Question 1 (15 Marks) Marks

(a) Find the modulus and principle argument of z if $z = 1 + i\sqrt{3}$. Hence find the smallest positive integer n such that z^n is a real number. 3

(b) (i) Find both complex roots of $\omega^2 = 3 + 4i$. 3

(ii) Hence find the complex roots of the equation $z^2 + (4 - 2i)z - 8i = 0$. 3

(c) Find all the complex numbers that satisfy $|z|^2 - iz = 36 + 4i$. 4

(d) Find $\int \frac{1}{x^2 + 2x + 5}\, dx$. 2

Question 2 (15 Marks) START A NEW PAGE Marks

(a) Hot liquid metal is poured into a cooling vat. The initial temperature, T_1, of the metal is $900°C$ and the initial temperature, T_2, of the vat is $36°C$. The temperatures of the metal and vat satisfy Newton's law of cooling according to the formulae

$\frac{dT_1}{dt} = -k(T_1 - T_2)$ and $\frac{dT_2}{dt} = 0.8k(T_1 - T_2)$ where $k > 0$.

(i) Show that $\frac{4}{5}\frac{dT_1}{dt} + \frac{dT_2}{dt} = 0$ and hence prove that $\frac{4}{5}T_1 + T_2 = 756$. 3

(ii) Find a formula for $\frac{dT_1}{dt}$ in terms of T_1 and hence show that $T_1 = 420 + Ae^{-\frac{9k}{5}t}$, where A and k are constants, satisfies this differential equation. 3

(iii) Four hours after the metal is poured into the vat, the metal has cooled to $500°C$. Find the temperature of the metal after a further 2 hours. Give your answer correct to the nearest degree. 3

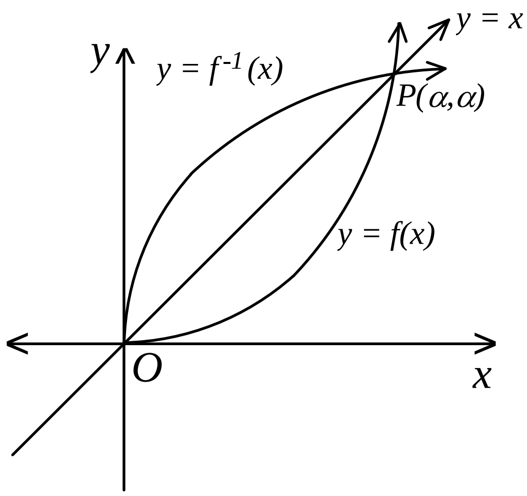
(b) (i) The area bounded by the graph of $y = x\sqrt{4 - x^2}$ and the x-axis is rotated one revolution about the y-axis to form a solid. Show, with the aid of a sketch of the curve $y = x\sqrt{4 - x^2}$, and using the method of Volume by Cylindrical Shells, that the volume, $V\ u^3$, of the solid is given by: 2

$$V = 4\pi \int_0^2 x^2 \sqrt{4 - x^2}\, dx.$$

(ii) Using the substitution $x = 2\sin\theta$, find the volume of the solid that is formed. 4

Question 3 (15 Marks) START A NEW PAGE Marks

(a) The diagram shows the graph of an increasing function $y = f(x)$ and its inverse $y = f^{-1}(x)$ for $x \geq 0$, and the line $y = x$. The graphs intersect at $(0,0)$ and $P(\alpha, \alpha)$. 4



Use the substitution $u = f^{-1}(x)$ to show that $\int_0^{\alpha} f^{-1}(x)\,dx = \int_0^{\alpha} u f'(u)\,du$ and hence show that the area , A square units, bounded by $y = f(x)$ and $y = f^{-1}(x)$ for $x \geq 0$ is given by

$$A = \int_0^{\alpha} \{x f'(x) - f(x)\}\,dx .$$

(b) For the curve $y = x\tan^{-1} x$.

