HURLSTONE AGRICULTURAL HIGH SCHOOL

MATHEMATICS – EXTENSION TWO

TRIAL EXAMINATION

2011

ASSESSMENT TASK 4

Examiners ~ G Huxley, G Rawson

GENERAL INSTRUCTIONS

- Reading Time – 5 minutes.
- Working Time – 3 hours.
- Attempt **all** questions.
- **All** necessary working should be shown in every question.
- This paper contains eight (8) questions.
- Marks may not be awarded for careless or badly arranged work.
- Board approved calculators may be used.
- **Each question is to be started in a new booklet.**
- This examination paper must **NOT** be removed from the examination room.

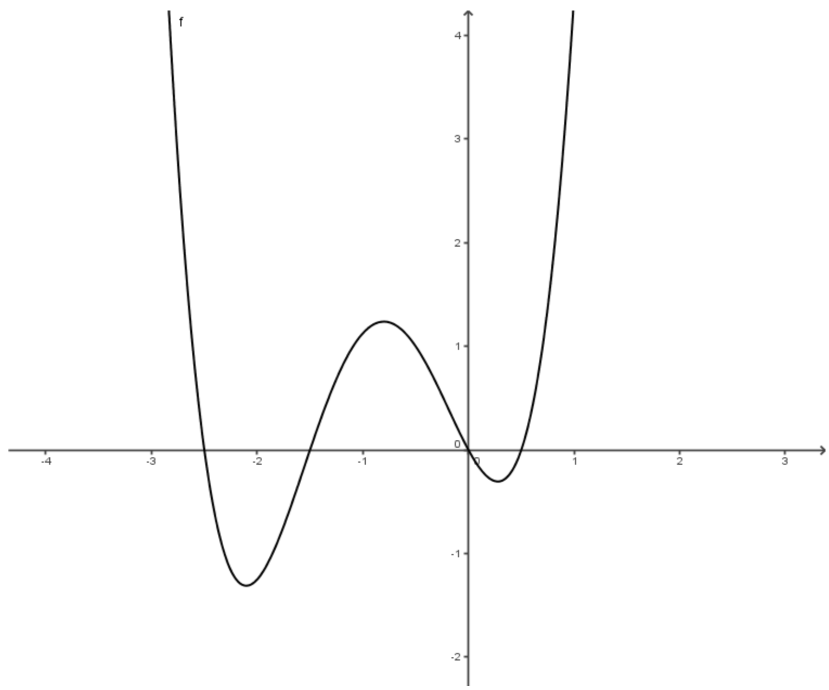
STUDENT NAME:______________________________

TEACHER: ______________________________

QUESTION ONE **15 marks** *Start a SEPARATE booklet.*

Marks

(a) The diagram shows the graph of $y = f(x)$



Draw separate one third page sketches of the following

(i) $y = \dfrac{1}{f(x)}$ **2**

(ii) $y^2 = f(x)$ **2**

(iii) $y = 2^{f(x)}$ **2**

(iv) $y = f\left(\dfrac{1}{x}\right)$ **2**

Question 1 continues on the next page

Marks

(b) Consider the curve $f(x) = \ln\left(2 + 2\cos(2x)\right), \quad -2\pi \le x \le 2\pi$.

(i) Show that the function *f* is even and the curve $y = f(x)$ is concave down for all values of *x* in its domain. **3**

(ii) Sketch, using a third of a page, the graph of the curve $y = f(x)$ **2**

(c) Find the coordinates of the points where the tangent to the curve $x^2 + 2xy + 3y^2 = 18$ is horizontal. **2**

<u>QUESTION TWO</u> **15 marks** *Start a SEPARATE booklet.*

Marks

(a) Using the substitution $u = e^x + 1$ or otherwise,

evaluate $\displaystyle\int_0^1 \frac{e^x}{(1+e^x)^2}\, dx$. **3**

(b) Find $\displaystyle\int \frac{1}{x \ln x}\, dx$. **1**

(c) (i) Find a, b, and c, such that

$$\frac{16}{(x^2+4)(2-x)} = \frac{ax+b}{x^2+4} + \frac{c}{2-x}.$$ **2**

(ii) Find $\displaystyle\int \frac{16}{(x^2+4)(2-x)}\, dx$. **2**

(d) Using integration BY PARTS ONLY, evaluate

$\displaystyle\int_0^1 \sin^{-1} x\, dx$. **3**

(e) Use the substitution $t = \tan\dfrac{\theta}{2}$ to show that :

$$\int_0^{\frac{\pi}{2}} \frac{d\theta}{4\sin\theta - 2\cos\theta + 6} = \frac{1}{2}\tan^{-1}\left(\frac{1}{2}\right).$$ **4**

QUESTION THREE **15 marks** *Start a SEPARATE booklet.*

Marks

(a) Find all the complex numbers $z = a + ib$, where a and b are real, such that $|z|^2 + 5\bar{z} + 10i = 0$. **3**

(b) $z_1 = 1 + i\sqrt{3}$ and $z_2 = 1 - i$ are two complex numbers.

