Student Number:	
Class:	

JAMES RUSE A H S GESTA NON VERBA

TRIAL HIGHER SCHOOL CERTIFICATE EXAMINATION 2014

MATHEMATICS EXTENSION 2

General Instructions:

- Reading Time: 5 minutes.
- Working Time: 3 hours.
- Write in black pen.
- Board approved calculators & templates may be used
- A Standard Integral Sheet is provided.
- In Question 11 - 16, show all relevant mathematical reasoning and/or calculations.
- Marks may not be awarded for careless or badly arranged working.

Total Marks 100

Section I: 10 marks

- Attempt Question 1 – 10.
- Answer on the Multiple Choice answer sheet provided.
- Allow about 15 minutes for this section.

Section II: 90 Marks

- Attempt Question 11 - 16
- Answer on lined paper provided. Start a new page for each new question.
- Allow about 2 hours & 45 minutes for this section.

The answers to all questions are to be returned in separate *stapled* bundles clearly labelled Question 11, Question 12, etc. Each question must show your Candidate Number.

Section I

10 marks Attempt Questions 1–10
Allow about 15 minutes for this section
Use the multiple-choice answer sheet for Questions 1–10.

1. If $z = 1+2i$ and $w = 3-i$, what is the value of $z - \overline{w}$?

(A) $3i - 2$

(B) $4 + 3i$

(C) $i - 2$

(D) $4 + i$

2. Which of the following is an expression for? $\int \frac{\cos^3 x + \sin^3 x}{\cos x + \sin x} dx$

(A) $x + \frac{1}{2}\cos 2x + c$

(B) $x - \frac{1}{2}\cos 2x + c$

(C) $x + \frac{1}{2}\sin^2 x + c$

(D) $x - \frac{1}{2}\sin^2 x + c$

3. The equation of the tangent to $xy^3 + 2y = 4$ at the point (2, 1) is

(A) x+8y=10

(B) x-8y=10

(C) x+8y=-10

(D) x-8y=-10

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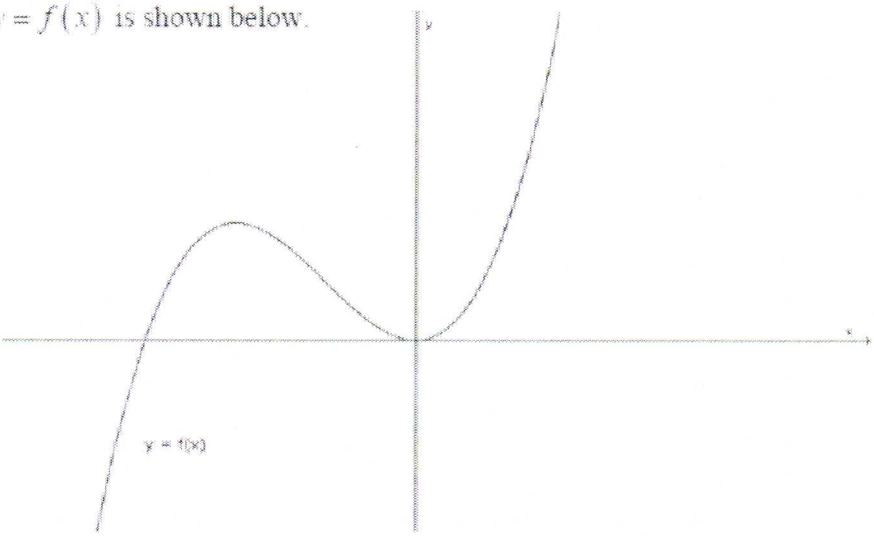
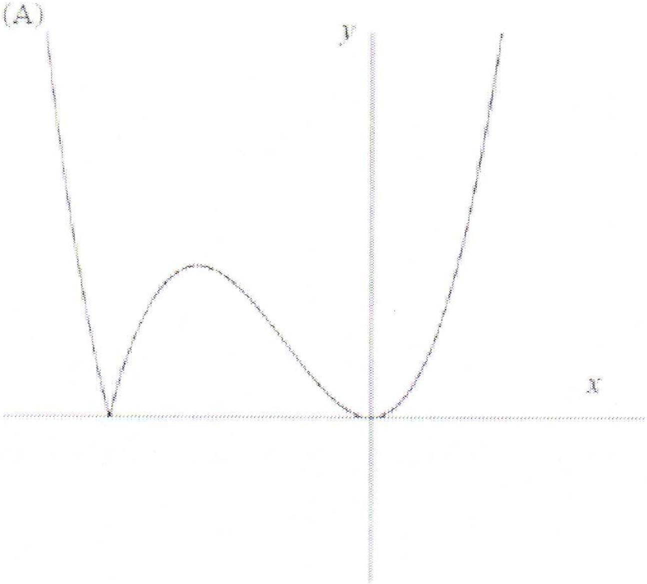
4.

