New South Wales

Higher School Certificate

4 Unit Mathematics

Examinations 1981-1989

Typeset with $\mathcal{A}\mathcal{M}\mathcal{S}$-TEX

NSW HSC 4 Unit Mathematics Examination 1981

1. **(i)** Evaluate: **(a)** $\int_0^\pi x \sin x \; dx$; **(b)** $\int_2^4 \frac{dx}{x^2-4x+8}$; **(c)** $\int_{-1}^1 \frac{4+x^2}{4-x^2} \; dx$.

(ii) Let n be a positive integer, and let $I_n = \int_1^2 (\log_e x)^n \; dx$. Prove that: $I_n = 2(\log_e 2)^n - nI_{n-1}$. Hence evaluate $\int_1^2 (\log_e x)^4 \; dx$ as a polynomial in $\log_e 2$.

2. **(i)** Sketch the curve $y = 2 + \frac{1}{x^2-1}$, $(x \neq \pm 1)$, showing the location and nature of all satationary points and the equations of all asymptotes.

(ii) **(a)** Factorise $1 + x + x^2 + x^3$.

(b) Prove that the equation $\frac{x^4}{4} + \frac{x^3}{3} + \frac{x^2}{2} + x + c = 0$ has no real root if $c > \frac{7}{12}$. How many real roots are there if $c \leq \frac{7}{12}$?

3. The ellipse E has cartesian (x, y) equation $\frac{x^2}{4} + \frac{y^2}{3} = 1$. Write down its eccentricity, the co-ordinates of its foci S and S', and the equation of each directrix. Sketch the curve and indicate on your diagram the foci and directrices. P is an arbitrary point on E.

(i) Prove that the sum of the distances $SP, S'P$ is independent of P.

(ii) Derive the equation of the normal line to E at P, and prove that it bisects the angle SPS'.

4. For the complex number $z = x + iy$:

(i) Find the curve in the Argand diagram for which $\Re(z^2) = 3$, and sketch the curve.

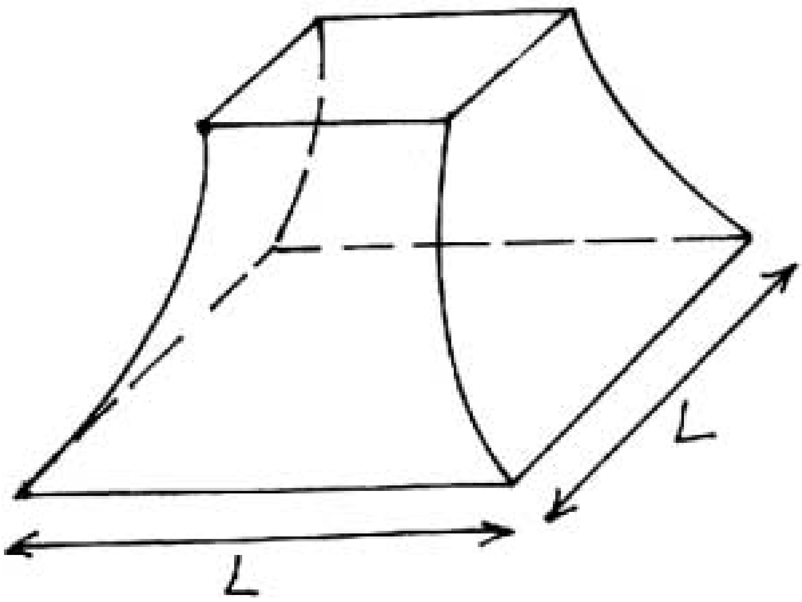
(ii) Find the curve such that $\Im(z^2) = 4$.

(iii) Solve completely the equation $z^2 = 3 + 4i$.

(iv) The region R of the Argand diagram consists of the set of all values of z such that both $0 < \Re(z^2) < 3$ and $0 < \Im(z^2) < 4$. Draw a sketch of the region R, showing the coordinates of the vertices.

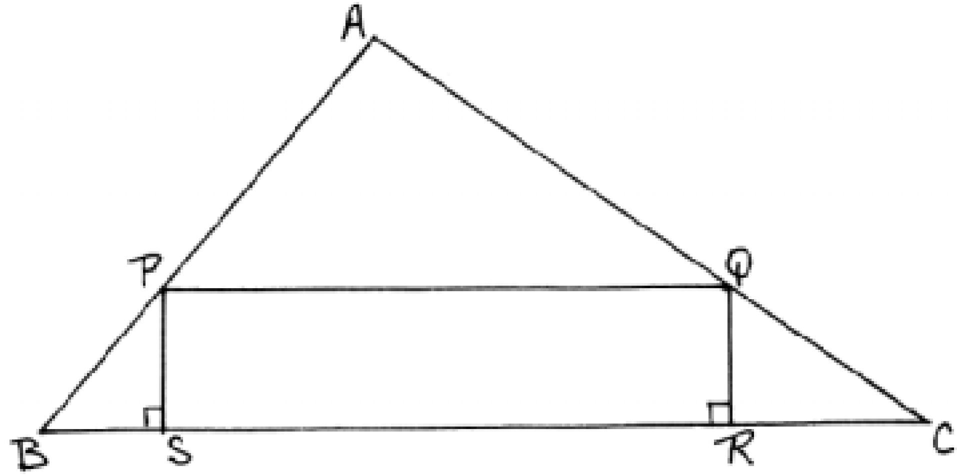
5. **(i)** By reference to an appropriate diagram, or otherwise, show that for $a > b > 0$, $\int_0^b \sqrt{a^2 - x^2} \; dx = \frac{1}{2} b\sqrt{a^2 - b^2} + \frac{1}{2} a^2 \sin^{-1} \frac{b}{a}$.

(ii)



A stone building of height H metres has the shape of a flat-topped square 'pyramid' with curved sides as shown in the figure. The cross-section at height h metres is a square with sides parallel to the sides of the base and of length $l(h) = \frac{L}{\sqrt{h+1}}$, where L is the side length of the square base in metres. Find the volume of the building given that $H = L = 30$. Give your answer correct to the nearest cubic metre.

(ii)



In the figure, PQ is parallel to BC, and $PQRS$ is a rectangle. Prove that the maximum area of $PQRS$ is half the area of the triangle ABC.

6. A point P is moving in a circular path around a centre O. Define the angular velocity of P with respect to O at time t.

Derive expressions for the tangential and normal components of the acceleration of P at time t.

A particle of mass m, attached by a light rod to a pivot point C, moves with constant speed in a horizontal circle whose centre O is distant h metres below C. Show that the time T seconds taken for one revolution of the particle is given by $T = 2\pi\sqrt{h/g}$ where g metres per second per second is the acceleration due to gravity.

Discuss the effect, if any, on the motion of the particle if

(i) Its mass is reduced; **(ii)** Its speed is doubled.

7. **(i)** Find all x such that $\sin x = \cos 5x$ and $0 < x < \pi$.

(ii) Nine persons gather to play football by forming two teams of four to play each other, the remaining person acting as referee. In how many different ways can the teams be formed?

If two particular persons are not to be in the same team, how many ways are there then to choose the teams?

(iii) Prove that, when x, y, z are real and not all equal,

$$x^2 + y^2 + z^2 > yz + zx + xy,$$

and deduce that, when also $x + y + z = 1$, then $yz + zx + xy < \frac{1}{3}$.

8. **(i)** **(a)** Given that ω is a complex root of the equation $x^3 = 1$, show that ω^2 is also a root of this equation.

(b) Show that $1 + \omega + \omega^2 = 0$, and $1 + \omega^2 + \omega^4 = 0$.

(c) Let α, β be real numbers. Find, in its simplest form, the cubic equation whose roots are $\alpha + \beta, \alpha\omega + \beta\omega^{-1}, \alpha\omega^2 + \beta\omega^{-2}$.

(ii) Using induction, show that for each positive integer n there are unique positive integers p_n and q_n such that $(1 + \sqrt{2})^n = p_n + q_n\sqrt{2}$.

Show also that $p_n^2 - 2q_n^2 = (-1)^n$.

NSW HSC 4 Unit Mathematics Examination 1982

1. **(i)** Prove that the curve: $y = \frac{4x}{1+x^2}$ has a minimum at $A(-1, -2)$, a maximum at $B(1, 2)$ and a point of inflexion at $O(0, 0)$. Sketch the curve.

(ii) The cubic curve $y = ax^3 + bx^2 + cx + d$ also has a minimum at A and a maximum at B. Obtain the values of the coefficients a, b, c, d, and deduce that O is also a point of inflexion of this curve.

(iii) Prove that the two curves have only the three points A, B, O, in common.

2. **(i)** Find the indefinite integrals: **(a)** $\int x(1 + x^2)^6 \, dx$;

(b) $\int (x - \frac{1}{x-2})^2 \, dx$, for $x > 2$.

(ii) Evaluate: $\int_0^{\pi/4} x \sin 2x \, dx$.

(iii) Let $I_n = \int_0^{\pi/4} \tan^n x \, dx$, where n is an integer, and $n \geq 3$.

Show that $I_n + I_{n-2} = \frac{1}{n-1}$. Hence evaluate I_7.

3. **(i)** For the complex number $z = 1 - \sqrt{3}i$, find $|z|, \arg z$. Also express each of $\overline{z}, z^2, 1/z$, and $\sqrt{z}$ in the form $a + ib$, where a, b are real numbers. Plot $z, \overline{z}, z^2, 1/z$ and both square roots of z on an Argand diagram, labelling them respectively with the letters A, B, C, D, E, F.

(ii) Sketch the region R in the Argand diagram consisting of those points z for which:

$|\arg z| < \frac{\pi}{3}, z + \overline{z} < 4$, and $|z| > 2$.

(iii) Solve the equation: $z^2 + 4z - 1 + 12i = 0$.

4. With respect to axes Ox, Oy, the line $x = 1$ is a directrix, and the point $(2, 0)$ a focus of a conic of eccentricity $\sqrt{2}$.

(a) Find the equation of the conic, show that it is a rectangular hyperbola, and sketch the curve, indicating its asymptotes, foci and directrices.

(b) Find the equation of the normal to the curve at any point P on it.

(c) The normal to the curve at P meets Ox, Oy in $(X, 0), (0, Y)$, respectively, and T is the point (X, Y). Show that as P varies on the curve, T always lies on the hyperbola $x^2 - y^2 = 8$.

