INTERNATIONAL GRAMMAR SCHOOL
Concordia per Diversitatem

MATHEMATICS EXTENSION 2

TRIAL HSC EXAMINATION

2003

Time allowed – 3 hours

(plus 5 minutes reading time)

DIRECTIONS

- Attempt ALL questions.
- All questions are of equal value.
- All necessary working should be shown in every question. Marks may be deducted for careless or poorly arranged work.
- Start each question in a new booklet.
- Board approved calculators may be used.
- A table of standard integrals is provided.

QUESTION 1 **MARKS**

(a) Evaluate $\int_2^3 \frac{3}{(1-x)^2} dx$. **2**

(b) Evaluate $\int_0^{\frac{\pi}{4}} \frac{\cos x}{\sin^2 x} dx$. **3**

(c) Find $\int \frac{1}{x^2+4x+5} dx$. **2**

(d) Evaluate $\int_0^1 \sin^{-1} x dx$. **4**

(e) Using the substitution $t = \tan\left(\frac{\theta}{2}\right)$, or otherwise, calculate **4**

$$\int_0^{\pi/2} \frac{1}{2+\cos\theta} d\theta .$$

QUESTION 2 *(Start a new booklet.)* **MARKS**

(a) If $z = -\sqrt{3} + i$, find:

(i) $\left| z^{-1} \right|$. **1**

(ii) $i^3 z$. **1**

(iii) $\left| \operatorname{Im} z^2 \right|$. **1**

(b) Simplify $\left(\cos\frac{\pi}{6} + i\sin\frac{\pi}{6}\right)^3$. **1**

QUESTION 2 *(continued.)* **MARKS**

(c) Give two separate sketches showing regions of the Argand diagram to represent complex numbers z which satisfy each of the following conditions.

(i) $-\frac{\pi}{6} < \arg(z-i) < \frac{\pi}{6}$. **2**

(ii) $\text{Re}\,z + \text{Im}\,z < 2$. **2**

(d) Let $z = \cos\theta + i\sin\theta$.

(i) Show that $z + z^{-1} = 2\cos\theta$. **1**

(ii) Simplify $z^2 - z^{-2}$. **2**

(e) Let $z_n = (1+i)^n$ where $0 < n < 20$.

(i) Express z_5 in modulus-argument form. **2**

(ii) For what values of n will z_n be purely imaginary ? **2**

QUESTION 3 *(Start a new booklet.)* **MARKS**

(a) If $x^3 + y^3 = 3xy$, use implicit differentiation to find $\frac{dy}{dx}$, expressing your answer in terms of x and y. **2**

(b) $y = f(x)$ and $y = g(x)$ are two functions where $g(x) = e^{f(x)}$.

(i) Show that for any stationary point on the graph of $y = f(x)$ there will be a corresponding stationary point on the graph of $y = g(x)$ with the same x coordinate. **2**

(ii) If $x = a$ corresponds to a stationary point for both functions, show that $f''(a)$ and $g''(a)$ will either both be zero or they will have the same sign. **2**

QUESTION 3 *(Continued.)* **MARKS**

(c) (i) Sketch together on the same coordinate axes graphs of $y = \sin x$ and $y = \sqrt{\sin x}$ for $0 \le x \le 2\pi$. **2**

(ii) Sketch together on the same coordinate axes graphs of $y = \cos x$ and $y = e^{\cos x}$ for $-\pi \le x \le \pi$. Give the coordinates of all turning points. **3**

(d) Let $\max(a, b)$ denote the maximum of the numbers a and b.

(i) Sketch the function $y = \max(2, x)$ over the interval $0 \le x \le 3$. **2**

(ii) Evaluate $\int_0^3 \max(2, x)\,dx$. **2**

QUESTION 4 *(Start a new booklet.)* **MARKS**

(a) (i) Write out the complex 5th roots of unity in modulus-argument form. **1**

(ii) If ω is a complex nth root of unity for $n > 2$, show that ω^2 is also an nth root of unity. **2**

(iii) If ω $(\neq 1)$ is a complex **5th** root of unity, show that **2**

$$\omega^8 + \omega^6 + \omega^4 + \omega^2 + 1 = 0.$$

(b) (i) $P(x)$ is a polynomial and α is one of repeated roots with multiplicity r $(r \ge 2)$. **2**
Show that the polynomial $P'(x)$ will also have α as a root with multiplicity $r - 1$.