(i) Express z_1, z_2 and $\frac{z_1}{z_2}$ in modulus-argument form. **3**

(ii) Find the smallest positive integer n such that $\frac{z_1^{\ n}}{z_2^{\ n}}$ is imaginary. For this value of n, write the value of $\frac{z_1^{\ n}}{z_2^{\ n}}$ in the form bi where b is a real number. **2**

(c) (i) On an Argand Diagram shade the region where both $|z-1| \leq 1$ and $0 \leq \arg z \leq \frac{\pi}{6}$. **3**

(ii) Find the perimeter of the shaded region. **2**

(d) On an Argand Diagram the points A, B, and C represent the complex numbers α, β, and γ respectively. ΔABC is equilateral, named with its vertices taken anticlockwise.

Show that $\gamma - \alpha = \left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right)(\beta - \alpha)$ **2**

QUESTION FOUR **15 marks** *Start a SEPARATE booklet.*

Marks

(a) (i) Show that $4x^2 + 9y^2 + 16x + 18y - 11 = 0$ represents an ellipse. **1**

(ii) Find the eccentricity and hence, the coordinates of its foci and the equations of its directrices. **2**

(b) The tangent to the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ is given by the equation $Ax + By + C = 0$.

Find the coordinates of the point of contact between the hyperbola and the tangent. **3**

(c) Show that the equation of the normal to the curve $xy = c^2$ at the point

$P\left(cp, \frac{c}{p}\right)$ is given by $p^3x - py = c\left(p^4 - 1\right)$. **3**

(d) The position of a particle moving in the Cartesian plane at a time *t* is given by the parametric equations.

$$x = 5\cos t$$
$$y = 12\sin t$$

(i) Eliminate *t* from the two equations above. **1**

(ii) Sketch the path of the particle in the *x*–*y* plane. **1**

(iii) Without using the area formula for an ellipse, show by integration that the area of the ellipse is 60π square units. **4**

<u>QUESTION FIVE</u> **15 marks** *Start a SEPARATE booklet.*

Marks

(a) Let $\alpha, \beta,$ and γ be the solutions of $x^3 - 4x^2 + 2x + 5 = 0$.

(i) Find $\alpha^2 + \beta^2 + \gamma^2$. **2**

(ii) Find $\alpha^3 + \beta^3 + \gamma^3$ **2**

(iii) Write an equation with roots $\alpha + 1, \beta + 1, \gamma + 1$. **2**

(b) Find a polynomial *P(x)* with real coefficients having $2i$ and $1\text{-}3i$ as zeroes. **3**

(c) (i) By considering $z^9 - 1$ as the difference of two cubes, or otherwise, write $1 + z + z^2 + z^3 + z^4 + z^5 + z^6 + z^7 + z^8$ as a product of two polynomials with real coefficients, one of which is a quadratic. **2**

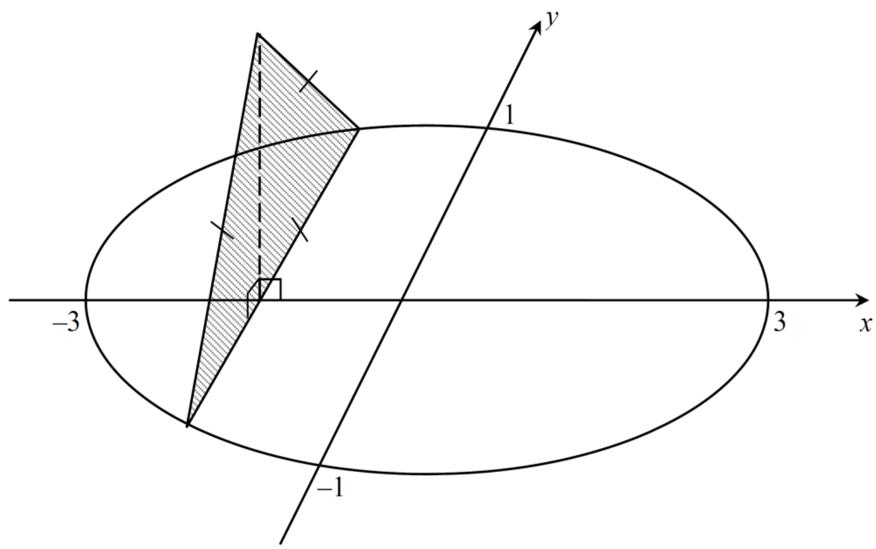
(ii) Solve $z^9 - 1 = 0$ and determine the six solutions of $z^6 + z^3 + 1 = 0$. **2**

(iii) Hence show that $\cos\frac{2\pi}{9} + \cos\frac{4\pi}{9} = \cos\frac{\pi}{9}$ **2**

<u>QUESTION SIX</u> **15 marks** *Start a SEPARATE booklet.*

Marks

(a) A solid shape has an elliptical base on the *xy*-plane as shown below.
Sections of the solid taken perpendicular to the *x*-axis are equilateral triangles.
The major and minor axes of the ellipse are of lengths 6 metres and 2 metres respectively.