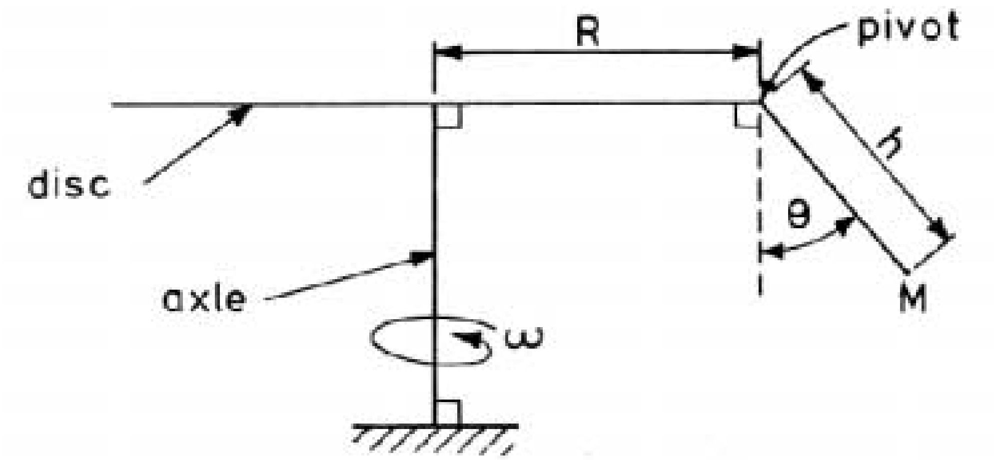
5. **(i)** Write down the volume of a right circular cylindrical shell of height h with inner and outer radii r, R, respectively. Hence show that for $\delta r = R - r$, this volume is $2\pi rh \, \delta r$, when δr is sufficiently small for $(\delta r)^2$ to be neglected.

The region bounded by the curves $y = \frac{1}{x+1}$, and $y = \frac{1}{x+2}$ between the ordinates at $x = 0$ amd $x = 2$ is rotated about the y axis, forming a solid of volume V units.

Show that: $V = 2\pi \int_0^2 \frac{x}{(x+1)(x+2)}\, dx$, and find V correct to three sig. fig.

(ii) A point P lies on the on the curve whose equation is $x^4 + y^4 = 1$. Prove that the distance of P from the origin is at most $2^{1/4}$.

6. A "chair-o-plane" at a fairground consists of seats hung from pivots attached the the rim of a horizontal circular disc, which is rotated by a motor driving its vertical axle. As the speed of rotation increases, the seats swing out. Represent a seat by a point mass M kg suspended by a weightless rod h metres below a pivot placed R metres from the axis of rotation. Assume that when the disc rotates about its axle with constant angular velocity ω radians per second, there is an equilibrium position in which the rod makes an angle θ with the vertical, as shown in the figure.



(a) Show that ω, θ satisfy the relation $(R + h\sin\theta)\omega^2 = g\tan\theta$.

(b) Use a graphical means to show that, for given ω there is just one value of θ in the range $0 \le \theta \le \frac{\pi}{2}$ which satisfies this relation.

(c) Given $R = 6, h = 2, \theta = 60°$, and assuming that $g = 10\,\mathrm{m.s}^{-2}$, find the speed of M relative to the ground.

7. (i) Find all x such that: $\cos x + \sin x = 1 + \sin 2x$, and $0 \le x \le 2\pi$.

(ii) Use De Moivre's theorem to express $\cos 5\theta, \sin 5\theta$ in powers of $\sin\theta$ and $\cos\theta$. Hence express $\tan 5\theta$ as a rational function of t, where $t = \tan\theta$.

Deduce that: $\tan\frac{\pi}{5}\tan\frac{2\pi}{5}\tan\frac{3\pi}{5}\tan\frac{4\pi}{5} = 5$.

8. (i) Given that $a_n = \sqrt{2 + a_{n-1}}$ for integers $n \ge 1$, and that $a_0 = 1$, prove that, for $n \ge 1, \sqrt{2} < a_n < 2$.

(ii) Eight players enter a knock-out singles tennis tournament in which each of the four first round winners plays one second round game to decide who enters the final.

Assuming that all players are equally likely to win a game, show that the probability that two particular entrants play each other in the tournament is $1/4$.

Also show that if sixteen persons enter the tournament, then the probability that the two players meet is $\frac{1}{8}$.

Prove that for a similar knock-out tournament for 2^n players, the probability that two players meet is 2^{1-n}.

NSW HSC 4 Unit Mathematics Examination 1983

1. **(i)** Draw a careful sketch of the curve $y = \frac{x^2}{x^2-1}$, indicating clearly any vertical or horizontal asymptotes, turning points or inflexions.

(ii) A function $f(x)$ is known to approach 0 as x appoaches ∞ and $-\infty$. Its derivative is given by $f'(x) = e^{-x^2}(x-1)^2(2-x)$.

From this information, describe the behaviour of $f(x)$ as x increases from $-\infty$ to $+\infty$. Include in your description an indication of those x where $f(x)$ is respectively increasing or decreasing and any points where f has a maximum or minimum value. Also explain why $f(x)$ must be positive for all real x.

Draw a sketch of a function $f(x)$ satisfying the given conditions.

2. **(i)** Find: **(a)** $\int_0^{\sqrt{3}} \frac{x+12}{x^2+9}\,dx$ **(b)** $\int_1^2 60x^3(1+x^2)^4\,dx$

(c) $\int_0^T xe^{-(x/2)}\,dx$, where T is a positive number.

(ii) Find the partial fraction decomposition of $\frac{16x}{x^4-16}$. Hence show that $\int_4^6 \frac{16x}{x^4-16}\,dx = \log_e(4/3)$, and also evaluate $\int_0^1 \frac{16x}{x^4-16}\,dx$.

3. Use mathematical induction to prove that for any real $\theta, \cos 6\theta + i\sin 6\theta = (\cos\theta + i\sin\theta)^6$.

Find the six sixth roots of -1, expressing each in the form $x+iy$ with x, y real.

Find also the four roots of the equation $z^4 - z^2 + 1 = 0$, and indicate their positions on an Argand diagram.

4. **(i)** Determine the (real) values of λ for which the equation $\frac{x^2}{4-\lambda} + \frac{y^2}{2-\lambda} = 1$ defines respectively an ellipse and an hyperbola.

Sketch the curve corresponding to the value $\lambda = 1$.

Describe how the shape of this curve changes as λ increases from 1 towards 2. What is the limiting position of the curve as 2 is approached?

(ii) P is a point on the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ with centre O. A line drawn from O, parallel to the tangent to the ellipse at P, meets the ellipse at Q.

Prrove that the area of the triangle OPQ is independent of the position of P.

5. **(i)** An egg-timer has the shape of an hour-glass and can be described mathematically as being obtained by rotating the curve $y = x + 6x^3$, $-1/\sqrt{2} \le x \le 1/\sqrt{2}$, about the y-axis.

Use the method of decomposition into cylindrical shells to calculate its volume, correct to three significant figures.

(ii) A plane curve is defined implicitly by the equation $x^2 + 2xy + y^5 = 4$.

This curve has a horizontal tangent at the point $P(X, Y)$. Show that X is the unique real root of the equation $X^5 + X^2 + 4 = 0$, and that $-2 < X < -1$.

6. An object of irregular shape and of mass 100 kilograms is found to experience a resistive force, in newtons, of magnitude one-tenth the square of its velocity in metres per second when it moves through the air.

If the object falls from rest under gravity, assumed constant of value 9.8 m.s^{-2}, calculate

(i) its terminal velocity;

(ii) the minimum height, to the nearest metre, of the release point above ground, if it attains a speed of 80% of its terminal velocity before striking the ground.

7. **(i)** A city council consists of 6 Liberal and 5 Labor aldermen, from whom a committee of 5 members is chosen at random. What is the probability that the Liberals have a majority on the committee?

(ii) Let α, β, γ be the roots of the equation $x^3 + qx + r = 0$, where $r \neq 0$. Obtain as functions of q, r in their simplest forms, the coefficients of the cubic equations whose roots are:

(a) $\alpha^2, \beta^2, \gamma^2$; **(b)** $\alpha^{-1}, \beta^{-1}, \gamma^{-1}$; **(c)** $\alpha^{-2}, \beta^{-2}, \gamma^{-2}$.

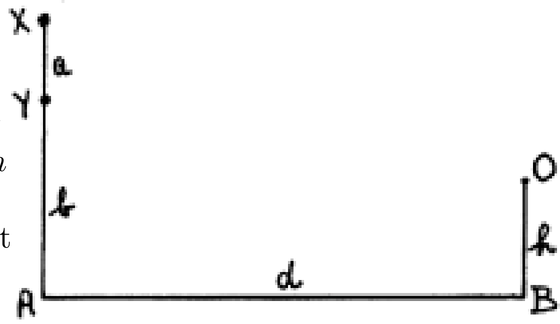
(iii) Given that $x + y = s$, prove that, for $x > 0, y > 0, s > 0$, $\frac{1}{x} + \frac{1}{y} \geq \frac{4}{s}$, and that $\frac{1}{x^2} + \frac{1}{y^2} \geq \frac{8}{s^2}$.

8. **(i)** Given $\sin x \sin y = ½(\cos A - \cos B)$, find A, B in terms of x, y. Hence prove that for any positive integer n,

$$\sin x + \sin 3x + \sin 5x + \cdots + \sin(2n-1)x = \sin^2 nx / \sin x.$$

(ii) In a triangle ABC, the point X on BC is such that AX bisects $\angle BAC$. Use the sine rule to prove that $AB/AC = BX/XC$.

In the figure, XY represents a vertical flagstaff of length a placed on top of a vertical tower AY of height b. An observer is at a point O, which is a vertical height h above B, and a horizontal distance d from A. Given that $\angle XOY = \angle YOA$, show that $(a-b)d^2 = (a+b)b^2 - 2b^2h - (a-b)h^2$.



NSW HSC 4 Unit Mathematics Examination 1984

1. **(i)** Show that **(a)** $4\int_e^{e^4} x\log_e x\ dx = 7e^8 - e^2$, **(b)** $\int_e^{e^4} \frac{dx}{x\log_e x} = 2\log_e 2$.