(ii) Hence show that $P(z) = z^n - 1$ has no repeated roots. **2**

(c) $1+i$ is one of the solutions to the equation $z^3-4z^2+6z-4=0$. Find the other solutions. **2**

(d) If $z=\cos\theta+i\sin\theta$, show that

(i) $\dfrac{2z}{1+z^2}=\sec\theta$. **2**

(ii) $\dfrac{1}{1+z}=\dfrac{1}{2}\left(1-i\tan\dfrac{\theta}{2}\right)$. **2**

QUESTION 5 *(Start a new booklet.)* **MARKS**

(a) The polynomial $x^3+2x^2-3x-2=0$ has roots $\alpha,\ \beta$ and γ. Find the equation with roots $\alpha^2\beta\gamma$, $\alpha\beta^2\gamma$ and $\alpha\beta\gamma^2$. **3**

(b) Prove the following by induction, where n is any positive integer: **4**

$$1\times 1!+2\times 2!+3\times 3!+\ldots+n\times n! \;=\; (n+1)!-1.$$

(c) Sketch the locus of the point P representing the complex number z on an Argand diagram, if $|z-3|=2|z|$. **3**

(d) A tangent to the hyperbola $\dfrac{x^2}{a^2}-\dfrac{y^2}{b^2}=1$ at the point $P(x_1,y_1)$ has the equation $\dfrac{xx_1}{a^2}-\dfrac{yy_1}{b^2}=1$ and crosses the nearest directrix at point T. Point S is the corresponding focus.

(i) Sketch a diagram to illustrate this information. **1**

(ii) State the coordinates of S and give the equation of the corresponding directrix. **1**

(iii) Prove that angle PST is a right angle. **3**

QUESTION 6 *(Start a new booklet.)* **MARKS**

(a) The line $y = mx + k$ crosses the ellipse $3x^2 + 5y^2 = 15$ at points P and Q. Point M is the midpoint of PQ.

(i) Show that the roots of the following equation represent the x coordinates of P and Q. **2**

$$\left(5m^2 + 3\right)x^2 + 10mkx + 5k^2 - 15 = 0.$$

(ii) Find expressions for the coordinates of point M in terms of m and k. **2**

(iii) Show that all chords PQ with gradient $-\frac{1}{5}$ have midpoints which lie on the line $y = 3x$. **2**

(b) Write down the value of the following definite integrals:

(i) $\int_{-3}^{3} x\sqrt{9 - x^2}\,dx$. **1**

(ii) $\int_{-3}^{3} \sqrt{9 - x^2}\,dx$. **1**

(c) (i) By expanding $(\cos\theta + i\sin\theta)^3$ in two different ways, show that $\cos 3\theta = 4\cos^3\theta - 3\cos\theta$. **3**

(ii) Hence solve the equation $8x^3 - 6x - 1 = 0$. **3**

(iii) Hence show that $\cos\frac{\pi}{9} = \cos\frac{2\pi}{9} + \cos\frac{4\pi}{9}$. **1**

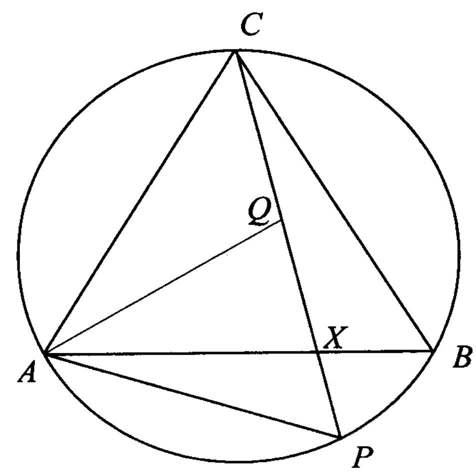
QUESTION 7 *(Start a new booklet.)* **MARKS**

(a) $P\left(ct, \frac{c}{t}\right)$ lies on the rectangular hyperbola $xy = c^2$.

