

## International Mathematical Olympiad, 2013

1. Prove that for any two positive integers  $k$  and  $n$ , there exist  $k$  positive integers  $m_1, m_2, \dots, m_k$  (not necessarily different) such that  $1 + \frac{2^k - 1}{n} = \left(1 + \frac{1}{m_1}\right) \left(1 + \frac{1}{m_2}\right) \cdots \left(1 + \frac{1}{m_k}\right)$ .
2. A configuration of 4027 points in the plane is called *Colombian* if it consists of 2013 red points and 2014 blue points, and no three of the points of the configuration are collinear. By drawing some lines, the plane is divided into several regions. An arrangement of lines is *good* for a Colombian configuration if the following two conditions are satisfied:
  - no line passes through any point of the configuration;
  - no region contains points of both colours.

Find the least value of  $k$  such that for any Colombian configuration of 4027 points, there is a good arrangement of  $k$  lines.

3. Let the excircle of triangle  $ABC$  opposite the vertex  $A$  be tangent to the side  $BC$  at the point  $A_1$ . Define the points  $B_1$  on  $CA$  and  $C_1$  on  $AB$  analogously, using the excircles opposite  $B$  and  $C$ , respectively. Suppose that the circumcentre of triangle  $A_1B_1C_1$  lies on the circumcircle of triangle  $ABC$ . Prove that triangle  $ABC$  is right-angled.

*The excircle of triangle  $ABC$  opposite the vertex  $A$  is the circle that is tangent to the line segment  $BC$ , to the ray  $AB$  beyond  $B$ , and to the ray  $AC$  beyond  $C$ . The excircles opposite  $B$  and  $C$  are similarly defined.*

4. Let  $ABC$  be an acute-angled triangle with orthocentre  $H$ , and let  $W$  be a point on the side  $BC$ , lying strictly between  $B$  and  $C$ . The points  $M$  and  $N$  are the feet of the altitudes from  $B$  and  $C$ , respectively. Denote by  $\omega_1$  the circumcircle of  $BWN$ , and let  $X$  be the point on  $\omega_1$  such that  $WX$  is a diameter of  $\omega_1$ . Analogously, denote by  $\omega_2$  the circumcircle of  $CWM$ , and let  $Y$  be the point on  $\omega_2$  such that  $WY$  is a diameter of  $\omega_2$ . Prove that  $X$ ,  $Y$  and  $H$  are collinear.
5. Let  $\mathbb{Q}_{>0}$  be the set of all positive rational numbers. Let  $f : \mathbb{Q}_{>0} \rightarrow \mathbb{R}$  be a function satisfying the following three conditions:
  - (i) for all  $x, y \in \mathbb{Q}_{>0}$ , we have  $f(x)f(y) \geq f(xy)$ ;
  - (ii) for all  $x, y \in \mathbb{Q}_{>0}$ , we have  $f(x+y) \geq f(x) + f(y)$ ;
  - (iii) There exists a rational number  $a > 1$  such that  $f(a) = a$ .

Prove that  $f(x) = x$  for all  $x \in \mathbb{Q}_{>0}$ .

6. Let  $n \geq 3$  be an integer, and consider a circle with  $n + 1$  equally spaced points marked on it. Consider all labellings of these points with the numbers  $0, 1, \dots, n$  such that each label is used exactly once; two such labellings are considered to be the same if one can be obtained from the other by a rotation of the circle. A labelling is called *beautiful* if, for any four labels  $a < b < c < d$  with  $a + d = b + c$ , the chord joining the points labelled  $a$  and  $d$  does not intersect the chord joining the points labelled  $b$  and  $c$ . Let  $M$  be the number of beautiful labellings and let  $N$  be the number of ordered pairs  $(x, y)$  of positive integers such that  $x + y \leq n$  and  $\gcd(x, y) = 1$ . Prove that  $M = N + 1$ .