The graph of $y = f(x)$ is shown below.



Which of the following graphs best represents $y = f(|x|)$?

(A)



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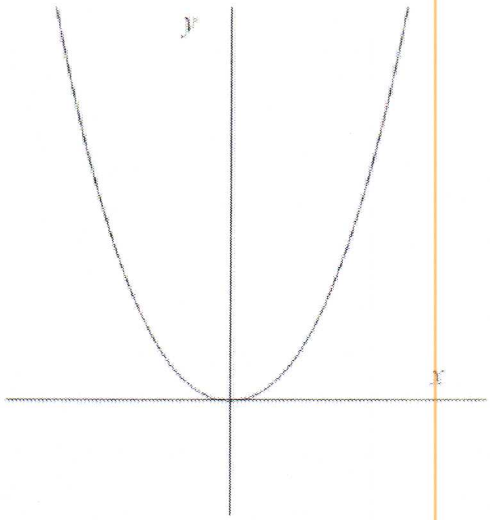
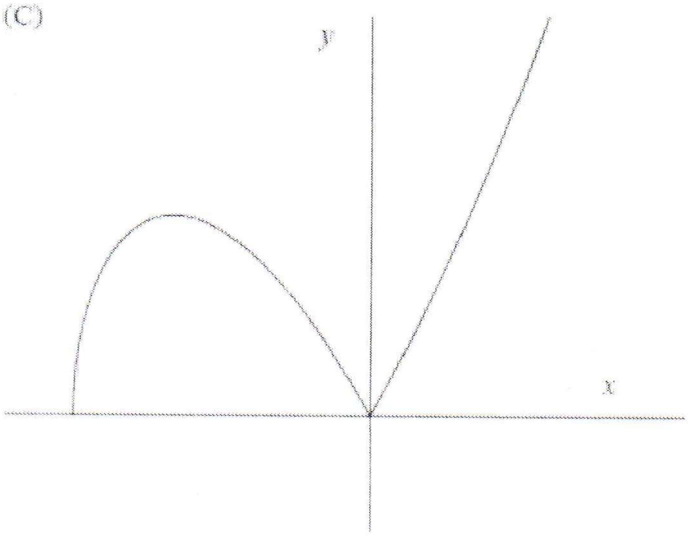
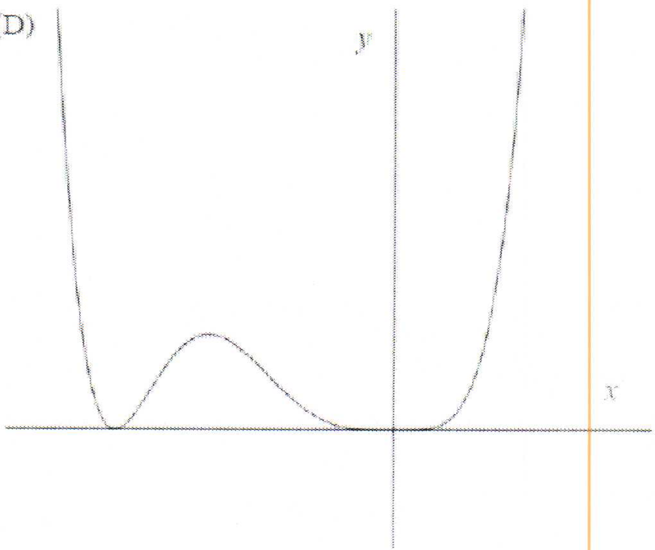


(C)



(D)



5. The point $P(a\cos\theta, b\sin\theta)$ lies on the ellipse $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}=1$ where a>b>0

What is the equation of the normal at P?

