INTERNATIONAL GRAMMAR SCHOOL
Concordia per Diversitatem

Name:.............................

INTERNATIONAL GRAMMAR SCHOOL

MATHEMATICS

Extension 2

YEAR 12

TRIAL EXAMINATION

31^{st} JULY, 2001

Time allowed —3 hours
(Plus 5 minutes reading time)

DIRECTIONS TO CANDIDATES

- Attempt ALL eight questions.
- ALL questions are of equal value.(8 @ 15 marks = 120 marks)
- All necessary working should be shown in every question. Marks may be deducted for careless or badly arranged work.
- Board-approved calculators may be used.
- Start each question on a *new page*. Number each question clearly.
- Label each page with your name.
- A table of Standard Integrals is attached.

YEAR 12 – TRIAL 2001 – EXTENSION 2

QUESTION 1 (Start a new page)

MARKS

a) Find $\int \frac{\sec^2 x}{\sqrt{1-\tan^2 x}}\,dx$ — 2

b) i) Find a, b and c such that — 2

$$\frac{16}{(x^2+4)(2-x)} = \frac{ax+b}{x^2+4} + \frac{c}{2-x}$$

ii) Find $\int \frac{16}{(x^2+4)(2-x)}\,dx$ — 2

c) Find $\int \frac{\ln x}{x^2}\,dx$ — 4

d) Use the substitution $t = \tan\frac{\theta}{2}$ to show that — 5

$$\int_0^{\frac{\pi}{2}} \frac{d\theta}{4\sin\theta - 2\cos\theta + 6} = \frac{1}{2}\tan^{-1}\left(\frac{1}{2}\right)$$

QUESTION 2 (Start a new page)

MARKS

a) The complex number Z moves such that $\operatorname{Im}\left(\frac{1}{\overline{Z}-i}\right) = 1$. — 3

Show that the locus of Z is a circle and find its centre and radius.

b) i) Find the square root of the complex number $5 - 12i$ — 2

ii) Given that $Z = \frac{1+\sqrt{5-12i}}{2+2i}$ and is purely imaginary, — 2

find Z^{400}

c) i) Shade the region in the argand diagram containing all points representing the complex numbers Z such that — 3

$|Z-1-i| \le 1$ and $-\frac{\pi}{4} \le \operatorname{Arg}(Z-i) \le \frac{\pi}{4}$

ii) Let ϕ be the complex number of minimum modulus satisfying the inequalities of i). — 1

Express ϕ in the form $x + yi$

d) Express $\phi = \frac{-1+i}{\sqrt{3}+i}$ in modulus / argument form. — 4

Hence, evaluate $\cos\frac{7\pi}{12}$ in surd form.

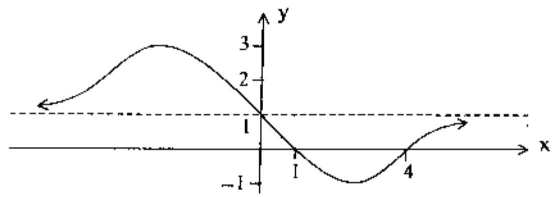
QUESTION 3 *(Start a new page)* — **MARKS**

a) Consider the equation $x^3 + 7x - 6i = 0$.

i) Given that this equation has no purely real root, show that none of the roots is a conjugate of any of the others. 1

ii) If $2i$ is one of the roots and the other two roots are purely imaginary, find the other two roots. 2

b)



The above diagram shows the graph of $y = f(x)$.
Sketch on separate diagrams the following curves, indicating clearly any turning points and asymptotes.

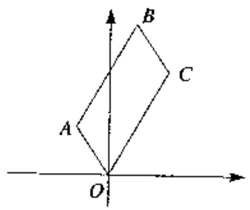
i) $y = \dfrac{1}{f(x)}$ 2

ii) $y = f(|x|)$ 2

iii) $y = \ln f(x)$ 2

iv) $y = \sin^{-1}(f(x))$ 3

c)



In the diagram above, $OABC$ is a parallelogram with $OA = \frac{1}{2}OC$.

The point A represents the complex number $-\frac{1}{2} + i\frac{\sqrt{3}}{2}$.

If $\angle AOC = 60°$, what complex number does C represent?

QUESTION 4 *(Start a new page)* — **MARKS**

a) Factorise $P(x) = x^4 - 5x^3 + 4x^2 + 2x - 8$ over
(i) R (all real numbers)
(ii) C (all complex numbers) 3

b) Write down all polynomials that have degree 4, 3 as a single zero and -1 as a zero of multiplicity 3. 1

c) If $\alpha, \beta, \wp$ are the roots of $x^3 - 2x^2 + x + 3 = 0$ evaluate:

(i) $\alpha^2 + \beta^2 + \wp^2$ (ii) $\alpha^3 + \beta^3 + \wp^3$ 3

d) If $\alpha, \beta, \wp$ are the roots of $x^3 + 2x^2 - 2x + 3 = 0$ form the equation whose roots are:

(i) $2\alpha, 2\beta, 2\wp$ (ii) $\alpha^2, \beta^2, \wp^2$ 4

e) The roots of the polynomial $P(x) = x^3 + ax^2 + bx + c = 0$ are in arithmetic progression. Show that the relationship between the coefficients of $P(x)$ is $2a^3 = 9ab - 27c$ 3

f) Prove that if α is a root of multiplicity r of $P(x)$ then it is a root of multiplicity $(r-1)$ of $P'(x)$. 1

QUESTION 5 *(Start a new page)* **MARKS**

a) i) Show that the equation of the chord of contact of the tangents from a point (x_0, y_0) to the rectangular hyperbola $xy = c^2$ is $xy_0 + x_0y = 2c^2$. **5**

ii) Hence find the chord of contact of the tangents from the point (2,1) to the hyperbola $xy = 4$ and determine the points of contact.

