{-x}\,dx$;

(b) $\int_1^2 x\log_e x\,dx$;

(c) $\int_0^2 \frac{1+x}{4+x^2}\,dx$;

(d) $\int_0^{\frac{1}{4}} \frac{dx}{\sqrt{1-4x^2}}$.

8. In the Cartesian plane, indicate, by shading, the region R whose points simultaneously satisfy the relations

$$0 \leqslant x \leqslant 2, \quad 0 \leqslant y \leqslant 4x^2 - x^4;$$

and determine the maximum value of y for points in R.

The region R is now rotated about the y-axis to form a solid of revolution and a plane perpendicular to the y-axis this solid i a section of area A. Prove that

$$A = 2\pi\sqrt{4-y}$$

where y is the distance of the plane from the origin.

Use this result to calculate the volume of the solid.

9. Show that, in a Cartesian (x, y) frame, a rotation $\mathcal{T}$ through an angle α about the origin (thereby transforming $\mathbf{r}$ into $\mathbf{r}'$) can be represented by the matrix equation $\mathbf{r}' = \mathbf{Tr}$ where

$$\mathbf{r} = \begin{bmatrix} x \\ y \end{bmatrix} \text{ and } \mathbf{T} = \begin{bmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{bmatrix}.$$

If $\mathcal{R}$ denotes a reflection in the line $y = x\tan\beta$ (thereby transforming $\mathbf{r}$ into $\mathbf{r}'$) find the matrix $\mathbf{R}$ such that $\mathbf{r}' = \mathbf{Rr}$.

Also prove that the rotation $\mathcal{T}$ above can be factorised into a product of two reflections in an infinite number of ways.

10. In the Cartesian plane prove that

$$x^2 + xy + y^2 = 6$$

represents an ellipse and find

(i) the length of its major axis (i.e., the longer axis);

(ii) the equation of the line in which the major axis lies;

(iii) the equations of its two tangents which are parallel to the x-axis.

Illustrate by a sketch.

NSW HSC 4 Unit Mathematics Examination 1977

1. (i) If a, b, c are given integers prove that the equation

$$ax + by = c$$

has solutions in integers if and only if c is divisible by the highest common factor of a and b.

(ii) Show that the equation

$$357x + 323y = 408$$

has integer solutions for x, y and find one such solution.

2. (i) Given that the three roots of the equation

$$8x^2 - 36x^2 + 38x - 3 = 0$$

are in arithmetic progression, find them.

(ii) Decompose

$$\frac{1}{(x+1)(x^2+4)}$$

into partial fractions and use this result to evaluate

$$\int_0^2 \frac{1}{(x+2)(x^2+4)}\ dx.$$

3. Prove that a particle projected with velocity V at an angle α with the horizontal describes a parabola, and find an expression for the focal length in terms of V, α, g where g is the acceleration due to gravity (assumed constant).

Several particles are projected from the same point simultaneously with the same initial speed U in various directions. Prove that at any subsequent time t they all lie on a sphere of radius Ut.

4. The function f is defined over the domain of real numbers by the relations

$$f(x) = x \text{ when } 0 \leqslant x \leqslant 1$$

and

$$f(x) = 0 \text{ for all other values of } x.$$

If k is a given constant evaluate

(i) $\int_0^k f(x)\ dx$;

(ii) $\int_{-1}^1 f(x)f(x-k)\ dx$.

5. (i) Given

$$w = \frac{2 - 3i}{1 + i}$$

determine

(a) $|w|$ (i.e., the modulus of w);

(b) $\overline{w}$ (i.e., the conjugate of w);

(c) $w + \overline{w}$.

(ii) Describe, in geometric terms, the locus (in the Argand plane) represented by $2|z| = z + \overline{z} + 4$.

6. (i) Explain the terms "convergent" and "absolutely convergent" as applied to an infinite series of real numbers.

(ii) Prove, quoting carefully any theorems or tests used, that:

(a) for $-1 < r < 1$ any α arbitrary, the series

$$1 + r\cos\alpha + r^2\cos 2\alpha + \cdots + r^n\cos n\alpha + \cdots$$

is convergent;

(b) the series

$$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots + \frac{(-1)^{n+1}}{n} + \cdots$$

is convergent, but not absolutely convergent. Also show that its sum lies between 5/6 and 7/12.