(A) $\dfrac{ax}{\cos\theta}-\dfrac{by}{\sin\theta}=a^2-b^2$

(B) $\dfrac{ax}{\cos\theta}+\dfrac{by}{\sin\theta}=a^2+b^2$

(C) $\dfrac{x}{a}\sec\theta-\dfrac{y}{b}\tan\theta=1$

(D) $\dfrac{x}{a}\cos\theta+\dfrac{y}{b}\sin\theta=1$

6. What is the multiplicity of the root x=1 of the equation $3x^5-5x^4+5x-3=0$

(A) 1

(B) 2

(C) 3

(D) 4

7. Without evaluating the integrals which one of the following will give an answer of zero?

(A) $\displaystyle\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}}\frac{\cos^3\theta+1}{\cos^2\theta}\,d\theta$

(B) $\displaystyle\int_{-1}^{1}(x^2-1)(1-x^2)^3\,dx$

(C) $\displaystyle\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\sin^7 x\cos x\,dx$

(D) $\displaystyle\int_{-2}^{2}\left|x^2-4\right|dx$

8. The base of a solid is the circle $x^2 + y^2 = 1$. Every cross section of the solid taken perpendicular to the x axis is a right angled, isosceles triangle with its hypotenuse lying in the base of the solid. Which of the following is an expression for the volume V of the solid?

(A) $\int_{-1}^{1} (1 - x^2)\, dx$

(B) $2\int_{-1}^{1} (1 - x^2)\, dx$

(C) $4\int_{-1}^{1} (1 - x^2)\, dx$

(D) $\frac{1}{2}\int_{-1}^{1} (1 - x^2)\, dx$

9. A particle of mass m is moving horizontally in a straight line. Its motion is opposed by a force of magnitude mk $(v+v^2)$ Newtons when its speed is v ms^{-1} (where k is a positive constant). At time t seconds the particle has displacement x metres from a fixed point O on the line and velocity v ms^{-1}. Which of the following is an expression for x in terms of v?

(A) $\frac{1}{k}\int \frac{1}{1+v} dv$

(B) $\frac{1}{k}\int \frac{1}{v(1+v)} dv$

(C) $-\frac{1}{k}\int \frac{1}{v(1+v)} dv$

(D) $-\frac{1}{k}\int \frac{1}{1+v} dv$

10 Four digit numbers are formed from the digits 1, 2, 3 and 4.
Each digit is used once only. The sum of all the numbers that can be formed is?

(A) 266,640

(B) 66,660

(C) 44,440

(D) 6,666

Section II

90 marks Attempt Questions 11–16

Allow about 2 hours and 45 minutes for this section

Answer each question on a SEPARATE sheet of paper. Extra paper is available.
In Questions 11–16, your responses should include relevant mathematical reasoning and/or calculations.

Question 11

a) Find $\int \frac{dx}{\sqrt{6x - x^2}}$ **2**

b) Show that $\int_2^4 \frac{dx}{x\sqrt{x-1}} = \frac{\pi}{6}$ **3**

c) Find $\int e^x \sin x \, dx$ **3**

d) Draw a diagram to illustrate the locus of the points z on the Argand diagram such that:

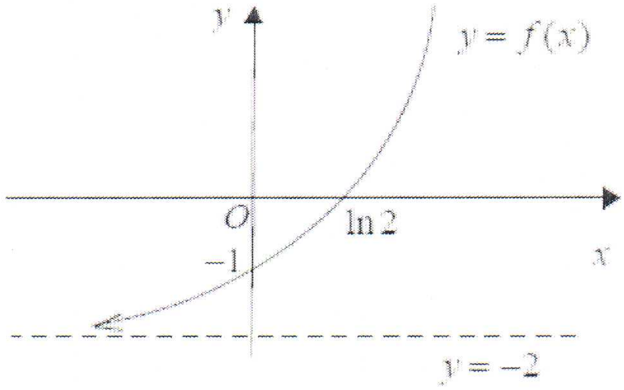
(i) $|z - \bar{z}| \leq 1$ and $|z - 1| \leq 2$ **2**

(ii) $Arg\left\{\frac{z-1}{z+1}\right\} = \frac{\pi}{4}$ **2**

e) Show by geometrical means or otherwise that if z_1 and z_2 are complex numbers such that $|z_1| = |z_2|$, then $\frac{z_1 + z_2}{z_1 - z_2}$ is purely imaginary. **3**

Question 12 (Start a new page)

(a) The diagram below shows the graph of $f(x) = e^x - 2$



On separate diagrams sketch the following graphs, in each case showing the intercepts on the axes and the equations of the asymptotes.