(i) Write down the equation of the ellipse. **1**

(ii) Show that the volume ΔV of a slice taken at $x = d$ is given by

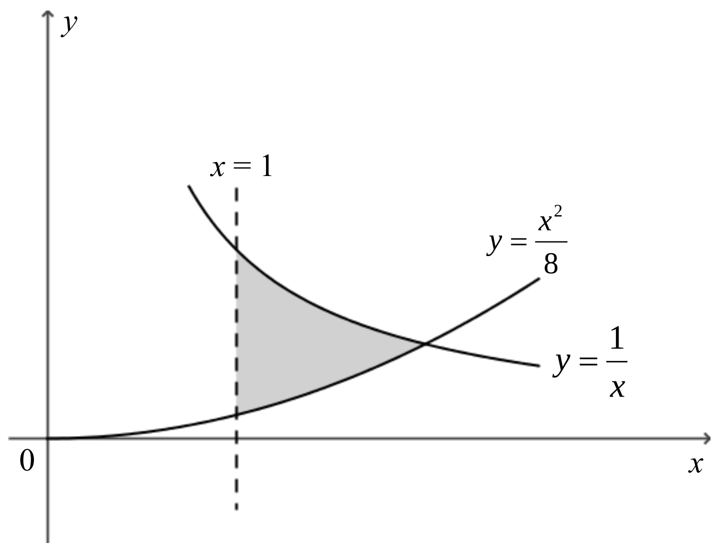
$$\Delta V \approx \frac{\sqrt{3}\left(9 - d^2\right)}{9} \Delta x$$ **2**

(iii) Find the volume of this solid. **3**

Question 6 continues on the next page

Marks

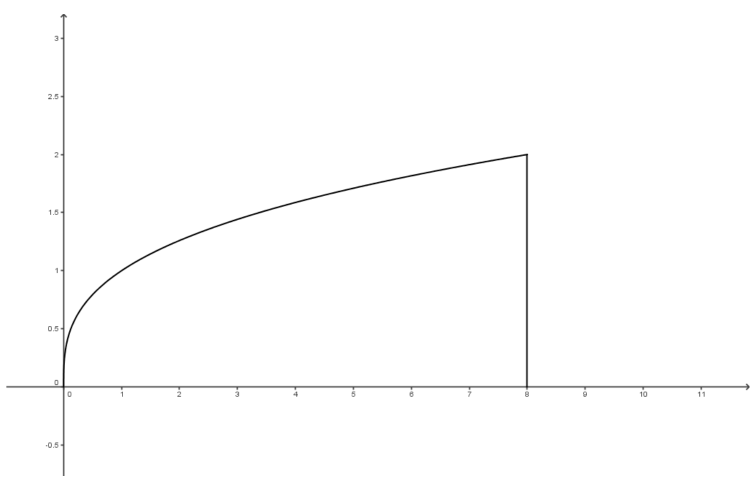
(b) The region bounded by $y = \frac{1}{x}$, $y = \frac{x^2}{8}$ and $x = 1$ is rotated about the line $x = 1$.



(i) Use the method of cylindrical shells to find an integral which gives the volume of the resulting solid of revolution. **3**

(ii) Find the volume of this solid of revolution. **2**

(c) The sketch below shows the region enclosed by the curve $y = x^{\frac{1}{3}}$, the x axis and the ordinate $x = 8$.



Find the volume generated when this region is rotated about the line $x = 8$. **4**

<u>QUESTION SEVEN</u> **15 marks** *Start a SEPARATE booklet.*

Marks

(a) (i) How many ways can a doubles tennis game be organised, given a group of four players? **1**

(ii) In how many ways can two games of doubles tennis be organised, given a group of eight players? **1**

(b) Use mathematical induction, or otherwise, to prove the following:

(i) $1.1!+2.2!+3.3!+...+n.n!=(n+1)!-1$, for $n\geq 1$. **3**

(ii) If $u_n=9^{n+1}-8n-9$, show that $u_{n+1}=9u_n+64n+64$, and hence show that u_n is divisible by 64 for $n\geq 1$. **4**