7. (i) Define $\sin^{-1} x$ and show that its derivative is

$$\frac{1}{\sqrt{1 - x^2}}.$$

(ii) Evaluate (giving numerical answers to three significant figures):

(a) $\int_0^1 \frac{dx}{\sqrt{9-4x^2}}$;

(b) $\int_0^\pi \sin^2(x/4)\ dx$;

(c) $\int_0^1 x^2 e^{-x}\ dx$.

8. Sketch (in the Cartesian x, y plane) the curve

$$3cy^2 = x(x - c)^2$$

where c is a positive constant.

Denoting that part of this curve which lies in the positive quadrant between $x = 0$ and $x = c$ by β prove that:

(i) the area enclosed between β and the x-axis is

$$\frac{4\sqrt{3}c^2}{45};$$

(ii) the length of β is $2c/\sqrt{3}$.

9. The matrix $\mathbf{A}$ is given by

$$\mathbf{A} = \begin{bmatrix} 5 & -4 \\ -4 & 5 \end{bmatrix}.$$

(i) Write down the characteristic equation of $\mathbf{A}$.

(ii) Prove that $\mathbf{A}^2 - 10\mathbf{A} + 9\mathbf{I} = \mathbf{0}$.

(iii) Find $\mathbf{A}^{-1}$.

(iv) Find the eigenvalues of $\mathbf{A}$.

(v) For the eigenvalues of $\mathbf{A}$ find the corresponding eigenvectors.

(vi) Deduce (without resorting to calculations similar to those done in the preceeding parts) the eigenvalues and eigenvectors of $\mathbf{A}^2$.

10. (i) Explain, with proof and the aid of a diagram, why

$$\begin{bmatrix} \cos\alpha & \sin\alpha \\ \sin\alpha & -\cos\alpha \end{bmatrix}$$

is called a reflection matrix.

(ii) The Cartesian equation of a given conic x is

$$ax^2 + 2hxy + by^2 = 1;$$

show that this equation can be written in the matrix form

$$\mathbf{r}^{\mathrm{T}}\mathbf{A}\mathbf{r} = \mathbf{1} \text{ where } \mathbf{r} = \begin{bmatrix} x \\ y \end{bmatrix}$$

and specify the matrix $\mathbf{A}$ explicitly in terms of a, h, b.

(iii) The reflection of the above conic in the line $y = mx$ is another conic s'; prove that the equation of s' can be expressed as

$$\mathbf{r}^{\mathrm{T}}\mathbf{S}\mathbf{A}\mathbf{S}\mathbf{r} = \mathbf{1}$$

and find the matrix $\mathbf{S}$ explicitly in terms of m.

NSW HSC 4 Unit Mathematics Examination 1978

1. (i) Use Euclid's algorithm to find the highest common factor of 1247 and 2537 and show that it can be expressed in the form

$$1247p + 2537q$$

in an infinite number of ways (with p, q as integers).

(ii) If a, b, c are given integers prove that the equation

$$ax + by = c$$

has integers solutions for (x, y) if and only if d divides c where d is the highest common factor of a, b.

If (x_0, y_0) is one such solution prove that any other solution must be expressible in the form

$$x = x_0 + \frac{nb}{d}, \ y = y_0 - \frac{na}{d}$$

where n is an integer.

2. (i) If α, β, γ are the three roots of

$$x^3 + px + q = 0$$

(p, q being constants) find the cubic equation whose roots are $\alpha^2, \beta^2, \gamma^2$.

(ii) When a polynomial $P(x)$ is divided by $x-2$ and by $x-3$ the respective remainders are 4 and 9; determine what the remainder must be when it is divided by $(x-2)(x-3)$.

3. (i) For a particle of mass m moving in a straight line under the action of a variable force prove that, over a given interval, the work done by the force is equal to the increase in kinetic energy.

