International Mathematical Olympiad 2015

1. We say that a finite set $\mathcal{S}$ of points in the plane is *balanced* if, for any two different points A and B in $\mathcal{S}$, there is a point C in $\mathcal{S}$ such that $AC = BC$. We say that $\mathcal{S}$ is *centre-free* if for any three different points A, B and C in $\mathcal{S}$, there is no point P in $\mathcal{S}$ such that $PA = PB = PC$.

   (a) Show that for all integers $n \geq 3$, there exists a balanced set consisting of n points.

   (b) Determine all integers $n \geq 3$ for which there exists a balanced centre-free set consisting of n points.

2. Determine all triples (a, b, c) of positive integers such that each of the numbers $ab - c$, $bc - a$ and $ca - b$ is a power of 2.

   (*A power of* 2 *is an integer of the form* 2^n, *where* n *is a non-negative integer.*)

3. Let ABC be an acute triangle with $AB > AC$. Let Γ be its circumcircle, H its orthocentre, and F the foot of the altitude from A. Let M be the midpoint of BC. Let Q be the point on Γ such that $\angle HQA = 90°$, and let K be the point on Γ such that $\angle HKQ = 90°$. Assume that the points A, B, C, K and Q are all different, and lie on Γ in this order. Prove that the circumcircles of triangles KQH and FKM are tangent to each other.

4. Triangle ABC has circumcircle Ω and circumcentre O. A circle Γ with centre A intersects the segment BC at points D and E, such that B, D, E, and C are all different and lie on the line BC in this order. Let F and G be the points of intersection of Γ and Ω, such that A, F, B, C, and G lie on Ω in this order. Let K be the second point of intersection of the circumcircle of triangle BDF and the segment AB. Let L be the second point of intersection of the circumcircle of triangle CGE and the segment CA. Suppose that the lines FK and GL are different and intersect at point X. Prove that X lies on the line AO.

5. Let $\mathbb{R}$ be the set of real numbers. Determine all functions $f : \mathbb{R} \to \mathbb{R}$ satisfying the equation $f(x + f(x + y)) + f(xy) = x + f(x + y) + yf(x)$ for all real numbers x and y.

6. The sequence $a_1, a_2, \ldots$ of integers satisfies the conditions:

   (i) $1 \leq a_j \leq 2015$ for all $j \geq 1$;

   (ii) $k + a_k \neq \ell + a_\ell$ for all $1 \leq k < \ell$.

   Prove that there exist two positive integers b and N such that $\left|\sum_{j=m+1}^{n}(a_j - b)\right| \leq 1007^2$ for all integers m and n satisfying $n > m \geq N$.