(i) Show that the equation of the normal at P is $t^2x - y - ct^3 + \frac{c}{t} = 0$. **2**

(ii) Hence find the co-ordinates of the other point where this normal cuts the hyperbola. **2**

(b) ABC is an equilateral triangle inscribed in a circle and P is another point on the circumference. PC crosses AB at X.



(i) Prove that $\Delta CXB \parallel\!| \Delta CBP$. **2**

(ii) Find the size of $\angle APB$, giving reasons. **2**

(iii) Q is a point which lies on PC such that $AP = QP$. Find $\angle AQC$, giving reasons. **2**

(iv) Show that $CP = AP + PB$. **2**

(c) When the polynomial $P(x)$ is divided by $(x-1)$ the remainder is 4, and when it is divided by $(x-2)$, the remainder is 5. Find $P(x)$. **3**

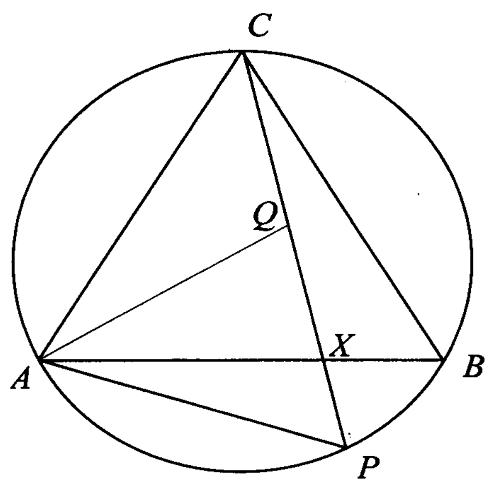
QUESTION 7 *(Start a new booklet.)* **MARKS**

(a) $P\left(ct, \frac{c}{t}\right)$ lies on the rectangular hyperbola $xy = c^2$.

(i) Show that the equation of the normal at P is $t^2x - y - ct^3 + \frac{c}{t} = 0$. **2**

(ii) Hence find the co-ordinates of the other point where this normal cuts the hyperbola. **2**

(b) ABC is an equilateral triangle inscribed in a circle and P is another point on the circumference. PC crosses AB at X.



(i) Prove that $\Delta CXB \parallel\!| \Delta CBP$. **2**

(ii) Find the size of $\angle APB$, giving reasons. **2**

(iii) Q is a point which lies on PC such that $AP = QP$. Find $\angle AQC$, giving reasons. **2**

(iv) Show that $CP = AP + PB$. **2**

(c) Find the number of ways to arrange 4 letters selected from the word DIVERSITY. **3**

QUESTION 8 *(Start a new booklet.)* **MARKS**

(a) If the functions $f(x)$ and $g(x)$ are such that $f(x) > g(x) \geq 0$ for $a \leq x \leq b$, by using a sketch (or otherwise) explain why $\int_a^b f(x)dx > \int_a^b g(x)dx$. **2**

(b) Let

$$u_n = \int_0^1 \left(1 - x^2\right)^{(n-1)/2} dx,$$

where $n = 0, 1, 2, \ldots$.

(i) Using integration by parts, or otherwise, show that $n u_n = (n-1)u_{n-2}$ for $n \geq 2$. **4**

(ii) Let $v_n = n u_n u_{n-1}$ for $n \geq 1$. Show by induction that $v_n = \frac{1}{2}\pi$ for all values of $n \geq 1$. **4**

(iii) Using part (a), or otherwise, show that $0 < u_n < u_{n-1}$ for $n \geq 1$. **2**

(iv) Hence prove that $\sqrt{\dfrac{\pi}{2n+2}} < u_n < \sqrt{\dfrac{\pi}{2n}}$ for $n \geq 1$. **3**