(ii) A mass of 20 kilograms hangs from the end of a rope and is hauled up vertically from rest by winding up the rope. The pulling force in the rope starts at 250 newtons and decreases uniformly by 10 newtons for every metre wound up. Find the velocity of the mass after 10 metres have been wound up. (Take the acceleration due to gravity as 9.8 m/s^2 and neglect the weight of the rope.)

4. (i) A, B, C are three points on horizontal ground forming a triangle with a right-angle at C and the lengths of the sides BC, CA are a, b respectively. Vertical borings are made at A, B, C and coal is struck at points P, Q, R at respective depths p, q, r. If θ is the inclination of the plane PQR to the horizontal prove that

$$\tan^2\theta = \frac{(q-r)^2}{a^2} + \frac{(r-p)^2}{b^2}.$$

(ii) The two spheres

$$x^2 + y^2 + z^2 + 2x - 2y = 2$$
$$x^2 + y^2 + z^2 = 1$$

cut in a circle β. Find the equation of the sphere that contains this circle β and also passes through the origin.

5. Define the modulus $|z|$ of a complex number z.

Given two complex numbers z_1, z_2 prove that:

(i) $|z_1 z_2| = |z_1||z_2|$;

(ii) the triangle represented (on the Argand diagram) by the complex numbers $0, 1, z_1$ is similar to the triangle represented by $0, z_2, z_1 z_2$;

(iii) $|z_1 - z_2|^2 + |z_1 + z_2|^2 = 2|z_1|^2 + 2|z_2|^2$ and interpret this as a theorem on a triangle giving the length of a median in terms of the lengths of the sides.

6. (i) Prove that the series

$$1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} + \cdots$$

is divergent by first proving that the sum of the first 2^n terms exceeds $n/2$.

Also prove that the sum of the first thousand term is less than 10.

(ii) Investigate for convergence the series whose n^{th} term is

(a) $\frac{x^n}{2n+1}$ (where x is positive);

(b) $\sqrt{n^2 + 1} - n$.

7. Evaluate (giving numerical answers to three significant figures):

(i) $\int_0^1 \frac{dx}{(x+1)(x+3)}$;

(ii) $\int_0^1 \sqrt{4 - x^2}\, dx$;

(iii) $\int_0^1 2^x\, dx$;

(iv) $\int_0^{\frac{1}{2}} \sin^{-1} x\, dx$.

8. In the cartesian plane sketch the curve

$$y = \frac{e^x - e^{-x}}{e^x + e^{-x}}$$

and prove that the lines $y = \pm 1$ are asymptotes.

Also, if k is a positive constant, find the area in the positive quadrant enclosed by the above curve and the three lines

$$y = 1, \quad x = 0, \quad x = k$$

and prove that this area is always less than $\log_e 2$, however large k may be.

9. In the cartesian plane the point P, whose co-ordinates are (x, y), is transformed to the point P', whose co-ordinates are (x', y'), by the affine transformation $\mathbf{r}' = \mathbf{A}\mathbf{r}$ where $\mathbf{A}$ is a matrix and

$$\mathbf{r} = \begin{bmatrix} x \\ y \end{bmatrix}, \quad \mathbf{r}' = \begin{bmatrix} x' \\ y' \end{bmatrix}.$$

Determine the matrix $\mathbf{A}$ in the following cases:

(i) P' is obtained from P by reflection in the line

$$y = x \tan\theta;$$

(ii) P' is the mid-point of the segment OP (O being the origin);

(iii) P' is the foot of the perpendicular from P on the line

$$ax + by = 0 \ \ \text{(where } a, b \text{ are constants).}$$

10. (i) In the cartesian plane the (x, y) co-ordinate frame is rotated about the origin anti-clockwise through an angle θ to give a new frame with co-ordinates denoted by (x', y'). Derive equations expressing the transformation between the two sets of co-ordinates (x, y) and (x', y').

(ii) Prove that (in cartesian co-ordinates)

$$2x^2 + 4xy - y^2 = 6$$

represents an hyperbola. Illustrate its shape and position by a clear diagram showing, in particular, its asymptotes and principal axes. Also, write down the equations of:

(a) the asymptotes;

(b) the principal axes.