(ii) Evaluate **(a)** $\int_0^{\sqrt{3}} \frac{x+1}{x^2+1}\ dx$, **(b)** $\int_{-4}^{4} \frac{x+6}{\sqrt{x+5}}\ dx$, **(c)** $\int_{-\pi/6}^{\pi/6} \sin 4x \cos 2x\ dx$.

2. **(i)** Sketch the graphs of **(a)** $(x+3)(y-2) = 1$, **(b)** $x^2 + y^2 + 1 = 2(x+y)$.

(ii) Consider the ellipse E whose equation is $\frac{x^2}{4} + \frac{y^2}{2} = 1$.

(a) Show that the equation of E may be given in the parametric form $x = 2\cos\theta, y = \sqrt{2}\sin\theta$.

(b) Assuming that the perimeter p of E is given by the formula $p = 2\int_0^{\pi} \sqrt{\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2}\ d\theta$, show that $p = 2\sqrt{2}\int_0^{\pi} \sqrt{2 - \cos^2\theta}\ d\theta$.

(c) Use seven evenly spaced ordinates from $\theta = 0$ to $\theta = \pi$ and Simpson's rule to estimate p. Round off your answer to two decimal places.

3. **(i)** Calculate the modulus and argument of the product of the roots of the equation $(5+3i)z^2 - (1-4i)z + (8-2i) = 0$.

(ii) Let $A = 1+i, B = 2-i$. Draw sketches to show the loci specified on the Argand diagram by

(a) $\arg(z - A) = \pi/4$, **(b)** $|z - A| = |z - B|$.

(iii) Show that the point representing $\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}$ on the Argand diagram lies on the circle of radius one with centre at the point which represents 1.

(iv) R is a positive real number and z_1, z_2 are complex numbers. Show that the points on the Argand diagram which represent respectively the numbers z_1, z_2, $\frac{z_1 - iRz_2}{1-iR}$, form the vertices of a right-angled triangle.

4. **(i)** Show that, if $(r\cos\theta, r\sin\theta), \left(s\cos\left(\theta+\frac{\pi}{2}\right), s\sin\left(\theta+\frac{\pi}{2}\right)\right)$ lie on the hyperbola $x^2/a^2 - y^2/b^2 = 1$ with centre O, then $\frac{1}{r^2} + \frac{1}{s^2} = \frac{1}{a^2} - \frac{1}{b^2}$.

Deduce that, if P, Q are points on the hyperbola such that OP is perpendicular to OQ, then

$\frac{1}{OP^2} + \frac{1}{OQ^2}$ is independent of the position of P and Q.

Give a sketch of a hyperbola and points O, P, Q related in this way.

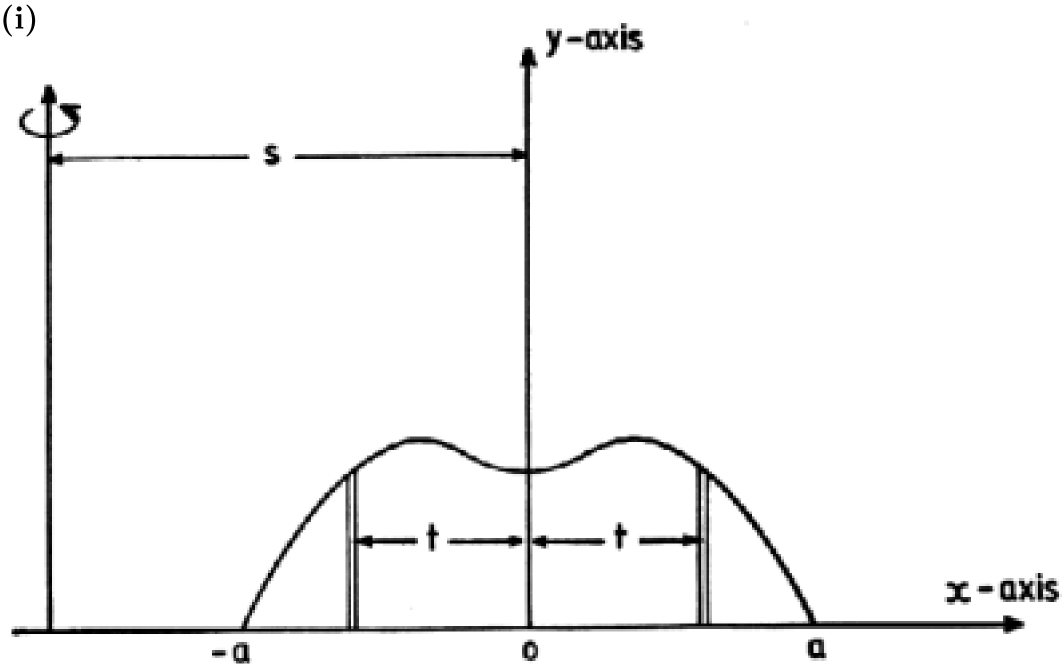
(ii) The line $L : ax + by = 1$ meets the rectangular hyperbola $H : xy = c^2$ in two distinct points, $X_1(x_1, y_1), X_2(x_2, y_2)$. $X_0(x_0, y_0)$ is the mid-point of X_1X_2.

(a) Find a quadratic equation whose roots are x_1, x_2.

(b) Show that the equation of L may be written $\frac{x}{x_0} + \frac{y}{y_0} = 2$.

(c) Deduce that every line L' (other than an asymptote) through the centre of a rectangular hyperbola bisects all chords parallel to a certain direction and describe how this direction is related to the direction of L'.

5. (i)



The diagram shows the area A between the smooth curve $y = f(x), -a \leq x \leq a$, and the x-axis. (Note that $f(x) \geq 0$ for $-a \leq x \leq a$ and $f(-a) = f(a) = 0$). The area A is rotated about the line $x = -s$ (where $s > a$) to generate the volume V. This volume is to be found by slicing A into thin vertical strips, rotating these to obtain cylindrical shells, and adding the shells. Two typical strips of width δt whose centre lines are distance t from the y-axis are shown.

(a) Show that the indicated strips generate shells of approximate volume $2\pi f(-t)(s-t)\delta t, 2\pi f(t)(s+t)\delta t$, respectively.

(b) Assuming that the graph of f is symmetrical about the y-axis, show that $V = 2\pi s A$.

(ii) Assuming the results of part **(i)**, solve the following problems.

(a) A doughnut shape is formed by rotating a circular disc of radius r about an axis in its own plane at a distance of s $(s > r)$ from the centre of the disc. Find the volume of the doughnut.

(b) The shape of a certain party jelly can be represented by rotating the area between the curve $y = \sin x, 0 \leq x \leq \pi$, and the x-axis about the line $x = -\pi/4$. Find the volume generated.

6. Two stones are thrown simultaneously from the same point in the same direction and with the same non-zero angle of projection (upward inclination to the horizontal), α, but with different velocities U, V metres per second $U < V$.

The slower stone hits the ground at a point P on the same level as the point of projection. At that instant the faster stone just clears a wall of height h metres above the level of projection and its (downward) path makes an angle β with the horizontal.

(a) Show that, while both stones are in flight, the line joining them has an inclination to the horizontal which is independent of time. Hence, express the horizontal distance from P to the foot of the wall in terms of h, α.

(b) Show that $V(\tan\alpha + \tan\beta) = 2U\tan\alpha$, and deduce that, if $\beta = {}^1\!/_2\,\alpha$, then $U < \frac{3}{4}V$.

7. **(i)** In how many ways can the five letters of the word CONIC be arranged in a line so that the two (indistinguishable) C's are separated by at least one other letter?

(ii) It is given that x, y, z are positive numbers. Prove that

(a) $x^2 + y^2 \geq 2xy$,

(b) $x^2 + y^2 + z^2 - xy - yz - zx \geq 0$.

Multiply both sides of the inequality **(b)** by $(x + y + z)$ to obtain

(c) $x^3 + y^3 + z + 3 \geq 3xzy$.

Deduce from **(c)**, or prove otherwise, that

(d) $(x + y + z)(x^{-1} + y^{-1} + z^{-1}) \geq 9$.

Suppose that x, y, z satisfy the additional constraint that $x + y + z$ is equal to 1.

Is it true that the minimum value of the expression $x^{-1} + y^{-1} + z^{-1}$ is equal to 9? Justify your answer.

8. **(i)** Write down expressions for $\sin(\alpha+\beta), \cos(\alpha+\beta)$ in terms of $\sin\alpha, \cos\alpha, \sin\beta$, $\cos\beta$.

Deduce that $\tan(\alpha + \beta) = \frac{\tan\alpha+\tan\beta}{1-\tan\alpha\tan\beta}$,

and $\tan(\alpha + \beta + \gamma) = \frac{\tan\alpha+\tan\beta+\tan\gamma-\tan\alpha\tan\beta\tan\gamma}{1-\tan\alpha\tan\beta-\tan\gamma\tan\alpha-\tan\beta\tan\gamma}$.

By means of the substitution $t = \tan\theta$, transform the equation
$\sin 4\theta + a\sin 2\theta + b\cos 2\theta + b = 0$
into a cubic equation in t. (a, b are real constants, $a \neq 2$). Suppose the roots of the transformed equation are $\tan\alpha, \tan\beta, \tan\gamma$. Show that $\alpha + \beta + \gamma$ is a multiple of π.

(ii) A woman travelling along a srraight flat road passes three points at intervals of 200 m. From these points she observes the angle of elevation of the top of the hill to the left of the road to be respectively $30°, 45°$, and again $45°$. Find the height of the hill.

NSW HSC 4 Unit Mathematics Examination 1985

1. (i) Find: **(a)** $\int_0^{3\pi} x\cos x\ dx$; **(b)** $\int_0^1 \frac{1}{x^2+4x+5}\ dx$.

(ii) Find real numbers A, B, C such that

$$\frac{x}{(x-1)^2(x-2)} \equiv \frac{A}{x-1} + \frac{B}{(x-1)^2} + \frac{C}{x-2}.$$

Hence show that $\int_0^{1/2} \frac{x}{(x-1)^2(x-2)}\ dx = 2\log_e\left(\frac{3}{2}\right) - 1$.

(iii) Use the substitution $x = a - t$, where a is a constant, to prove that

$$\int_0^a f(x)\ dx = \int_0^a f(a-t)\ dt.$$

Hence, or otherwise, show that $\int_0^1 x(1-x)^{99}\ dx = \frac{1}{10100}$.

