International Grammar School

2008

TRIAL HIGHER SCHOOL CERTIFICATE EXAMINATION

Mathematics

Extension 2

General Instructions

- Reading Time- 5 minutes
- Working Time – 3 hours
- Write using a blue or black pen
- Approved calculators may be used
- A table of standard integrals is provided at the back of this paper.
- All necessary working should be shown for every question.

Total marks (**120**)

- Attempt Questions 1-8
- All questions are of equal value

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Answer each question in a SEPARATE writing booklet. Extra writing booklets are available.

Question 1 (15 marks) Use a SEPARATE answer booklet **Marks**

a) Find $\displaystyle\int \frac{dx}{\sqrt{16-9x^2}}$ **2**

b) Find $\displaystyle\int 5\cos x \sin^2 x \, dx$ **2**

c) Evaluate $\displaystyle\int_1^e x \ln x \, dx$ **3**

d) Evaluate $\displaystyle\int_2^3 \frac{dx}{x^2-1}$ **4**

e) Using the substitution $t = tan\frac{\theta}{2}$ or otherwise find **4**

$$\int_0^{\frac{\pi}{2}} \frac{d\theta}{2+\cos\theta}$$

End of Question 1

Question 2 (15 marks) Use a SEPARATE writing booklet. **Marks**

a) Let $A = 3 + 4i$ and $B = 2 - 2i$. Find in the form $x + iy$ (x and y real).

i) $\dfrac{A}{B}$ **2**

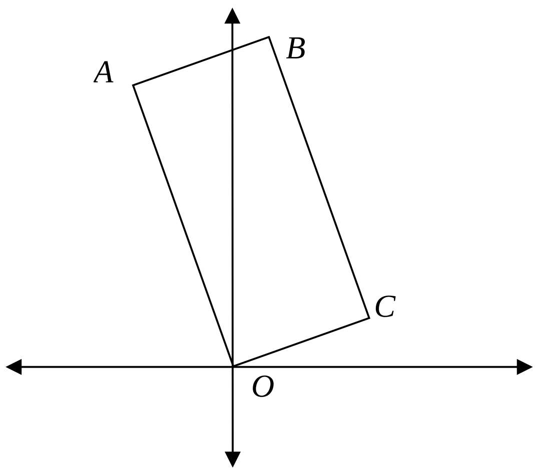
ii) $\sqrt{A}$ **3**

iii) $A - \overline{B}$ **1**

b) i) Write $1 + \sqrt{3}i$ in the form $r(\cos\theta + i\sin\theta)$ **2**

ii) Hence write $\left(1 + \sqrt{3}i\right)^6$ showing that it is real. **2**

c)



The points $OABC$ are the vertices of a rectangle on the Argand diagram with $|OA| = 2|OC|$. If OC represents the complex number $p + iq$, write down the complex numbers represented by:

i) OA **1**

ii) OB **1**

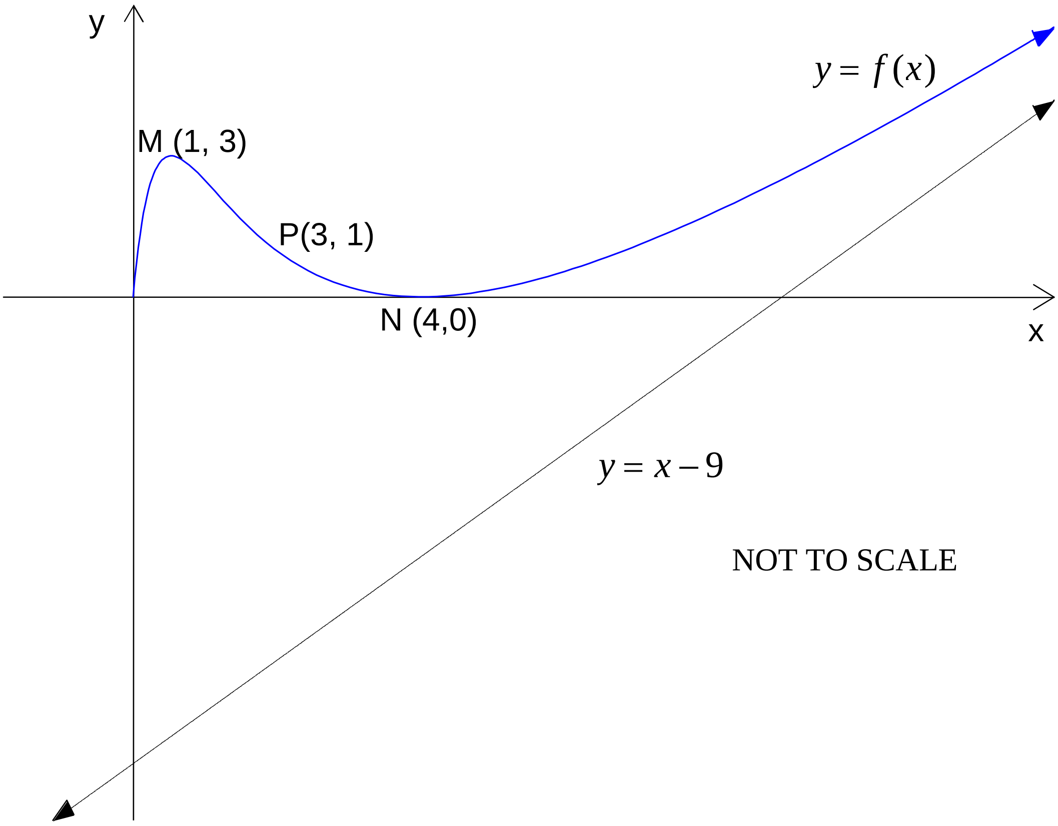
iii) BC **1**

iv) AC **2**

End of Question 2

Question 3 (15 marks) Use a SEPARATE writing booklet. **Marks**

a)



The diagram shows the graph of $y = f(x)$ for $x \geq 0$.
M(1, 3) and N(4, 0) are stationary points of $y = f(x)$ and P(3, 1) is a point of inflexion of $y = f(x)$. The line $y = x - 9$ is an asymptote as $x \to \infty$. Draw separate one third page sketches showing any special features for the following:

i) $f'(x)$ **2**

ii) $\dfrac{1}{f(x)}$ **2**

iii) $-\left(f(x)\right)^2$ **2**

Question 3 continues on the next page

Question 3 continued **Marks**

b) Determine the gradient of the tangent to the curve $x^2 + 2xy - y^2 = 17$ at the point $(3,\ 2)$ **2**

c) The zeros of $x^3 - 3x^2 - 2x + 4$ are α, β and γ

i) Find a cubic polynomial whose zeros are α^2, β^2 and γ^2 **2**

ii) Hence or otherwise find the value of $\alpha^2 + \beta^2 + \gamma^2$ **1**

iii) Determine the value of $\alpha^3 + \beta^3 + \gamma^3$ **2**

d) The equation $P(x) = x^3 + 3x^2 - 24x + k = 0$ has a double root. Find the possible values of k. **2**

End of Question 3

Question 4 (15 marks) Use a SEPARATE writing booklet. **Marks**

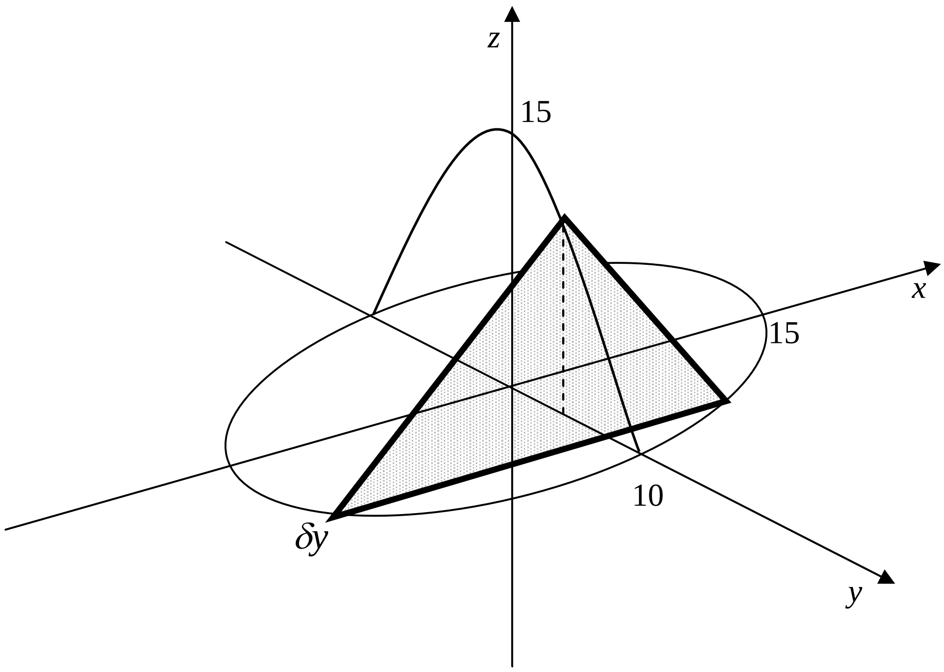
a) i) Show that a reduction formula for $I_n = \int (\ln x)^n dx$ **3**

is $I_n = x(\ln x)^n - nI_{n-1}$

ii) Hence evaluate $\int_1^{e^4} (\ln x)^3 \, dx$ **4**

b) The arc of the curve $y = 6x - x^2 - 8$ where $y \geq 0$ is rotated about the line $x = 1$. By applying the technique of cylindrical shells determine the exact volume of the solid formed **4**

c)