NSW HSC 4 Unit Mathematics Examination 1979

Question 1.

Let
$$f(x) = x^3 + px + q = (x-\alpha)(x-\beta)(x-\gamma),$$
where p, q are real and α, β, γ are (not necessarily distinct) real numbers.

(i) If $f(\alpha) = f'(\alpha) = 0$, prove that $(x-\alpha)^2$ is a divisor of $f(x)$.

(ii) If $\alpha = \beta, \alpha \neq \gamma$, so that $f(x) = (x-\alpha)^2(x-\gamma)$, prove that

(a) $p \neq 0$;

(b) $4p^3 + 27q^2 = 0$;

(c) $\alpha = \frac{-3q}{2p}$.

(iii) Solve completely the equation $x^3 - 27x - 54 = 0$.

Question 2.

Given that the highest common factor d of two integers a, b is expressible as
$$d = al + bm$$
where l, m are integers, prove that:

(i) if c, d are integers, p is a prime, and p divides cd, then p divides at least one of c and d;

(ii) the equation
$$ax + by = c,$$
where a, b, c are given integers, has integer solutions x, y if and only if the highest common factor of a and b is a divisor of c;

(iii) if $\frac{a}{b}$ is a rational number in reduced form, then there exist infinitely many distinct rational numbers $\frac{r}{s}$ such that
$$|\frac{a}{b} - \frac{r}{s}| \leq \frac{1}{s}.$$

Question 3.

Prove that the range on a horizontal plane of a particle projected upwards at an angle α to the plane with velocity V metres per second is $\frac{V^2 \sin 2\alpha}{g}$ metres, wher g metres per second per second is the acceleration due to gravity.

A garden sprinkler sprays water symmetrically about its vertical axis at a constant speed of V metres per second. The initial direction of the spray varies continuously between angles of 15° and 60° to the horizontal.

Prove that, from a fixed position O on level ground, the sprinkler will wet the surface of an annular region with centre O and with internal and external radii $\frac{V^2}{2g}$ metres and $\frac{V^2}{g}$ metres respectively.

Deduce that by locating the sprinkler appropriately relative to a rectangular garden bed of size 6 metres by 3 metres, the entire bed may be watered provided that

$$\frac{V^2}{2g} \geq 1 + \sqrt{7}.$$

Question 4.

(i) Write down the binomial expansion of $(1+x)^n$ (n a positive integer).

As the result of an experiment, a curve of the form $y = f(x)$ is drawn. It is suspected that $f(x)$ is expressible in the form $f(x) = (1+ax)^n$, where a is real and n is a positive integer.

For values of x so small that third (and higher) powers of x may be ignored, the parabola

$$y = 1 - 6x + 16x^2$$

is practically identical with the given curve. Assuming the curves coincide for these values of x, what are the values of a and n?

(ii) It is known that a set of n (≥ 3) points (x, y), each with $x > 0$, satisfies either a relation of the form

$$y = Ax^\alpha \ (A,\ \alpha \text{ real constants})$$

or a relation of the form

$$y = Bc^x \ (B,\ c \text{ real constants},\ c > 0).$$

Assuming ordinary (squared) graph paper and logarithm tables were available to you, explain how you would determine which form of the relation is the correct one.

Question 5.

(i) Define modulus and conjugate of a complex number $z = x + iy$ (x, y real). Prove that $|z|^2 = z\overline{z}$, and that, for any two complex numbers z_1, z_2,

$$(\overline{z_1 z_2}) = (\overline{z_1})(\overline{z_2}).$$

Deduce that

$$|z_1 z_2| = |z_1||z_2|.$$

(ii) Draw neat, labelled sketches (not on graph paper) to indicate each of the subsets of the Argand diagram described below.

(a) $\{z : |z| \geq 1 \text{ and } 0 \leq \arg z \leq \frac{\pi}{3}\}$.

(b) $\{z : z + \overline{z} > 0\}$.

(c) $\{z : |z - 1| < |z + 1|\}$.

(d) $\{z : |z^2 - \overline{(z^2)}| < 4\}$.

