

53rd International Mathematical Olympiad, July 10-11, 2012, Argentina

1. Given triangle ABC the point J is the centre of the excircle opposite the vertex A . This excircle is tangent to the side BC at M , and to the lines AB and AC at K and L , respectively. The lines LM and BJ meet at F , and the lines KM and CJ meet at G . Let S be the point of intersection of the lines AF and BC , and let T be the point of intersection of the lines AG and BC . Prove that M is the midpoint of ST .

(The *excircle* of ABC opposite the vertex A is the circle that is tangent to the line segment BC , to the ray AB beyond B , and to the ray AC beyond C .)

2. Let $n \geq 3$ be an integer, and let a_2, a_3, \dots, a_n be positive real numbers such that $a_2 a_3 \cdots a_n = 1$. Prove that $(a_2 + 1)^2 (a_3 + 1)^3 \cdots (a_n + 1)^n > n^n$.
3. The *liar's guessing game* is a game played between two players A and B . The rules of the game depend on two positive integers k and n which are known to both players.

At the start of the game A chooses integers x and N with $1 \leq x \leq N$. Player A keeps x secret, and truthfully tells N to player B . Player B now tries to obtain information about x by asking player A questions as follows: each question consists of B specifying an arbitrary set S of positive integers (possibly one specified in some previous question), and asking A whether x belongs to S . Player B may ask as many questions as he wishes. After each question, player A must immediately answer it with *yes* or *no*, but is allowed to lie as many times as she wants; the only restriction is that, among any $k + 1$ consecutive answers, at least one answer must be truthful.

After B has asked as many questions as he wants, he must specify a set X of at most n positive integers. If x belongs to X , then B wins; otherwise, he loses. Prove that:

1. If $n \geq 2^k$, then B can guarantee a win.
2. For all sufficiently large k , there exists an integer $n \geq (1.99)^k$ such that B cannot guarantee a win.
4. Find all functions $f : \mathbb{Z} \rightarrow \mathbb{Z}$ such that, for all integers a, b, c that satisfy $a + b + c = 0$, the following equality holds:

$$f(a)^2 + f(b)^2 + f(c)^2 = 2f(a)f(b) + 2f(b)f(c) + 2f(c)f(a).$$

(Here \mathbb{Z} denotes the set of integers.)

5. Let ABC be a triangle with $\angle BCA = 90^\circ$, and let D be the foot of the altitude from C . Let X be a point in the interior of the segment CD . Let K be the point on the segment AX such that $BK = BC$. Similarly, let L be the point on the segment BX such that $AL = AC$. Let M be the point of intersection of AL and BK . Show that $MK = ML$.
6. Find all positive integers n for which there exist non-negative integers a_1, a_2, \dots, a_n such that

$$\frac{1}{2^{a_1}} + \frac{1}{2^{a_2}} + \cdots + \frac{1}{2^{a_n}} = \frac{1}{3^{a_1}} + \frac{2}{3^{a_2}} + \cdots + \frac{n}{3^{a_n}} = 1.$$