A cake is made with base in the shape of an ellipse, with semi-major axis 15 cm and semi-minor axis 10 cm. Slices of the cake parallel to the major axis of the base are isosceles triangles, whose vertices trace out a semi-elliptical path with the same semi-major axis and semi-minor axis lengths as in the diagram below.

i) Show that the volume of a 'typical' triangular slice is given by:

$$V_{slice} \approx xz\delta y \ \ cm^3$$ **1**

ii) Find the exact volume of the cake in cm^3. **3**

End of Question 4

Question 5 (15 marks) Use a SEPARATE writing booklet. **Marks**

a) The line $y = mx + a$ intersects the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ at two points which have x coordinates x_1 and x_2. **3**

i) Express x_2 in terms of *m, a, b* and x_1. **1**

ii) Hence or otherwise show that the line is a tangent to the ellipse at the point where $\frac{-a^3m}{b^2 + a^2m^2}$.

b) A parabola has parametric equations $x = 2at$ and $y = at^2$.

i) Find the equation of the normal to the parabola at the point where $t = p$. **2**

ii) Hence show that, through the point (x_1, y_1) , it is possible to draw up to three normals to the parabola. **2**

c) Given the complex number $z = cos\,\theta + i\,sin\,\theta$

i) Use De Moivres Theorem and the binomial expansion find an expression for $cos\,4\theta$ in terms of $cos\,\theta$ **3**

ii) Also, using $z^n + \frac{1}{z^n} = 2\,cos\,n\theta$ determine an expansion for $cos^4\,\theta$ in terms of $cos\,n\theta$ **2**

iii) Hence evaluate $\int_0^{\frac{\pi}{2}} cos^4\theta\ d\theta$ **2**

End of Question 5

Question 6 (15 marks) Use a SEPARATE writing booklet. **Marks**

a) On a suitably labelled Argand diagram sketch the region determined by
$[\text{Re}(z)]^2 + \text{Im}\, z < 0$ **2**

(b) Consider the function

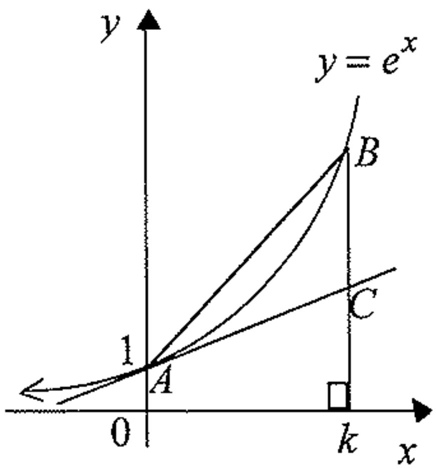
$$f(x) = \begin{cases} \dfrac{e^x - 1}{x}, & x \neq 0 \\ 1 \quad , & x = 0 \end{cases}$$

(i) Use differentiation to show that $e^{-x} + x - 1 \geq 0$ for all values of x. Hence show that $f(x)$ is an increasing function for $x \neq 0$ **3**

(ii) Show that $f(x)$ is continuous at $x = 0$. **2**

(iii) Sketch the graph of $y = f(x)$. **1**

(c)



The curve $y = e^x$ cuts the y axis at A. B is a second point on the curve such that $x = k$, where $k > 0$. The tangent to the curve $y = e^x$ at A cuts the vertical line $x = k$ at the point C.

(i) By considering areas, show that $\frac{1}{2}k(k+2) < e^k - 1 < \frac{1}{2}k(1+e^k)$. Hence deduce that $2.5 < e < 3$. **4**

(ii) Show that the curve $y = e^x$ bisects the area of ΔABC for some value of k such that $2 < k < 3$. Taking $k = 2.7$ as a first approximation, apply Newton's method once to obtain a second approximation. Give your answer to one decimal place. **3**