2. (i) (a) Find the turning points of the cubic polynomial $p(x) = x^3 - x^2 - 5x - 1$, and without attempting to solve the equation, show that the equation $p(x) = 0$ has three distinct real roots, two of which are negative.

(b) Sketch the graph of $p(x)$.

(c) Starting with the approximation $x = 0$, use one application of Newton's method to estimate a root of the equation $p(x) = 0$.

(d) What initial approximation would you use to estimate the positive root of $p(x) = 0$ by Nowton's method? State briefly your reasons for this choice.

(ii) (a) Sketch the function $g(x) = xe^{-x}$, for $x \geq -1$.

(b) Given $g(x)$ as in **(a)** above, the function $f(x)$ is given by the rule

$$f(x) = \begin{cases} g(x-2), & x \geq 1, \\ g(-x), & x \leq 1. \end{cases}$$

Find the zeros of this function, and the maximum and minimum values.

Draw a sketch of the graph of $y = f(x)$.

3. (i) Reduce the complex expression $\frac{(2-i)(8+3i)}{(3+i)}$ to the form $a + ib$, where a, b are real numbers.

(ii) The complex number z is given by $z = -\sqrt{3}_i$.

(a) Write down the values of $\arg z$ and $|z|$.

(b) Hence, or otherwise, show that $z^7 + 64z = 0$.

(iii) On the Argand diagram, let $A = 3 + 4i, B = 9 + 4i$.

(a) Draw a clear sketch to show the important features of the curve defined by

$$|z - A| = 5.$$

Also, for z on this curve, find the maximum value of $|z|$.

(b) On a separate diagram, draw a clear sketch to show the important feartures of the curve defined by

$$|z - A| + |z - B| = 12.$$

For z on this curve, find the greatest value of $\arg z$.

4. **(i)** The normal at the point P on the parabola $4ay = x^2$ intersects the y axis at Q. The directrix intersects the y axis at Y, and S is the focus of the parabola. The mid-point of QS is R. Show that $YR^2 - RP^2 = 4a^2$.

(ii) **(a)** Show that for all values of θ the point $P(3\cos\theta, 2\sin\theta)$ lies on an ellipse, and find its equation.

(b) Find the equation of the tangent to this ellipse at P.

(c) Show that the point $Q(-3\sin\theta, 2\cos\theta)$ also lies on the same ellipse.

(d) The tangents at P, Q to the ellipse intersect at T. Find the coordinates of T, and verify that T lies on the curve $4x^2 + 9y^2 = 72$.

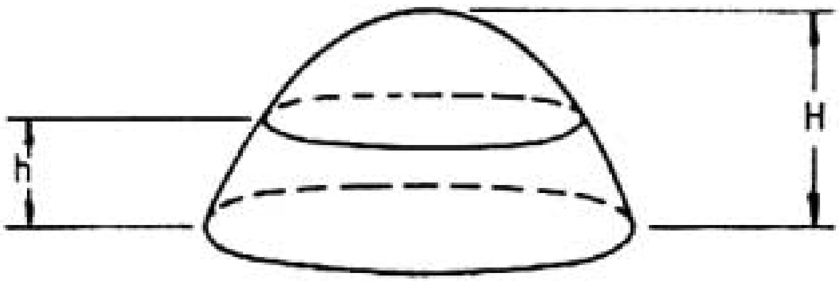
5. **(i)** A thin wire of length L is cut into two pieces, out of which a circle and a closed square are to be formed, so that the sum of the areas of the circle and square so formed is a minimum. Show that this minimum value is

$$\frac{L^2}{4(\pi + 4)}$$

(ii) **(a)** Using the substitution $x = a\sin\theta$, or otherwise, verify that $\int_0^a (a^2 - x^2)^{\frac{1}{2}} dx = \frac{1}{4}\pi a^2$.

(b) Deduce that the area enclosed by the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ is πab.

(c)



The diagram shows a mould of height H. At height h above the horizontal base, the horizontal cross-section of the mound is elliptical in shape, with equation $\frac{x^2}{a^2} + \frac{y^2}{b^2} =$

λ^2, where $\lambda = 1 - \frac{h^2}{H^2}$, and x, y are appropriate coordinates in the plane of the cross-section. Show that the volume of the mound is $\frac{8\pi abH}{15}$.

6. A particle of mass 10 kg is found to experience a resistive force, in newtons, of one-tenth of the square of its velocity in metres per second, when it moves through the air.

The particle is projected vertically upwards from a point O with a velocity of u metres per second, and the point A, vertically above O, is the highest point reached by the particle before it starts to fall to the ground again. Assuming the value of g is $10\,\mathrm{m.s}^{-2}$,

(a) find the time the particle takes to reach A from O;

(b) show that the height OA is $50\log_e[1 + 10^{-3}u^2]$ metres;

(c) show that the particle's velocity $w\,\mathrm{m.s}^{-1}$ when it reaches O again is given by $w^2 = u^2(1 + 10^{-3}u^2)^{-1}$.

7. **(i)** Given that $\sin x + \sin y = 2\sin A\cos B$, find values for A and B in terms of x and y.

Solve the equation $\sin\theta + \sin 2\theta + \sin 3\theta + \sin 4\theta = 0$, giving all solutions in the interval $0 \le \theta \le 2\pi$.

(ii) **(a)** A, B, C are three points lying on a given circle, and P is another point in the same plane. Write down two different angle tests to determine whether A, B, C, P are concyclic (i.e., P also lies on the given circle).

(b) In an acute-angled triangle with vertices L, M, N, the foot of the perpendicular from L to MN is P, and the foot of the perpendicular from N to LM is Q. The lines LP, QN intersect at H.

(α) Draw a clear diagram showing the given information.

(β) Prove that $\angle PHM = \angle PQM$.

(γ) Prove that $\angle PHM = \angle LNM$.

(δ) Produce MH to meet LN at R. Prove that $MR \perp LN$.

(c) What general result about triangles is proved in **(b)**?

8. **(i)** **(a)** In how many ways can 4 persons be grouped into two pairs to play a set of doubles tennis?

(b) The eight members of a tennis club meet to play two simultaneous sets of doubles tennis on two separate but otherwise identical courts. In how many different ways can the members of the club be selected for these two sets of tennis?

(ii) **(a)** Show that for $k \ge 0, 2k + 3 > 2\sqrt{(k+1)(k+2)}$.

(b) Hence prove that for $n \geq 1$,

$$1 + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \cdots + \frac{1}{\sqrt{n}} > 2\big[\sqrt{n+1} - 1\big].$$

(c) Is the statement that, for all positive integers N,

$$\sum_{k=1}^{N} \frac{1}{\sqrt{k}} < 10^{10} \text{ true?}$$

Give reaons for your answer.

NSW HSC 4 Unit Mathematics Examination 1986

1. **(i)** Evaluate $\int_0^1 \frac{x}{\sqrt{2-x}}\, dx$.

(ii) Use integration by parts to show that $\int_0^1 \tan^{-1} x\, dx = \frac{1}{4}\pi - \frac{1}{2}\log_e 2$.

(iii) Find numbers A, B, C such that $\frac{x^2}{4x^2-9} \equiv A + \frac{B}{2x-3} + \frac{C}{2x+3}$. Hence evaluate $\int_0^1 \frac{x^2}{4x^2-9}\, dx$.

(iv) Using the substitution $t = \tan(1/2\,\theta)$, or otherwise, show that

$$\int_0^{\pi/3} \frac{1}{1+\sin\theta}\, d\theta = \sqrt{3} - 1.$$

2. The functions $S(x), C(x)$ are defined by the formulae $S(x) = \frac{1}{2}(e^x - e^{-x})$, and $C(x) = \frac{1}{2}(e^x + e^{-x})$.

(i) **(a)** Verify that $S'(x) = C(x)$,

(b) Show that $S(x)$ is an increasing function for all real x.

(c) Prove that $\{C(x)\}^2 = 1 + \{S(x)\}^2$.

(ii) **(a)** $S(x)$ has an inverse function, S^{-1}, for all values of x. Briefly justify this statement.

(b) Let $y = S^{-1}(x)$. Prove that $\frac{dy}{dx} = \frac{1}{\sqrt{1+x^2}}$.

(c) Hence, or otherwise, show that $S^{-1}(x) = \log_e \left[x + \sqrt{1+x^2}\,\right]$.

(d) Show that $\int_0^1 \frac{dx}{\sqrt{x^2+2x+2}} = \log_e \left\{\frac{2+\sqrt{5}}{1+\sqrt{2}}\right\}$.

3. **(i)** The point $P(x_1, y_1)$ lies on the hyperbola of equation $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$.

(a) Find the equation of the normal at P.

(b) The normal at P meets the x-axis at G, and N is the foot of the perpendicular from P to the x-axis. Show that $NG : ON = b^2 : a^2$, where O is the origin $(0, 0)$.

(ii) **(a)** Show that if a is a multiple root of the polynomial equation $f(x) = 0$ then $f(a) = f'(a) = 0$.

(b) The polynomial $\alpha x^{n+1} + \beta x^n + 1$ is divisible by $(x-1)^2$. Show that $\alpha = n$, and $\beta = -(1+n)$.

(c) prove that $1 + x + \frac{x^2}{2!} + \cdots + \frac{x^n}{n!}$ has no multiple roots for any $n \geq 1$.

4. **(i)** Given that $z_1 = 3 - i, z_2 = 2 + 5i$, express in the form $a + ib$, where a, b are real,

(a) $(\overline{z_1})$ **(b)** z_1/z_2 **(c)** $|(\overline{z_1}/z_2)|$

(ii) Given that for the complex number $z, |z| = 2, \arg z = 2\pi/5$, write in the form $a + ib$, where a, b are real:

(a) z **(b)** z^7.

(iii) (a) Draw a sketch of the region of the Argand diagram consisting of the set of all values of z for which $1 \le |z| \le 4$ and $\pi/4 \le \arg z \le 3\pi/4$.

(b) **(α)** The curve in the Argand diagram for which $|z - 2| + |z - 4| = 10$ is an ellipse. Write down the coordinates of the centre, and the lengths of the major and minor axes of this ellipse.