(c) (i) Let $z=\cos\theta+i\sin\theta$. Show that $2\cos\theta=z+z^{-1}$. **1**

(ii) Hence or otherwise show that $16\cos^4\theta=2\cos 4\theta+8\cos 2\theta+6$. **2**

(iii) Use the substitution $x=2\sin\theta$ to evaluate $\int_0^2 (4-x^2)\,dx$. **3**

QUESTION EIGHT **15 marks** *Start a SEPARATE booklet.*

Marks

(a) The region R is bounded by the curve $y = \dfrac{x}{x+1}$, the x-axis and the vertical line $x = 3$. **3**

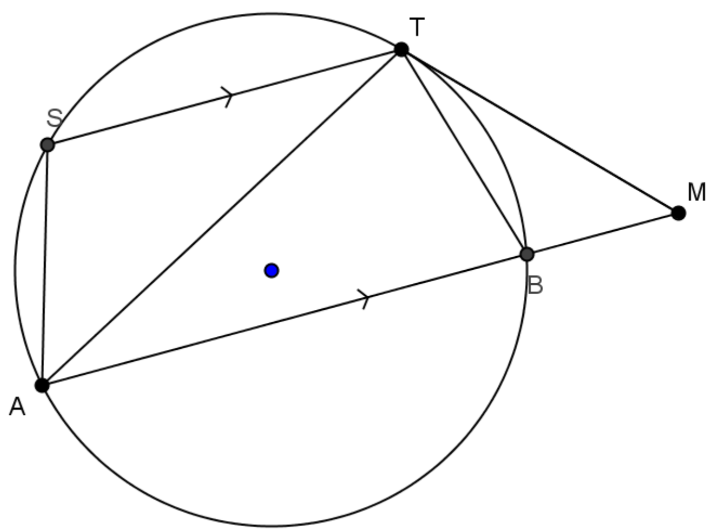
Find the exact volume generated when R is rotated about the x-axis.

(b) (i) $I_n = \int x^n e^{ax}\, dx$, where a is a constant. **2**

Prove that $I_n = \dfrac{x^n e^{ax}}{a} - \dfrac{n}{a} I_{n-1}$.

(ii) Hence find the value of $\int_0^1 x^3 e^{2x}\, dx$. **3**

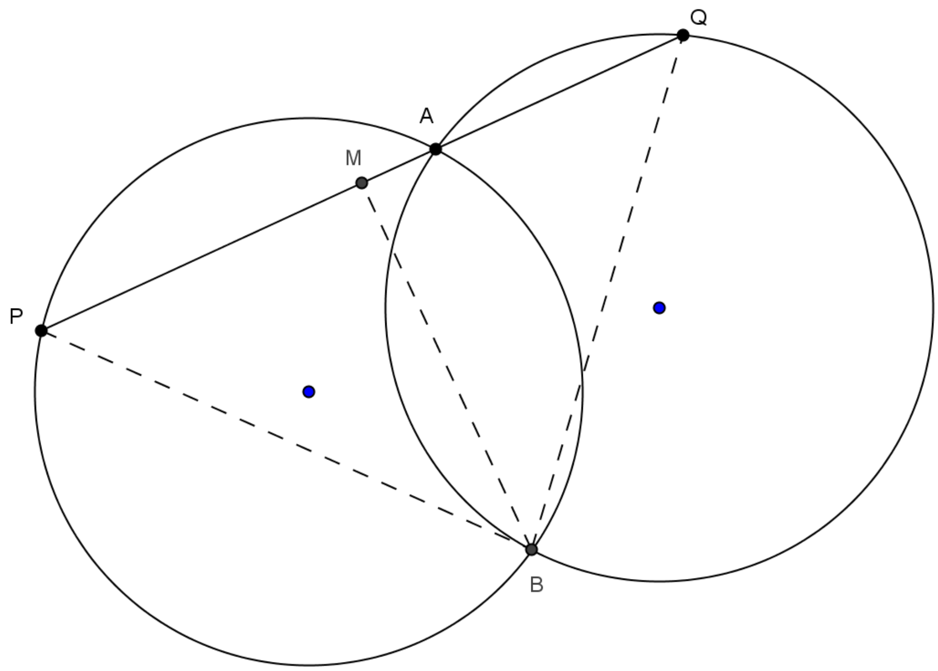
(c)



If $ST \parallel AB$ and TM is a tangent, prove that $\Delta TMB \parallel\!\mid \Delta TAS$. **3**

Question 8 continues on the next page

(d) Two circles of equal radii intersect at A and B. A variable line through A meets the two circles again at P and Q.



(i) Give the reason why $\angle QPB = \angle PQB$ **1**

(ii) M is the midpoint of PQ. Prove that $BM \perp PQ$ **2**

(iii) What is the locus of M as the line PAQ varies? **1**

END OF EXAMINATION

Standard Integrals Sheet