End of Question 6

Question 7 (15 marks) Use a SEPARATE writing booklet. **Marks**

a) Given that $z^5 - 1 = 0$

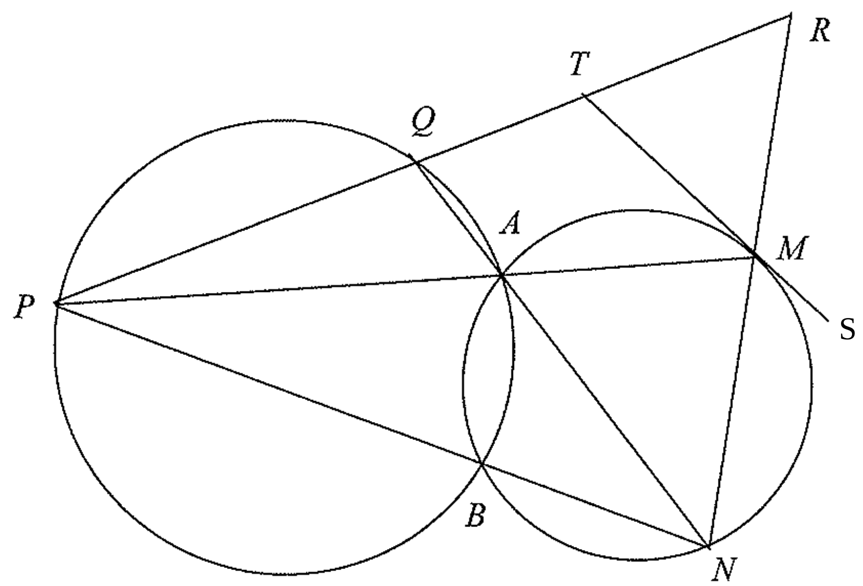
i) Solve for Z over the complex field in the form $cos\theta + i\,sin\theta$. **3**

i) Hence express $z^5 - 1$ as the product of linear and quadratic factors. **2**

iii) Write down the complex roots of $z^4 + z^3 + z^2 + z + 1 = 0$. **1**

iv) Without evaluating, show that $cos\frac{2\pi}{5} + cos\frac{4\pi}{5} = -\frac{1}{2}$ **2**

b)



In the diagram, the two circles intersect at A and B. P is a point on one circle. PA and PB produced meet the other circle at M and N respectively. NA produced meets the first circle at Q. PQ and NM produced meet at R. The tangent at M to the second circle meets PR at T.
Copy or trace the diagram into your answer booklet.

(i) Show that QAMR is a cyclic quadrilateral. **3**

(ii) Show that TM=TR. **4**

End of Question 7

Question 8 (15 marks) Use a SEPARATE writing booklet **Marks**

a) i) Show that for all values of x and y: **1**

$$\sin(x+y)-\sin(x-y)=2\cos x\sin y$$

ii) Use mathematical induction to show that for all positive integers n: **4**

$$\cos x+\cos 2x+\cos 3x+\cdots\cos nx=\frac{\sin\left(n+\frac{1}{2}\right)x-\sin\frac{1}{2}x}{2\sin\frac{1}{2}x}$$

iii) Hence show that: **4**

$$\cos 2x+\cos 4x+\cos 6x+\cdots+\cos 16x \;=\; 8.\cos 9x.\cos 4x.\cos 2x.\cos x$$

b) Show that a relationship between the coefficients of $p(x)=x^3+ax^2+bx+c=0$ is $2a^3-9ab-27c=0$, if the roots are three consecutive terms of an arithmetic series. **4**

c) Solve the differential equation $\frac{dy}{dx}=2y$ for y given that when $x=1$, $y=1$ **2**

End of Examination

STANDARD INTEGRALS

$$\int x^n \, dx = \frac{1}{n+1} x^{n+1}, \quad n \neq -1; \quad x \neq 0, \text{ if } n < 0$$

$$\int \frac{1}{x} dx = \ln x, \quad x > 0$$

$$\int e^{ax} \, dx = \frac{1}{a} e^{ax}, \quad a \neq 0$$

$$\int \cos ax \, dx = \frac{1}{a} \sin ax, \quad a \neq 0$$

$$\int \sin ax \, dx = -\frac{1}{a} \cos ax, \quad a \neq 0$$

$$\int \sec^2 ax \, dx = \frac{1}{a} \tan ax, \quad a \neq 0$$

$$\int \sec ax \, \tan ax \, dx = \frac{1}{a} \sec ax, \quad a \neq 0$$

$$\int \frac{1}{a^2 + x^2} dx = \frac{1}{a} \tan^{-1} \frac{x}{a}, \quad a \neq 0$$

$$\int \frac{1}{\sqrt{a^2 - x^2}} dx = \sin^{-1} \frac{x}{a}, \quad a > 0, \quad -a < x < a$$

$$\int \frac{1}{\sqrt{x^2 - a^2}} dx = \ln\left(x + \sqrt{x^2 - a^2}\right), \quad x > a > 0$$

$$\int \frac{1}{\sqrt{x^2 + a^2}} dx = \ln\left(x + \sqrt{x^2 + a^2}\right)$$

NOTE : $\ln x = \log_e x, \quad x > 0$