b) i) Show that the condition for the line $y = mx + c$ to be a tangent to the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ is $c^2 = a^2m^2 + b^2$. **5**

ii) Hence show that the pair of tangents from the point (3,4) to the ellipse $\frac{x^2}{16}+\frac{y^2}{9}=1$ are at right angles to one another.

c) i) Show that the equation of the normal to the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ at the point $P(a\sec\theta, b\tan\theta)$ is $a\sin\theta\, x + by = (a^2+b^2)\tan\theta$. **5**

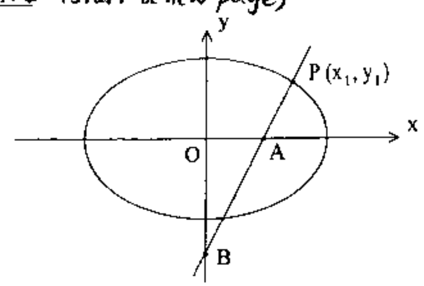
ii) The normal at the point $P(a\sec\theta, b\tan\theta)$ on the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ meets the x-axis at G.

PN is the perpendicular from P to the x-axis.

Prove that $OG = e^2 \cdot ON$, where O is the origin.

QUESTION 6 *(Start a new page)* **MARKS**

a)



The point $P(x_1, y_1)$, where $x_1 > 0$ and $y_1 > 0$, lies on the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$. The normal at P intersects the x axis at A and the y axis at B.

i) Show that the equation of the normal is $\frac{a^2x}{x_1}-\frac{b^2y}{y_1}=a^2-b^2$ **3**

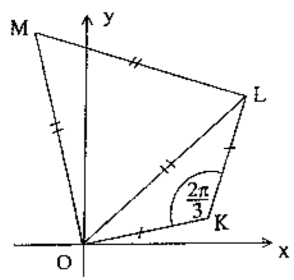
ii) Explain why the point A cannot be the focus of the ellipse. **2**

iii) Find the ratio in which A divides the interval BP internally. **2**

iv) Find the midpoint M of AB in terms of x_1 and y_1. **1**

v) Given that H divides the interval OM in the ratio 4:1, show that the locus of H is an ellipse. **3**

b)



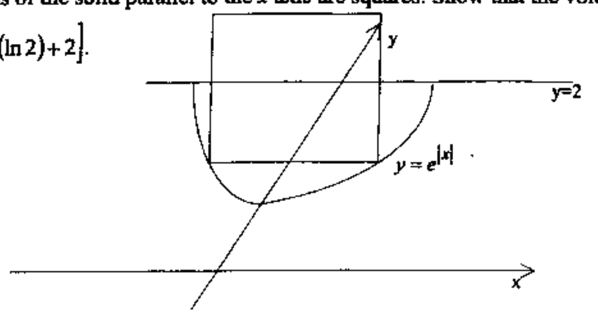
The points K and M in a complex plane represent the complex numbers α and β respectively. The triangle OKL is isosceles and $\angle OKL = \frac{2\pi}{3}$. The triangle OLM is equilateral. **4**

Show that $3\alpha^2+\beta^2=0$

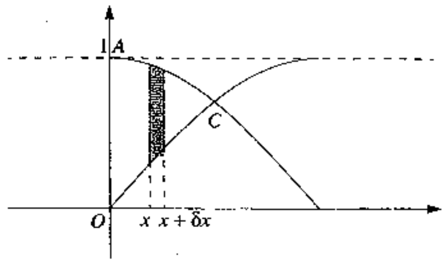
QUESTION 7 (Start a new page)

MARKS

a) The base of a solid is formed by the segment cut off by the line $y = 2$ of the curve $y = e^{|x|}$. Cross sections of the solid parallel to the x axis are squares. Show that the volume is given by $4\left[2(\ln 2)^2 - 4(\ln 2) + 2\right]$. **8**



b) The diagram below shows part of the graphs of $y = \cos x$ and $y = \sin x$. The graph of $y = \cos x$ meets the y axis at A, and the C is the first point of intersection of the two graphs to the right of the y axis.



The region OAC is to be rotated about the line $y = 1$.

(i) Write down the coordinates of the point C. **1**

(ii) The shaded strip of width δx shown in the diagram is rotated about the line $y = 1$. Show that the volume δV of the resulting slice is given by **3**

$$\delta V = \pi(2\cos x - 2\sin x + \sin^2 x - \cos^2 x)\delta x.$$

(iii) Hence evaluate the total volume when the region OAC is rotated about the line $y = 1$. **3**

QUESTION 8 (Start a new page)

MARKS

a) Let $I_n = \int_{\frac{\pi}{6}}^{\frac{\pi}{2}} \operatorname{cosec}^n x \, dx$, where n is a positive integer.

i) Using integration, show that 4

$$(n-1)\, I_n = 2^{n-2}\sqrt{3} + (n-2)\, I_{n-2}$$

ii) Evaluate $J = \int_0^{\frac{\pi}{3}} \sec^4 x \, dx$ **3**

b) Consider the polynomial $x^5 - i = 0$

i) Show that $1 - ix - x^2 + ix^3 + x^4 = 0$ for $x \neq i$ 2

ii) Show that 4

$$(x - i)\left(x^2 - 2i\sin\frac{\pi}{10}x - 1\right)\left(x^2 + 2i\sin\frac{3\pi}{10}x - 1\right) = 0$$

iii) Show that $\sin\frac{\pi}{10}\sin\frac{3\pi}{10} = \frac{1}{4}$ 2