(i) $y = (f(x))^2$ 1

(ii) $y = \log_e f(x)$ 1

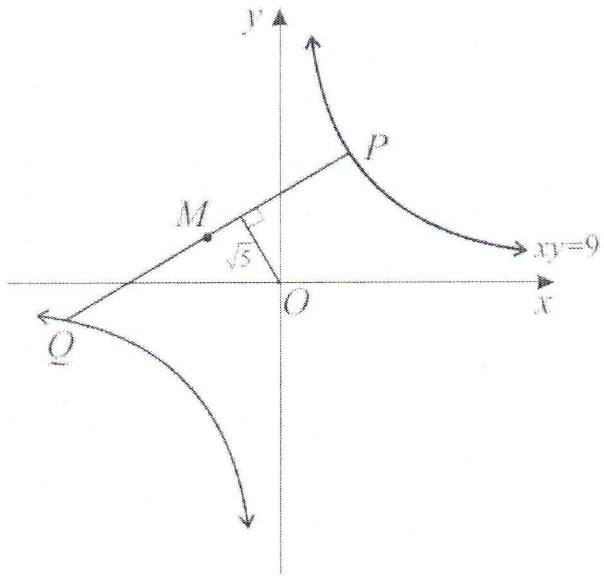
(iii) $y = \dfrac{1}{f(x)}$ 2

(iv) $y^2 = |f(x)|$ 2

(b) (i) Show that $4x^2 + 9y^2 + 16x + 18y - 11 = 0$ represents an ellipse. 1

(ii) Find the eccentricity and hence the coordinates of its foci and the equation of its directrices. 2

(c)



In the diagram above, $P\left(3p, \frac{3}{p}\right)$ and $Q\left(3q, \frac{3}{q}\right)$ are variable points on the rectangular hyperbola
$xy = 9$. The perpendicular distance from the origin to the chord PQ is $\sqrt{5}$ units. Let M be the midpoint of the chord PQ.

(i) Show that the chord PQ has the equation $x+pqy=3(p+q)$ **2**

(ii) Using the perpendicular distance formula, or otherwise, show that $9(p+q)^2 = 5(1+p^2q^2)$ **1**

(iii) Show that the locus of M has the Cartesian equation **3**

$$y^2 = \frac{5x^2}{4x^2 - 5}$$

Question 13 (Start a new page)

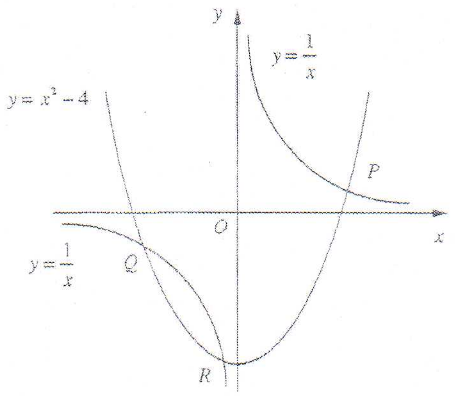
(a) (i) Use the substitution $x=10\sqrt{2}\sin\theta$ to show that $\int_{-10}^{10}\sqrt{200-x^2}\,dx = 100+50\pi$ **3**

(ii) Use a geometrical argument to verify the result in part (i) **1**

(iii) A mould for a model railway tunnel is made by rotating the region bounded by the curve $y=\sqrt{200-x^2}$ and the x-axis between the lines x=-10 and x=10 through 180° about the line x=100 (where all the measurements are in cm).
Use the method of cylindrical shells to show that the volume $V\text{cm}^3$ of the tunnel is given by $\pi\int_{-10}^{10}(100-x)\sqrt{200-x^2}\,dx$. **4**

Hence find the volume of the tunnel in m^3 correct to 2 significant figures.

