

Hurstville Boys High School
Trial Higher School Certificate Examination 1988
4 Unit Mathematics

Question 1.

(i) Find these indefinite integrals:

(a) $\int \frac{x-5}{x^2-x-2} dx$

(b) $\int \frac{x+3}{x^2+4} dx$

(c) $\int te^{-t} dt$

(d) $\int \frac{dx}{x \log_e x}$

(ii) Use the substitution $t = \tan \frac{x}{2}$ to evaluate $\int_0^{\pi/2} \frac{dx}{1+\sin x}$

(iii) Use the substitution $x = 3 \sin \theta$ to evaluate $\int_0^{3/\sqrt{2}} \sqrt{9-x^2} dx$

Question 2.

(i) Sketch the curve $y = 2 + \frac{1}{x^2-1}$ ($x \neq \pm 1$) showing the location and nature of all stationary points and the equations of all asymptotes.

(ii) Prove that the condition that the line $ax + by + c = 0$ is a tangent to the circle $x^2 + y^2 = R^2$ is that $R^2(a^2 + b^2) = c^2$.

(iii) Let n be a positive integer, and let $I_n = \int_1^2 (\log_e x)^n dx$. Show that $I_n = 2(\log_e 2)^n - nI_{n-1}$. Hence evaluate $\int_1^2 (\log_e x)^4 dx$ a polynomial in $\log_e 2$.

Question 3.

The hyperbola H has cartesian (x, y) equation $\frac{x^2}{5} - \frac{y^2}{5} = 1$. Write down its eccentricity, the co-ordinates of its foci S and S' , the equation of each directrix, and the equation of the asymptotes. Sketch the curve and indicate on your diagram the foci, directrices, and asymptotes. P is an arbitrary point $(\sqrt{5} \sec \theta, \sqrt{5} \tan \theta)$. Show that P lies on H and prove that the tangent to H at P has equation $\frac{x \sec \theta}{\sqrt{5}} - \frac{y \tan \theta}{\sqrt{5}} = 1$. This tangent cuts the asymptotes in L and M . Prove that $LP = PM$ and the area of $\triangle OLM$ is independent of the position of P on H . (O is the origin.)

Question 4.

- (i) Express the roots of $z^2 - (1 - i)z + 7i - 4 = 0$ in the form $x + iy$.
- (ii) Express $z = \frac{1+2i}{1-i} + \frac{1}{i}$ in the form $r(\cos \theta + i \sin \theta)$ where $r = |z|$ and $\theta = \arg z$.
- (iii) On the argand diagram show clearly the region defined by
- (a) $|z - 3i| \leq 2$
- (b) $|z - 2| < |z + 2|$
- (iv) Find the complex cube roots of -1 , expressing them in the form $r(\cos \theta + i \sin \theta)$. Show the roots on the argand diagram.

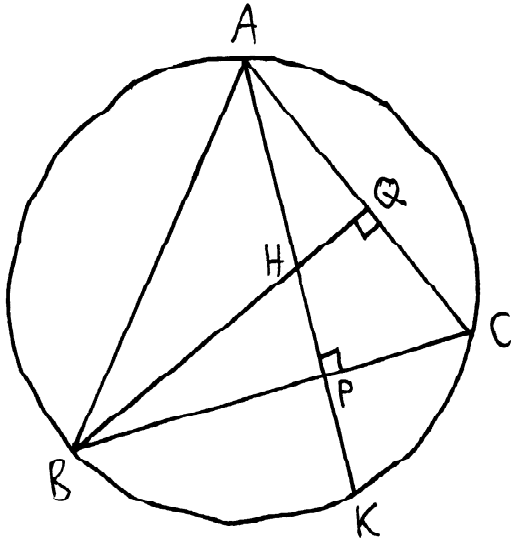
Question 5.

- (i) (a) Find the first and second derivative of $y = e^{-\frac{1}{2}x^2}$
- (b) Sketch the curve $y = e^{-\frac{1}{2}x^2}$, showing clearly any stationary points and points of inflexion.
- (ii) The area between the curve $y = e^{-\frac{1}{2}x^2}$ the x and y axes and the line $x = \sqrt{2}$ is rotated about the y axis. Using the method of cylindrical shells (parallel to y axis), find the volume of the solid of revolution so formed.
- (iii) The base of a certain solid is the region between the x axis and the curve $y = \sin x$ between $x = 0$ and $x = \frac{\pi}{2}$. Each plane perpendicular to the x axis is an equilateral triangle with one side on the base of the solid. Find the volume of the solid.

Question 6.

- (i) Solve $\cos 2x - \sin x = 1$ for $0 \leq x \leq 2\pi$.
- (ii) As the result of an experiment, a curve of the form $y = f(x)$ is drawn. It is suspected that $f(x)$ is expressible in the form $f(x) = (1 + ax)^n$ where “ a ” is real and “ n ” is a positive integer. For values of x so small that third and higher powers of x can be neglected: $y = 1 - 6x + 16x^2$ is practically identical with the given curve. Assuming the curves coincide for these values of x , what are the values of “ a ” and “ n ”?

(iii) The altitudes AP and BQ of an acute angled triangle meet at H . AP produced cuts the circle through A, B, C at K . Prove that $HP = PK$.



Question 7.

(i) (a) State the factor theorem for polynomials.

(b) If $P(x) = x^3 + 5x^2 + 9x + 6$ resolve $P(x)$ into irreducible factors over the field of complex numbers.

(ii) If α, β, γ are the roots of the cubic equation $x^3 + qx + r = 0$, find the value of (in terms of q, r)

(a) $\alpha + \beta + \gamma$

(b) $\alpha\beta + \alpha\gamma + \beta\gamma$

(c) $\alpha\beta\gamma$

Hence find the value of

$$(\beta - \gamma)^2 + (\gamma - \alpha)^2 + (\alpha - \beta)^2$$

(iv) Prove by mathematical induction that

$$3^n > 1 + 2n \quad \text{for } n > 1.$$

Question 8.

(i) Sketch the following curves (without the use of calculus) for $-2\pi \leq x \leq 2\pi$:

(a) $y = |\sin x|$

(b) $y = \sin |x|$

(ii) A ball is projected from a point on the ground distance a units from the foot of a vertical wall of height b units. The ball is projected at an elevation with speed v . Find how high above the wall the ball will pass.

If the ball just clears the wall, prove that the greatest height reached is

$$\frac{1}{4} \left[\frac{a^2 \tan^2 \theta}{a \tan \theta - b} \right].$$