<u>Question 6.</u>

(i) (a) Prove that

$$1 + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \cdots + \frac{1}{\sqrt{n}} \geq \sqrt{n}$$

for every positive integer n. Deduce that $\sum \frac{1}{\sqrt{n}}$ diverges.

(b) Write down a true statement connecting the statements "$u_n \to 0$ as $n \to \infty$" and "$\sum u_n$ converges".

(c) State (without proof) a result on the convergence or divergence of

$$\sum_{n=1}^{\infty} \frac{1}{n^p} \ (p > 0).$$

(ii) Determine whether the series $\sum\limits_{n=2}^{\infty} \frac{n^3 - n^2}{n^4 - 1}$ converges or diverges.

(iii) If $\sum \left(u_n - \frac{1}{n^2}\right)$ converges, prove that $\sum u_n$ converges.

<u>Question 7.</u>

(i) Show that $\frac{d}{dx} \log(x + \sqrt{1 + x^2}\ ) = \frac{1}{\sqrt{1+x^2}}$. Hence, or otherwise, evaluate $\int_{-\pi/2}^{\pi/2} \frac{\cos x}{\sqrt{1+\sin^2 x}}\, dx$.

(ii) (a) Find

$$\int_0^1 \frac{dx}{e^x + 1}.$$

(b) Assuming $e = 2.7183$, evaluate $\int_1^e (\log x)^2 \, dx$ to three decimal places.

(iii) A curve is defined by the equation

$$y = \frac{a}{2}\left[e^{x/a} + e^{-x/a}\right] \ (a > 0).$$

Show that $1 + (y')^2 = y^2/a^2$. Show also that the arc length s between points $(0, a)$ and (x, y) of the curve is given by

$$s = \sqrt{y^2 - a^2}.$$

Question 8.

The mean value theorem of the differential calculus states that if the function f is differentiable on the interval $a \leq x \leq b$, then

$$f(b) - f(a) = (b - a)f'(\xi)$$

for a value ξ satisfying $a << b$.

Indicate by means of a clear sketch the geometrical significance of this result.

Using either this theorem or the mean value theorem of the integral calculus, prove that if $0 < x < y$, then

$$\frac{y-x}{y} < \log y - \log x < \frac{y-x}{x}.$$

Deduce that, if $0 < a < c < b$, then

$$\log a < \log c + \frac{a-c}{c},$$
$$\log b < \log c + \frac{b-c}{c}.$$

Prove that if $0 < \lambda < 1$, and if $a < b$, then $a < \lambda a + (1-\lambda)b < b$. Deduce that if $0 < a < b$, and if $0 < \lambda < 1$, then

$$a^{\lambda} b^{1-\lambda} < \lambda a + (1-\lambda)b.$$

Question 9.

An equilateral triangle has vertices $A(0, 2), B(-\sqrt{3}, -1)$ and $C(\sqrt{3}, -1)$, so that its centroid is at the origin. Describe in geometrical terms the six linear transformations of the plane (including the identity transformation) which are symmetry operations for this triangle (i.e., which transform the triangle onto itself).

Write down the 2×2 matrix corresponding to each of these symmetry operations.

Evaluate any four matrix products $\mathbf{AB}$, where $\mathbf{A} \neq \mathbf{1}, \mathbf{B} \neq \mathbf{1}, \mathbf{A} \neq \mathbf{B}$, and $\mathbf{A}, \mathbf{B}$ are chosen from matrices in your list just written down. Verify that the products obtained by you occur in your list, and explain why this is so.

Question 10.

(i) Determine the real values of k for which the matrix

$$\mathbf{A} = \begin{bmatrix} 1 & 2 \\ 2 & 2k-1 \end{bmatrix}$$

has:

(a) non-zero eigenvalues of the same sign;

(b) non-zero eigenvalues of opposite sign.

(ii) Describe the type of graph represented by the equation $x^2+4xy+(2k-1)y^2=1$ for each of the cases $k=3, k=\frac{5}{2}, k=1$.

(iii) For the case $k=1$, find the principal axes and the foci of the corresponding curve.

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Question 1.