(b)



The curves $y=x^2-4$ and $y=\dfrac{1}{x}$ intersect at the points P, Q, R where $x=\alpha, x=\beta, x=\gamma$

(i) Show that α,β,γ are the roots of the equation $x^3-4x-1=0$ **1**

(ii) Find a polynomial equation with integer coefficients which has roots $\alpha^2,\beta^2,\gamma^2$. **2**

(iii) Find a polynomial equation with integer coefficients which has roots $\dfrac{1}{\alpha^2},\dfrac{1}{\beta^2},\dfrac{1}{\gamma^2}$. **2**

(iv) Hence find the numerical value of $OP^2+OQ^2+OR^2$ **2**

Question 14 (Start a new page)

(a) Figure 1 below shows a scale model of the volcano Mt Snaefellsjökull
The base of the model is elliptical in shape with the axes 60cm by 40cm reducing uniformly to a circle of radius 12cm at the top.
The hollow core of the model has circular cross sections with a circle of radius 6cm at the base rising uniformly to a circle of radius 12cm at the top.
The model is 24cm high.
Figure 2 shows the top view of the cross sectional area of the volcano

Figure 1

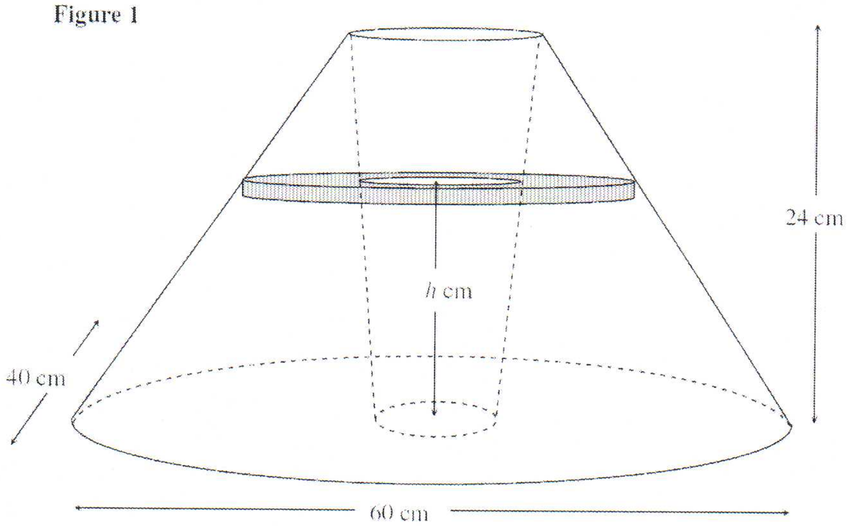
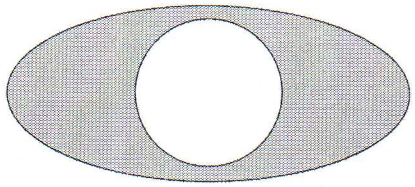


Figure 2



(i) Show that at height h, the length of the semi-major axis is given by **2**

$$a = 30 - \frac{3}{4}h$$

(ii) Show that the cross sectional slice at height h is given by **3**

$$A = \frac{\pi}{16}(9024 - 448h + 3h^2)$$

You can assume the area of an ellipse with semi-major axis a and semi-minor axis b is given by πab

(iii) Find the volume of the scale model of Mt Snaefellsjökull **1**

Question 14 continued on next page

Question 14 continued

(b) (i) Show that if n is an even positive integer, then **2**

$$(1+x)^n + (1-x)^n = 2\sum_{k=0}^{n/2} \binom{n}{2k} x^{2k}$$

(ii) An alphabet consists of three letters A, B and C

(I) Show that the number of words of 4 letters containing exactly 2B's is **1**

$$\binom{4}{2} \times 2^2$$

(II) Hence, or otherwise, show that if n is an even positive integer then the number of words of n letters with zero or an even number of B's is given by **2**

$$\frac{1}{2}(3^n + 1)$$

(c) Three men Bill, Garry and Jason observe a vertical tower.
Bill stands due North of the tower, and sees its top at an angle of elevation of $\alpha°$
Garry stands due East of the tower, and sees its top at an angle of elevation of $\beta°$
Jason stands on a line from Bill to Garry, exactly half way between them.