(β) On a separate Argand diagram, show the region for which z satisfies the inequalities $z + \overline{z} \le 6$ or $|z - 2| + |z - 4| \le 10$.

5. **(i) (a)** The coordinates of the vertices of a triangle ABC are $(0, 2), (1, 1), (-1, 1)$ respectively and H is the point $(0, H)$. The line through H, parallel to the x-axis meets AB and AC at X and Y. Find the length of XY.

(b) The region enclosed by the triangle ABC, defined in **(a)** above, is rotated about the x-axis. Find the volume of the solid of revolution so formed.

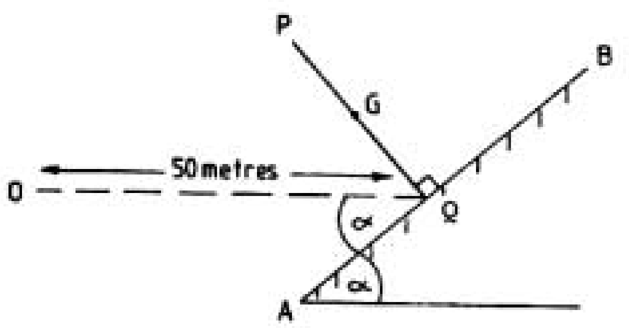
(ii) A particle travels in a straight line away from a fixed wall. At time t secs its acceleration $a\,\text{m.s}^{-2}$ is given by $a = t \sin t$.

Determine the velocity $v\,\text{m.s}^{-1}$, and the distance from the wall x metres, as functions of t, given that $v = V$ and $x = 0$, at $t = 0$.

6. **(a)** A cyclist is travelling with constant speed v metres per second around a circular track of radius r metres. Draw a diagram of the forces on the cyclist, and show that the cyclist must lean inwards towards the centre of the circle at an angle θ, where $\tan \theta = v^2/(rg)$ and θ is the angle to the vertical of the line from the mass centre to the point of wheel contact.

(b) The figure represents the front view of a cyclist PQ, with mass centre G, riding on a circular competition track AB which is banked at an angle α to the horizontal.

The track is so designed that when the cyclist is travelling at a constant speed of 40 km per hour around a horizontal circle of centre O and radius $OQ = 50$ metres, then PQ and AB are perpendicular. Assuming the value of g to be $10\,\text{m.s}^{-1}$, show that $\alpha = 14°$, to the nearest degree.

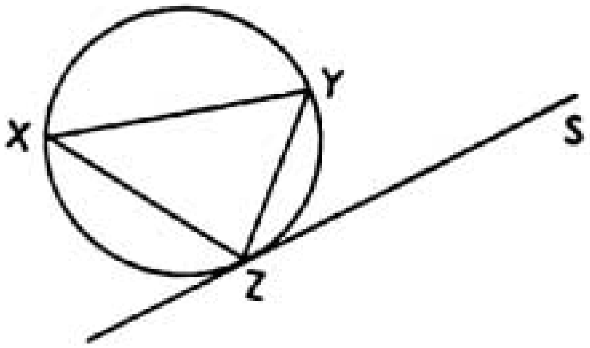


(c) Also calculate the force of the wheels on the track at Q in the direction AB, given that a cyclist of total mass 80 kg is travelling around the same circle at 50 km per hour.

7. (i) Find all x such that $\cos 2x = \sin 3x$, and $0 \leq x \leq \pi/2$.

(ii) Appropriate diagrams should accompany each of your solutions to this section.

(a) In the figure, ZS is the tangent to the circle at Z, and X, Y are any two points on the circle. By drawing the diameter through Z, or otherwise, prove that $\angle YZS = \angle ZXY$.



(b) In the given figure, XY produced meets ZS at P. The lengths PX, PY and PZ are x, y, z respectively. Prove that $z^2 = xy$.

(c) Two unequal circles intersect at L, M. The common tangent AB touches the circles at A, B. Prove that $\angle LM$ produced bisects AB.

8. (i) Let α, β, γ be the roots of the cubic equation $x^3 + px^2 + q = 0$, where p, q are real. The equation $x^3 + ax^2 + bx + c = 0$ has roots $\alpha^2, \beta^2, \gamma^2$. Find a, b, c as functions of p, q.

(ii) A committee of 4 women and 3 men are to be seated at random around a circular table with 7 seats. What is the probability that all the women will be seated together?

(iii) The function $f(x)$ is given, for $x > 0$, by $f(x) = 2\log_e x - \frac{x^2-1}{x}$.

(a) Show that the only zero of $f(x)$ occurs at $x = 1$.

(b) Let $g(x) = \frac{x \log_e x}{x^2-1}$, for $x > 0$ and $x \neq 1$. Show that $0 < g(x) < \frac{1}{2}$.

NSW HSC 4 Unit Mathematics Examination 1987

1. **(i)** Prove that: **(a)** $\int_1^2 \frac{t+1}{t^2}\,dt = \frac{1}{2} + \log_e 2$; **(b)** $\int_4^6 \frac{4\,dx}{(x-1)(x-3)} = 2\log_e\left(\frac{9}{5}\right)$.

(ii) **(a)** Use the substitution $x = \frac{2}{3}\sin\theta$ to prove that $\int_0^{2/3} \sqrt{4-9x^2}\,dx = \pi/3$.

(b) Hence, or otherwise, find the area enclosed by the ellipse $9x^2 + y^2 = 4$.

(iii) **(a)** Given that $I_n = \int_0^{\pi/2} \cos^n x\,dx$, prove that $I_n = \left(\frac{n-1}{n}\right)I_{n-2}$, where n is an integer and $n \geq 2$.

(b) Hence evaluate $\int_0^{\pi/2} \cos^5 x\,dx$.

2. The hyperbola h has equation $9x^2 - 16y^2 = 144$.

(a) Write down its eccentricity, the coordinates of its foci S and S', the equation of each directrix and the equation of each asymptote. Sketch the curve and indicate on your diagram the foci, directrices and asymptotes.

(b) $P(x_1, y_1)$ is an arbitrary point on h.

(α) Prove that the equation of the tangent ℓ at P is: $9x_1x - 16y_1y = 144$.

(β) Find the coordinates of the point G at which ℓ cuts the x-axis.

(γ) Hence prove that $\frac{SP}{S'P} = \frac{SG}{S'G}$.

3. A function $f(x)$ is defined by $f(x) = \frac{\log_e x}{x}$ for $x > 0$.

(a) Prove that the graph of $f(x)$ has a relative turning point at $x = e$ and a point of inflexion at $x = e^{3/2}$.

(b) Discuss the behaviour of $f(x)$ in the neighbourhood of $x = 0$ and for large values of x.

(c) Hence draw a clear sketch of $f(x)$ indicating on it all these features.

(d) Draw separate sketches of the graphs of:

(α) $y = \left|\frac{\log_e x}{x}\right|$; **($\beta$)** $y = \frac{x}{\log_e x}$. [Hint: There is no need to find any further derivatives to answer this part.]

(e) What is the range of the function $y = \frac{x}{\log_e x}$?

4. **(i)** Find the complex square roots of $7 + 6\sqrt{2}i$ giving your answers in the form $x + iy$, where x and y are real.

(ii) Let $z_1 = 4 + 8i$ and $z_2 = -4 - 8i$.

(a) Draw a neat sketch of the locus specified by $|z - z_1| = |z - z_2|$.

(b) Show that the locus specified by $|z - z_1| = 3|z - z_2|$ is a circle. Give its centre and radius.

(iii) **(a)** Let $OABC$ be a square on an Argand diagram where O is the origin. The points A and C represent the complex numbers z and iz respectively. Find the complex number represented by B.

(b) The square is now rotated about O through $45°$ in an anticlockwise direction to $OA'B'C'$. Find the complex numbers represented by the points A', B' and C'.

5. **(i)** **(a)** On a number plane shade in the region representing the inequality $(x-2R)^2+y^2 \leqq R^2$.

(b) Show that the volume of a right circular cylindrical shell of height h with inner and outer radii x and $x+\delta x$ respectively is $2\pi x h\,\delta x$ when δx is sufficiently small for terms involving $(\delta x)^2$ to be neglected.

(c) The region $(x-2R)^2+y^2 \leqq R^2$ is rotated about the y-axis forming a solid of revolution called a torus. By summing volumes of cylindrical shells show that the volume V of the torus is given by: $V = 4\pi^2 R^3$.

(ii) Five letters are chosen from the letters of the word CRICKET. These five letters are then placed alongside one another to form a five letter arrangement. Find the number of distinct five letter arrangements which are possible, considering all possible choices.

6. A particle of unit mass moves in a straight line against a rsistance numerically equal to $v+v^3$, where v is its velocity. Initially the particle is at the origin and is travelling with velocity Q, where $Q > 0$.

(a) Show that v is related to the displacement x by the formula $x = \tan^{-1}\left(\frac{Q-v}{1+Qv}\right)$.

(b) Show that the time t which has elapsed when the particle is travelling with velocity v is given by $t = \frac{1}{2}\log_e\left(\frac{Q^2(1+v^2)}{v^2(1+Q^2)}\right)$.

(c) Find v^2 as a function of t.

(d) Find the limiting values of v and x as $t \to \infty$.

7. **(i)** ABC is an isosceles triangle with $AB = AC$. Let Q be a point on the base BC between B and C. AQ produced meets the circle through the points A, B, C at P.

(a) Prove that triangle BQP is similar to triangle AQC.

(b) Show that $BP.CQ = PQ.AC$.

(c) Prove that $\frac{1}{BP} + \frac{1}{CP} = \frac{1}{PQ} \cdot \frac{BC}{AC}$

(ii) **(a)** Prove using mathematical induction that for $n \geqq 1$,

$$1 + \frac{1}{2^2} + \frac{1}{3^2} + \cdots + \frac{1}{n^2} \leqq 2 - \frac{1}{n}.$$

(b) Prove that $1.45 \leqq 1 + \frac{1}{2^2} + \frac{1}{3^2} + \cdots + \frac{1}{99^2} \leqq 1.99$.

8. (i) Write down the general solution of the equation $\sin 2\theta + \cos 5\theta = 0$.