(i) Decompose $\frac{4}{1-x^4}$ into partial fractions and hence show that

$$\int_0^{1/2} \frac{4}{1-x^4}\, dx = \log 3 + 2\tan^{-1}\tfrac{1}{2}.$$

(ii) α, β, γ are non-zero and the roots of the cubic equation

$$x^3 + px + q = 0.$$

Find the equation whose roots are $\frac{\alpha}{\beta\gamma}, \frac{\beta}{\gamma\alpha}, \frac{\gamma}{\alpha\beta}$.

Question 2.
(i) The function $f(x)$, defined for all real x, is called even if $f(x) = f(-x)$, for all x, and odd if $f(x) = -f(-x)$, for all x.

(a) Determine which of these functions is even, odd, or neither:

$$x;\ \frac{\cos x}{1+x^2};\ xe^x;\ xe^{-x^2}.$$

(b) Show that if $f(x)$ is even and is differentiable for all x, then $f'(x)$ is odd.

(c) If $f(x)$ is odd and is continuous for all x, and $b > a > 0$, use a sketch to explain why $\int_{-a}^{b} f(x)\, dx = \int_a^b f(x)\, dx$.

(ii) The polynomial $P(x)$ is given by $P(x) = x^5 - 5cx + 1$, where c is a real number.

(a) By considering the turning points, prove that if $c < 0, P(x)$ has just one real root, which is negative.

(b) Prove that $P(x)$ has three distinct real roots if and only if

$$c > \left(\frac{1}{4}\right)^{4/5}.$$

Question 3.

Assume that for any integers a, b and c, if $a|c$, $\ b|c$ and if a and b are relatively prime, then $(ab)|c$.

Given that n is a positive integer:

(i) show that $n(n-1)(n+1)$ is always divisible by 6;

(ii) show that $n(n-1)(n+1)$ is divisible by 30 unless n is of the form $n = 5m \pm 2$ for some integer m;

(iii) deduce that $n^5 - n$ is divisible by 30;

(iv) hence, or otherwise, show that n^5 always has the same units digit as n.

Question 4.

From an origin on a horizontal plane, a particle is projected upwards with speed V and angle of elevation α. Show that the equation of the particle's path is

$$y = x \tan \alpha - \frac{gx^2 \sec^2 \alpha}{2V^2}.$$

where x, y are horizontal, vertical co-ordinates in the plane of the path of the particle and g is the acceleration due to gravity. Hence, or otherwise, deduce that the range on the horizontal plane is $\frac{V^2 \sin 2\alpha}{g}$.

A vertical wall is a distance d from the origin, and the plane of the wall is perpendicular to the plane of the particle's trajectory.

(i) Show that if $d < V^2/g$, the particle will strike the wall provided that

$$\beta < \alpha < \frac{\pi}{2} - \beta,$$

where $\beta = \frac{1}{2} \sin^{-1}(gd/V^2)$.

(ii) Show also that the maximum height the particle can reach on the wall is

$$\frac{V^4 - g^2 d^2}{2gV^2}.$$

Question 5.

(i) The complex number w is given by $w = -1 + \sqrt{3}i$.

(a) Show that $w^2 = 2\overline{w}$.

(b) Evaluate $|w|$ and $\arg w$.

(c) Show that w is a root of the equation $w^3 - 8 = 0$.

(ii) Sketch the region of the Argand diagram whose points z satisfy the inequalities $|z - \overline{z}| \leq 4$ and $-\frac{\pi}{3} \leq \arg z \leq \frac{\pi}{3}$.

(iii) The complex number z is a function of the real number t given by the rule

$$z = \frac{t - i}{t + i}, \quad 0 \leq t \leq 1.$$

Evaluate $|z|$ and hence describe the locus of z in the Argand diagram as t varies from 0 to 1.

Question 6.

(i) (a) Write down the sum of n terms of the geometric series

$$1 + x + x^2 + x^3 + \cdots$$

and hence show that this series is convergent if and only if $|x| < 1$.

(b) State the comparison test for convergence of a series. Illustrate your answer by constructing a series and proving that it converges or diverges by comparing it with the geometric series.