If Jason observes the top of the tower at an angle of elevation of $\theta°$.

Show that $\cot\theta° = \frac{1}{2}\sqrt{\cot^2\alpha° + \cot^2\beta°}$

Question 15 (Start a new page)

(a) Let $I_n = \int_1^2 \left(1 - \frac{1}{x}\right)^n dx$ for $n = 1,2,3,.......$

(i) Show that $\frac{1}{n+1} I_{n+1} = \frac{1}{n} I_n - \frac{1}{n(n+1)2^n}$ for $n = 1,2,3,.....$ **3**

(ii) Hence show that $\frac{1}{n+1} I_{n+1} = I_1 - \sum_{r=1}^{n} \frac{1}{r(r+1)2^r}$ **2**

(i) Show that $\sum_{r=1}^{n} \frac{1}{r(r+1)2^r} = (1 - \log_e 2) - \frac{1}{n+1} I_{n+1}$ and hence find the limiting sum of the series $\frac{1}{1 \text{x} 2 \text{x} 2^1} + \frac{1}{2 \text{x} 3 \text{x} 2^2} + \frac{1}{3 \text{x} 4 \text{x} 2^3} + ...$ **3**

(b) Let α be the complex root of the polynomial $z^7 = 1$ with the smallest positive argument.

Let $\theta = \alpha + \alpha^2 + \alpha^4$ and $\phi = \alpha^3 + \alpha^5 + \alpha^6$

(i) Show that $\theta + \phi = -1$ and $\theta\phi = 2$ **3**

(ii) Write a quadratic equation whose roots are θ *and* ϕ **2**

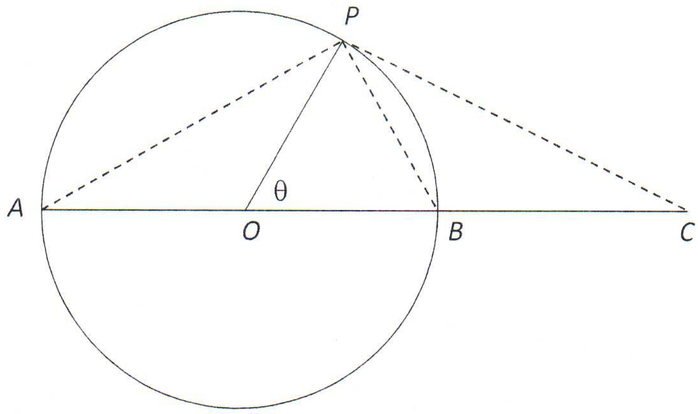
Hence show that $\theta = -\frac{1}{2} + \frac{i\sqrt{7}}{2}$ and $\phi = -\frac{1}{2} - \frac{i\sqrt{7}}{2}$

(iii) Show that $\cos\frac{2\pi}{7} + \cos\frac{4\pi}{7} - \cos\frac{\pi}{7} = -\frac{1}{2}$ **2**

Question 16 (Start a new page)

(a) The diagram below shows a point P rotating in a circle of radius 1 metre, whose centre is at O.

AB is a diameter produced to C such that $OC = 2$ metres.



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The angular velocity of P about O is given by $\dot{\theta} = \pi$ rad/sec. ($\theta = \angle POB$)

(i) Find the angular velocity of P about A and about B. **2**

(ii) If $\angle PCO = \alpha$, show that $\sin(\alpha + \theta) = 2\sin\alpha$. **1**

(iii) Hence find the angular velocity of P about C at the instant when $\theta = \frac{\pi}{2}$. **4**

(b) The circular bend on a bike track has a constant radius of 20 metres and is banked at a constant angle of 30° to the horizontal. A bicycle rider can safely negotiate the bend if the maximum sideways thrust F, up or down the slope is at most one-tenth of the normal reaction N. By resolving the forces vertically and horizontally, show that the range of speeds V, correct to two decimal places and in metres per second, at which the bend can be safely negotiated, is $9.50 \le V \le 11.99$. Take $g = 10\text{m}/\text{s}^2$. **8**