(ii) (a) A polynomial $R(x)$ is given by $R(x) = x^7 - 1$. Let $\rho \neq 1$ be that complex root of $R(x) = 0$ which has the smallest positive argument. Show that:

($\boldsymbol{\alpha}$) $R(x) = (x-1)(1 + x + x^2 + x^3 + x^4 + x^5 + x^6)$,

($\boldsymbol{\beta}$) $1 + \rho + \rho^2 + \rho^3 + \rho^4 + \rho^5 + \rho^6 = 0$.

(b) Let $\theta = \rho + \rho^2 + \rho^4$ and $\phi = \rho^3 + \rho^5 + \rho^6$.

($\boldsymbol{\alpha}$) Prove that $\theta + \phi = -1$ and $\theta\phi = 2$.

($\boldsymbol{\beta}$) Show that $\theta = \frac{-1+i\sqrt{7}}{2}$ and $\phi = \frac{-1-i\sqrt{7}}{2}$.

(c) Given that
$$\begin{aligned} T(x) &= 1 + x + x^2 + x^3 + x^4 + x^5 + x^6 \\ &= \{(x-\rho)(x-\rho^2)(x-\rho^4)\}\{(x-\rho^3)(x-\rho^5)(x-\rho^6)\} \end{aligned}$$
write the polynomial $T(x)$ as a product of two cubics with coefficients involving θ, ϕ and rational numbers.

NSW HSC 4 Unit Mathematics Examination 1988

1. **(a)** Find the exact value of:-

(i) $\int_0^1 \frac{2x}{1+2x}\,dx$; **(ii)** $\int_1^e \frac{(\log_e x)^2}{x}\,dx$; **(iii)** $\int_0^{\frac{1}{2}} \cos^{-1} x\,dx$.

(b) Use the substitution $t = \tan\frac{\theta}{2}$ to find the exact value of $\int_0^{\pi/2} \frac{d\theta}{\sin\theta + 2}$.

2. **(a)** Draw a neat sketch of the function $f(x) = (x-2)(6-x)$. State the co-ordinates of its vertex and of its points of intersection with both co-ordinate axes.

(b) Hence or otherwise draw a neat sketch of the function $g(x) = \frac{16}{(x-2)(6-x)}$. Clearly indicate on your sketch the equations of the vertical asymptotes and the co-ordinates of any stationary points.

(c) The lines with equations $x = 3$ and $x = 5$ cut the graph of $y = g(x)$ at P and Q respectively. Mark on your sketch the co-ordinates of P and Q. Shade the region R bounded by $y = g(x)$ and the line PQ.

(d) Prove that the area of R is $\left(\frac{32}{3} - 8\log_e 3\right)$ square units.

3. **(a)** On a perticular island, twenty per cent of all turtles survive for four weeks after hatching. Fifteen turtles hatch on the same day and are tagged for a study.

(i) **(α)** all fifteen turtles will survive the four weeks;

(β) none of the turtles survives the four weeks.

(ii) Write down expressions for the probability that:

(α) no more than three turtles survive the four weeks;

(β) at least three turtles survive the four weeks.

(b) The population P of a town increases at a rate proportional to the number by which the town's population exceeds 1000. This can be expressed by the differential equation $\frac{dP}{dt} = k(P - 1000)$ where t is the time in years and k is a constant.

(i) By differentiation show that $P = 1000 + Ae^{kt}$, where A is a constant, is a solution of this equation.

(ii) The population of the town was 2500 at the start of 1970 and 3000 at the start of 1985. Find its population at the start of the year 2000.

(iii) During which year will the population reach 4000?

4. **(a)** **(i)** Express $z = \sqrt{2} - i\sqrt{2}$ in a modulus-argument form.

(ii) Hence write z^{22} in the form $a + ib$, where a and b are real.

(b) (i) On an Argand diagram shade in the region R containing all points representing complex numbers z such that $1 < |z| < 2$ and $\frac{\pi}{4} < \arg z < \frac{\pi}{2}$.

(ii) In R mark with a dot the point K representing a complex number z. Clearly indicate on your diagram the points M, N, P and Q representing the complex numbers $\overline{z}, -z, \frac{1}{z}$ and $2z$ respectively.

(c) Show that the locus specified by $3|z-(4+4i)| = |z-(12+12i)|$ is a circle. Write down its radius and the co-ordinates of its centre. Draw a neat sketch of the circle.

5. The hyperbola H has equation $xy = 16$.

(a) Sketch this hyperbola and indicate on your diagram the positions and co-ordinates of all points at which the curve intersects the axes of symmetry.

(b) $P(4p, \frac{4}{p})$, where $p > 0$, and $Q(4q, \frac{4}{q})$, where $q > 0$, are two distince arbitrary points on H. Find the equation of the chord PQ.

(c) Prove that the equation of the tangent at P is $x + p^2 y = 8p$.

(d) The tangents at P and Q intersect at T. Find the co-ordinates of T.

(e) The chord PQ produced passes through the point $N(0, 8)$.

(i) Find the equation of the locus of T.

(ii) Give a geometrical description of this locus.

6. A body of mass one kilogram is projected vertically upwards from the ground at a speed of 20 metres per second. The particle is under the effect of both gravity and a resistance which, at any time, has a magnitude of $\frac{1}{40}v^2$, where v is the magnitude of the particle's velocity at that time.

In the following questions take the acceleration due to gravity to be 10 metres per second per second.

(a) While the body is travelling upwards the equation of motion is $\ddot{x} = -(10 + \frac{1}{40}v^2)$.

(i) Taking $\ddot{x} = v\frac{dv}{dx}$, calculate the greatest height reached by the particle.

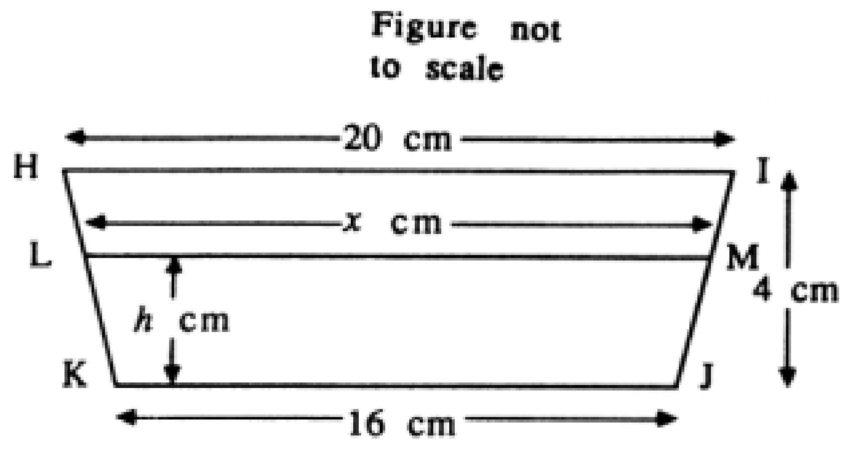
(ii) Taking $\ddot{x} = \frac{dv}{dt}$, calculate the time taken to reach this greatest height.

(b) Having reached its greatest height the particle falls to its starting point. The particle is still under the effect of both gravity and a resistance which, at any time, has a magnitude of $\frac{1}{40}v^2$.

(i) Write down the equation of motion of the particle as it falls.

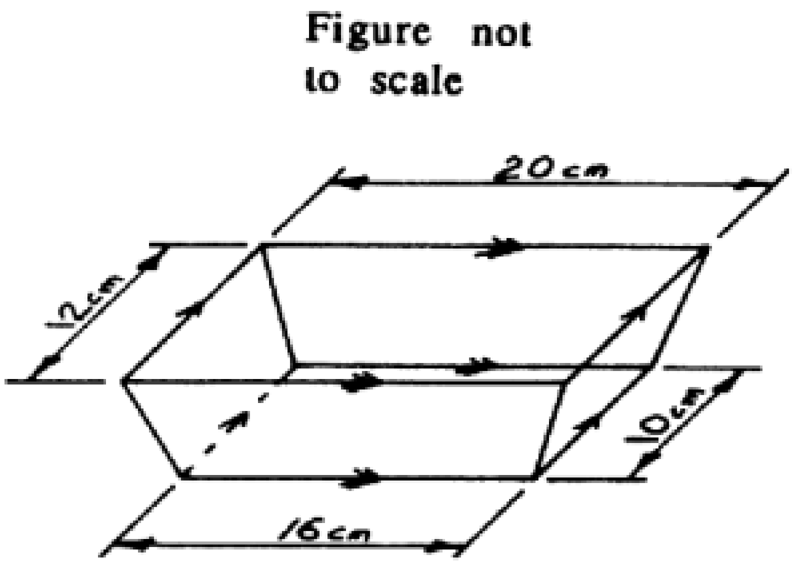
(ii) Find the speed of the particle when it returns to its starting point.

7. (a) (i)



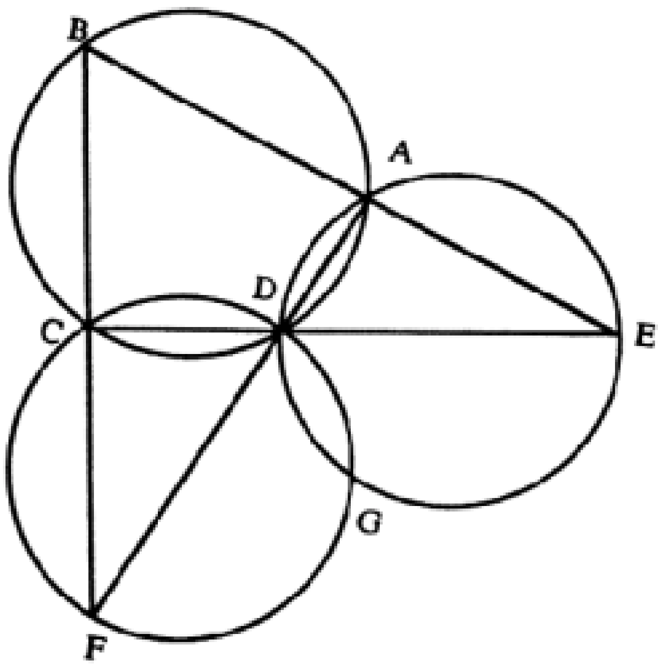
A trapezium $HIJK$ has parallel sides $KJ = 16$ cm and $HI = 20$ cm. The distance between these sides is 4 cm. L lies on HK and M lies on IJ such that LM is parallel to KJ. The shortest distance from K to LM is h cm and LM has length x cm. Prove that $x = 16 + h$.