(ii) (a) Find A, B such that

$$\frac{1}{x(x+1)} \equiv \frac{A}{x} + \frac{B}{x+1}.$$

(b) The sum S_n of the first n terms of a series is

$$S_n = \frac{1}{1.2} + \frac{1}{2.3} + \frac{1}{3.4} + \cdots + \frac{1}{n(n+1)}.$$

Show that

$$S_n = 1 - \frac{1}{n+1}.$$

Hence prove that the series converges, and find its sum.

Deduce that the series

$$1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \cdots$$

converges, and that its sum is less than 2.

Question 7.

(i) Evaluate

(a) $\int_0^{\pi/2} x \cos x \; dx$

(b) $\int_0^1 \frac{x^2}{x+1} \; dx$

(c) $\int_0^{\pi/2} \frac{1-\tan x}{1+\tan x} \; dx$

(ii) Use the substitution $x = a - y$ to show that

$$\int_0^a f(x) \; dx = \int_0^a f(a-x) \; dx.$$

Hence, or otherwise, evaluate

(a) $\int_0^1 x(1-x)^{20} \; dx$

(b) $\int_0^{\pi/2} \cos^2 x \; dx$.

Question 8.

(i) Evaluate $\int_1^x \frac{du}{\sqrt{u}}$ for $x > 1$. Assuming that $f(x) \leq g(x)$ for all x in $a \leq x \leq b$, explain briefly why

$$\int_a^b f(x) \; dx \leq \int_a^b g(x) \; dx,$$

whenever both integrals exist. Hence, given that for $x > 0$

$$\log x = \int_1^x \frac{du}{u},$$

prove that $\frac{\log x}{x} \to$ as $x \to \infty$. Deduce that $x \log x \to 0$ as $x \to 0$.

(ii) Draw a careful sketch of the curve defined by

$$y = \begin{cases} x^2(\log x)^2 & \text{if } x > 0, \\ 0 & \text{if } x = 0, \end{cases}$$

paying attention to the slope of the curve and the turning points.

Question 9.

For each of the matrices $\mathbf{M}$ given below describe exactly the effect of the linear transformation of the Cartesian (x, y) plane which maps the point with position vector $\mathbf{r} = \begin{bmatrix} x \\ y \end{bmatrix}$ to the point with position vector $\mathbf{r}' = \begin{bmatrix} x' \\ y' \end{bmatrix}$, where

$$\mathbf{r}' = \mathbf{M}\mathbf{r}.$$

In particular, state whether the transformation is singular or non-singular, and in the non-singular cases determine the inverse matrix $\mathbf{M}^{-1}$.

(i) $\mathbf{M} = \mathbf{M}_1 = \begin{bmatrix} -2 & 0 \\ 0 & -2 \end{bmatrix} + 2 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

(ii) $\mathbf{M} = \mathbf{M}_2 = \begin{bmatrix} \frac{-\sqrt{3}}{2} & \frac{-3}{2} \\ \frac{3}{2} & \frac{-\sqrt{3}}{2} \end{bmatrix}$

(iii) $\mathbf{M} = \mathbf{M}_3 = \begin{bmatrix} \frac{-3}{\sqrt{2}} & \frac{3}{\sqrt{2}} \\ \frac{3}{\sqrt{2}} & \frac{3}{\sqrt{2}} \end{bmatrix}$

(iv) $\mathbf{M} = \mathbf{M}_4 = \begin{bmatrix} 4 & 2 \\ 6 & 3 \end{bmatrix}$

(v) $\mathbf{M} = \mathbf{M}_5 = \mathbf{M}_2\mathbf{M}_3$.

Question 10.

Determine the exact ranges of values (if any) of the non-negative real number r for which the equation

$$x^2 + 2(r-1)xy + (2-r^2)y^2 = 1$$

represents respectively a circle, an ellipse, an hyperbola, and a rectangular hyperbola with respect to Cartesian (x, y) co-ordinates.

Draw a careful sketch of the curve corresponding to the value $r = 2$, showing all important features. Identify those values (if any) of r for whichy the equation represents any other type of locus and sketch the locus for each such value.