(ii)



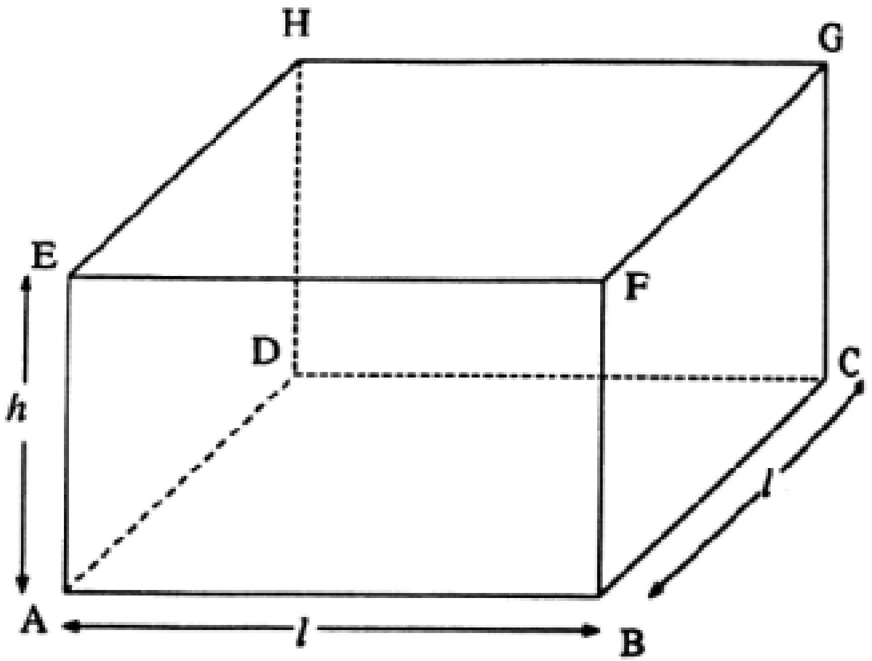
The diagram above is of a cake tin with a rectangular base with sides of 16 cm and 10 cm. Its top is also rectangular with dimensions 20 cm and 12 cm. The tin has depth 4 cm and each of its four side faces is a trapezium. Find its volume.

(b)



$ABCD$ is a cyclic quadrilateral. BA and CD are both produced and intersect at E. BC and AD produced intersect at F. The circles EAD, FCD intersect at G as well as at D. Prove that the points E, G and F are collinear.

8. (a)



A building is in the shape of a square prism with base edge ℓ metres and height h metres. It stands on level ground. A base diagonal AC is produced to a point

K. From K it is found that the angles of elevation of F and G are $30°$ and $45°$ respectively. Prove that $\frac{h}{\ell} = \frac{\sqrt{2}+\sqrt{10}}{4}$.

(b) Newton's method may be used to determine numerical approximations to the real roots of the equation $x^3 = 2$.

Let $x_1 = 2, x_2, x_3, \ldots, x_n, \ldots$ be a series of estimations obtained by iterative applications of Newton's method.

(i) Show that $x_{n+1} = \frac{2}{3}\left(x_n + \frac{1}{x_n^2}\right)$.

(ii) Show algebraically that $x_{n+1} - \sqrt[3]{2} = \frac{\left(x_n - \sqrt[3]{2}\right)^2\left(2x_n + \sqrt[3]{2}\right)}{3x_n^2}$.

(iii) Given that $x_n > \sqrt[3]{2}$, show that $x_{n+1} - \sqrt[3]{2} < \left(x_n - \sqrt[3]{2}\right)^2$.

(iv) Show that x_{12} and $\sqrt[3]{2}$ agree to at least 267 decimal places.

NSW HSC 4 Unit Mathematics Examination 1989

1. (a) Evaluate $|2+3i|$.

(b) Given that a and b are real numbers, express in the form $x+iy$, where x and y are real:

(i) $(a+bi)(\overline{5+i})$; **(ii)** $\frac{a+bi}{3+4i}$.

(c) Find the complex square roots of $10-24i$, giving your answers in the form $x+iy$, where x and y are real.

(d) On an Argand diagram shade in the region containing all points representing complex numbers z such that $2 \leqq \Re(z) \leqq 4$ and $-1 \leqq \Im(z) \leqq 3$.

(e) Find in modulus-argument form all complex numbers z such that $z^3 = -1$ and plot them on an Argand diagram.

(f) On separate diagrams draw a neat sketch of the locus specified by:

(i) $\arg[z-(1+\sqrt{3}i)] = \frac{\pi}{3}$; **(ii)** $z^2 - z^{-2} = 16i$.

2. (a) Evaluate: **(i)** $\int_0^{\sqrt{3}} \frac{2}{x^2+9}\,dx$; **(ii)** $\int_1^3 x^2 \ln x\,dx$; **(iii)** $\int_{\sqrt{3}}^2 \sqrt{4-x^2}\,dx$.

(b) (i) Write $\frac{4x^2-5x-7}{(x-1)(x^2+x+2)}$ in the form $\frac{A}{x-1} + \frac{Bx+C}{x^2+x+2}$.

(ii) Hence evaluate $\int_{-1}^0 \frac{4x^2-5x-7}{(x-1)(x^2+x+2)}\,dx$.

3. (a) The ellipse $\mathcal{E} : \left(\frac{x}{5}\right)^2 + \left(\frac{y}{3}\right)^2 = 1$ has foci $S(4,0)$ and $S'(-4,0)$.

(i) Sketch the ellipse $\mathcal{E}$ indicating its foci S, S' and its dirctrices.

(ii) Show that the tangent at $P(x_1,y_1)$ on the ellipse $\mathcal{E}$ has equation $9x_1x+25y_1y = 225$.

(iii) The line joining $P(x_1,y_1)$ to $Q(x_2,y_2)$ passes through S. Show that $4(y_2-y_1) = x_1y_2 - x_2y_1$.

(iv) It is also known that $Q(x_2,y_2)$ lies on $\mathcal{E}$. Show that the tangents at P and Q on the ellipse intersect on the directrix corresponding to S.

(v) Find the equation of the normal to $\mathcal{E}$ at P and decide under what circumstances, if any, it passes through S or S'.

(b) A public opinion survey of a certain parliamentary proposition finds 47% of the population in favour, 38% opposed and 15% undecided. Three persons are selected at random. Using the expansion

$$(p+q+r)^3 = p^3+q^3+r^3+3p^2q+3q^2r+3pq^2+3qr^2+3rp^2+3r^2p+6pqr$$

or otherwise, find the probability that:

(i) one person is in favour, one opposed and one is undecided;

(ii) exactly two persons are opposed;

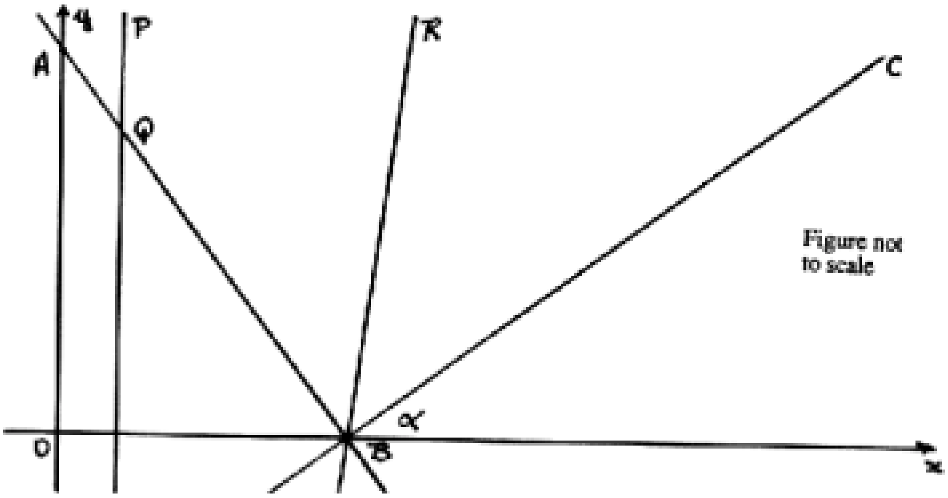
(iii) at least two persons are of the same opinion, either in favour, or opposed or undecided.

4. (a) (i) Write an expression for $\sin(\alpha+\beta), \cos(\alpha+\beta)$ in terms of $\sin\alpha, \sin\beta, \cos\alpha$ and $\cos\beta$.

(ii) Show that $\tan(\alpha+\beta) = \frac{\tan\alpha+\tan\beta}{1-\tan\alpha\tan\beta}$.

(iii) Hence find $\tan\left(\alpha+\frac{\pi}{4}\right)$ in terms of $\tan\alpha$.

(b)



The lines AB and BC in the diagram above have equations $3y = -4x + 20$ and $4y = 3x - 15$ respectively and meet at $B(5, 0)$. BC makes an angle α with the x axis. The line PQ has equation $x = 1$ and meets the line AB in Q. BR is the bisector of $\angle ABC$.

(i) Show that AB is perpendicular to BC and then copy the diagram.

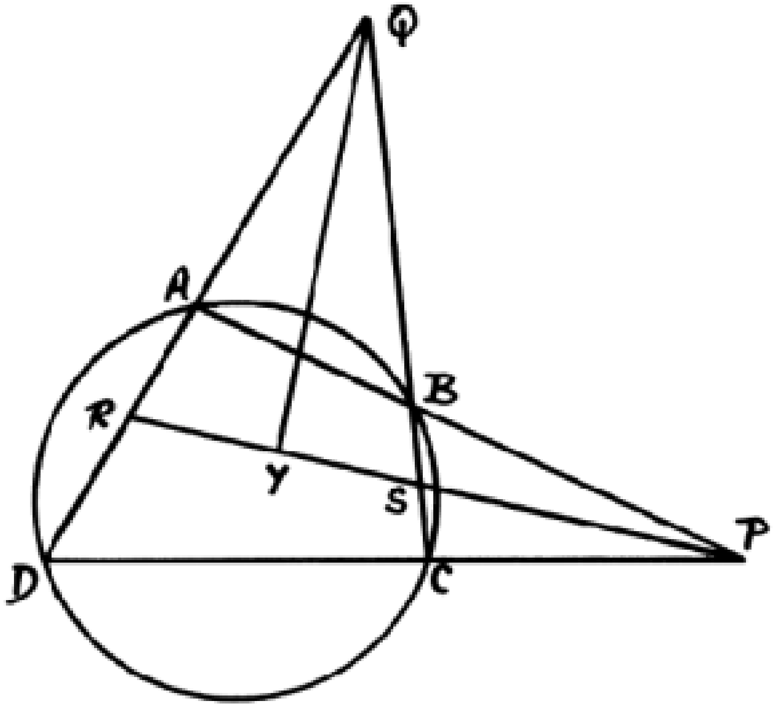
(ii) Use **(a)** to show that BR has equation $y = 7x - 35$.

(iii) The bisector of $\angle PQB$ has slope $1/3$ and meets BR at S. Calculate the co-ordinates of S.

(iv) Draw SM and SN perpendicular to AB and BC, meeting AB at M and BC at N, respectively. Prove that $SM = SN$.

(v) Show that S is the centre of a circle tangential to PQ, AB and BC and write down the equation of the circle.

5. (a)



Let $ABCD$ be a cyclic quadrilateral; AB and DC produced meet at P; DA and CB produced meet at Q as in the diagram.

Let PR be the internal bisector of $\angle APD$ meeting AD at R and BC at S.

Let QY be the internal bisector of $\angle DQC$ meeting PR at Y as in the diagram.

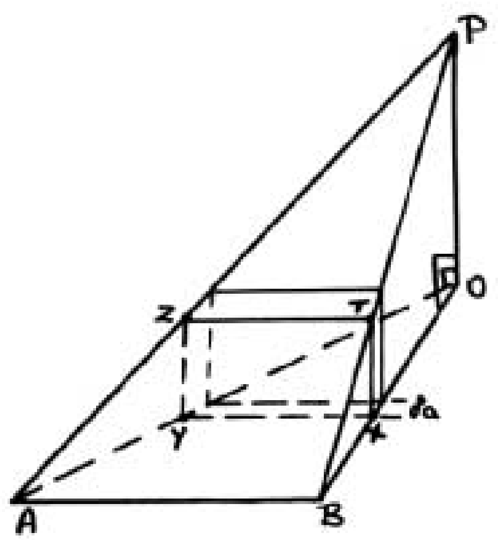
Copy the diagram and prove that: **(i)** $\angle QRS = \angle QSR$; **(ii)** $QY \perp PR$.

(b) Let ABO be an isosceles triangle, $AO = BO = r,\ AB = b$.

Let $PABO$ be a triangular pyramid with height $OP = h$ and OP perpendicular to the plane of ABO as in the diagram.

Consider a slice $\mathcal{S}$ of the pyramid of width δa as in the diagram.

The slice $\mathcal{S}$ is perpendicular to the plane of ABO at XY with $XY||AB$ and $XB = a$. Note that $XT||OP$.



(i) Show that the volume of $\mathcal{S}$ is $\left(\frac{r-a}{r}\right)b\left(\frac{ah}{r}\right)\delta a$, when δa is small. (You may assume that the slice is approximately a rectangular prism of base $XYZT$ and height δa.)

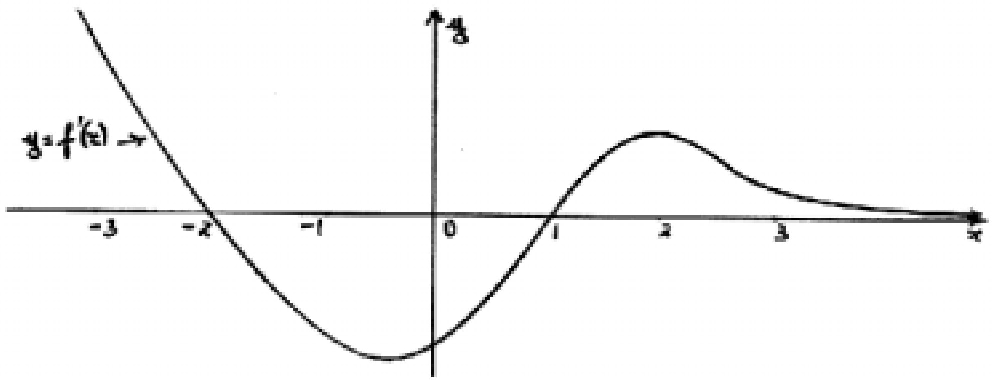
(ii) Hence show that the pyramid $PABO$ has volume $\frac{1}{6}hbr$.

(iii) Suppose now that $\angle AOB = \frac{2\pi}{n}$ and that n identical pyramids $PABO$ are arranged about O as centre with common vertical axis OP to form a solid $\mathcal{C}$. Show that the volume V_n of $\mathcal{C}$ is given by $V_n = \frac{1}{3}r^2hn\sin\frac{\pi}{n}$.

(iv) Note that when n is large, the solid $\mathcal{C}$ approximates a right circular cone. Using the fact that $\frac{\sin x}{x} \to 1$ as $x \to 0$, find $\lim_{n\to\infty} V_n$.

Hence verify that a right circular cone of radius r and height h has volume $\frac{1}{3}\pi r^2h$.

6. (a)



The function $f(x)$ has derivative $f'(x)$ whose graph appears in the diagram. You are given that $f'(-2) = f'(1) = 0$, $f'(x) \to \infty$ as $x \to -\infty$ and $f'(x) \to 0$ as $x \to \infty$.

(i) Sketch the graph of $f(x)$ showing its behaviour at its stationary points. You are given that $f(0) = 0$ and $f(3) > 0$.

(ii) Describe the behaviour of $f(x)$ as $x \to \pm\infty$.

(b) (i) Sketch the graph of $g(x) = x^4 - 4x^3 + 4x^2 - \frac{1}{2}$ showing that it has four real zeros.

(ii) On different diagrams, sketch the curves:

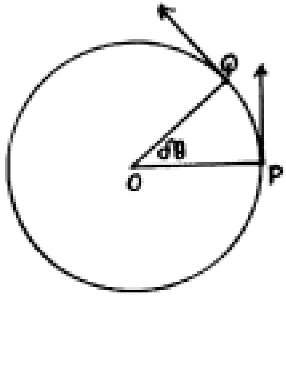
(α) $y = |g(x)|$; **(β)** $y^2 = g(x)$.

(iii) (α) Indicate the nature of the curve $y = g(x)$ at a zero of $g(x)$.

(β) Calculate the slope of the curve $y^2 = g(x)$ at any point x and describe the nature of the curve at a zero of $g(x)$.

7. (a) A particle of mass m kg moves in a horizontal circle with centre O and radius r metres, with uniform speed v metres per second.

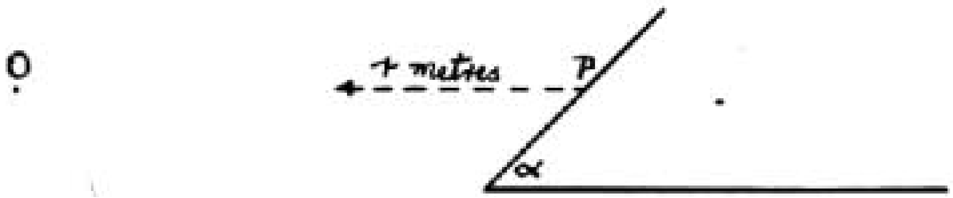
At time t, the particle is at point P while at time $t + \delta t$, it is at point Q with $\angle POQ = \delta\theta$ as in the diagram.



(i) Calculate the complement of the velocity at Q in the direction PO.

(ii) Hence show that the particle is subject to a force of $\frac{mv^2}{r}$ newtons directed towards O.

(iii) Suppose that the circle lies on a track banked at an angle α to the horizontal as in the figure below.



Draw a diagram of all the forces on the moving particle P and show that the resultant of these forces is normal to the track precisely when $\tan\alpha = \frac{v^2}{rg}$. Here g m.s^{-2} is the acceleration due to gravity.

(b) **(i)** Find real numbers a and b such that

$$x^4 + x^3 + x^2 + x + 1 = (x^2 + ax + 1)(x^2 + bx + 1).$$

(ii) Given that $x = \cos\frac{2\pi}{5} + i\sin\frac{2\pi}{5}$ is a solution of $x^4 + x^3 + x^2 + x + 1 = 0$, find the exact value of $\cos\frac{2\pi}{5}$.

8. **(a)** Find all values θ with $0 \leqq \theta \leqq 2\pi$ such that $\sin\theta - \sqrt{3}\cos\theta = 1$.

(b) The difference between a real number r are the greatest integer less than or equal to r is called the fractional part of $r, F(r)$. Thus $F(3.45) = 0.45$. Note that for all real numbers $r, 0 \leqq F(r) < 1$.

(i) Let $a = 2136\log_{10} 2$.

Given that $\quad F(a) = 7.0738\cdots \times 10^{-5}$

observe that $\quad F(2a) = 14.1476\cdots \times 10^{-5}$

$\quad F(3a) = 21.2214\cdots \times 10^{-5}$

(α) Use your calculator to show that $\log_{10} 1.989 < F(4223a) < \log_{10} 1.990$.

(β) Hence calculate an integer M such that the ordinary decimal representation of 2^M begins with 1989. Thus 2^M=1989 ... .

(ii) Let r be a real number and let m and n be non-zero integers with $m \neq n$.

(α) Show that if $F(mr) = 0$, then r is rational.

(β) Show that if $F(mr) = F(nr)$, then r is rational.

(iii) Suppose that b is an irrational number. Let N be a positive integer and consider the fractional parts $F(b), F(2b), \ldots, F((N+1)b)$.

($\boldsymbol{\alpha}$) Show that these $N+1$ numbers $F(b), \ldots, F((N+1)b)$ are all distinct.

($\boldsymbol{\beta}$) Divide the interval $0 \leqq x < 1$ into N subintervals each of length $1/N$ and show that there must be integers m and n with $m \neq n$ and $1 \leqq m,\ n \leqq N+1$ such that $F((m-n)b) < 1/N$.

(iv) Given that $\log_{10} 2$ is irrational, choose any integer N such that $1/N < \log_{10} \frac{1990}{1989}$; note that in **(i)** , $F(a) < \log_{10} \frac{1990}{1989}$.

Use **(iii)** to decide whether there exists another integer M such that $2^M = 1989\ldots$.