(i) For what values of x is $y = x\tan^{-1} x$ an increasing function? 1

(ii) Show that $y = x\tan^{-1} x$ is concave up for all real values of x. 2

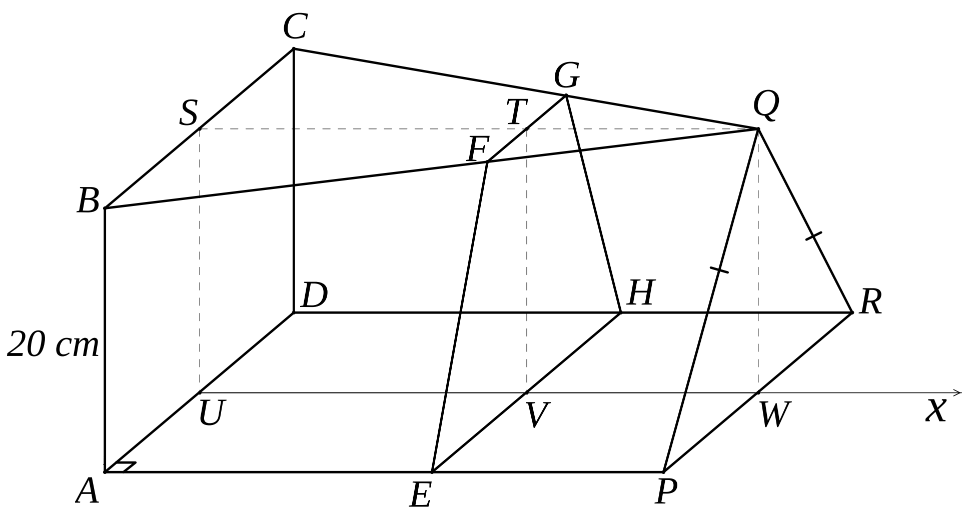
(iii) Sketch the graph of $y = x\tan^{-1} x$ for $-2.5 \leq x \leq 2.5$, clearly showing any stationary points. 2

(iv) If $f(x) = x\tan^{-1} x$, $x \geq 0$, show on a new diagram the graphs of $y = f(x)$ and its inverse $y = f^{-1}(x)$, and the line $y = x$. Clearly label the coordinates of their points of intersection. 3

(v) With the aid of the result from Q3(a), find the area of the region bounded by $y = f(x)$ and $y = f^{-1}(x)$. 3

Question 4 (15 Marks) START A NEW PAGE Marks

(a)



$UV = h$ cm and $UW = 100$ cm

The above solid has a square face (*ABCD*) at one end and a triangular face (*PQR*) that is parallel to it at the other end. The solid is 100 cm long. The square face has sides of length 20 cm and is perpendicular to the base (*ADRP*) while the triangular face is isosceles with $QP = QR$. The triangular face *BCQ* and rectangular face *ADRP* are parallel.

(i) If the length $UV = h$ cm, show that the area, A cm^2, of the cross-section *EFGH* is given by $A = 400 - 2h$. **2**

(ii) Hence calculate the volume of the above solid. **3**

(b) The total surface area, A cm^2, of a cylinder is given by $A = 2\pi r(r+h)$, where r cm is the radius of the base and h cm is the height. Find the rate at which the surface area is changing if the radius of the base is increasing at 0.2 cm/minute and the height is decreasing at 0.5 cm/minute when the base radius equals 16 cm and the height equals 20 cm. **3**

(c) (i) If $y = x^k + (c-x)^k$, where $c > 0$, $k > 0$, $k \neq 1$, show that y has a single stationary value between $x = 0$ and $x = c$, and show that this stationary value is a maximum if $0 < k < 1$ and a minimum if $k > 1$. **3**

(ii) Hence show that if $a > 0$, $b > 0$, $a \neq b$, then $\dfrac{a^k + b^k}{2} < \left(\dfrac{a+b}{2}\right)^k$ if $0 < k < 1$ and $\dfrac{a^k + b^k}{2} > \left(\dfrac{a+b}{2}\right)^k$ if $k > 1$. **4**

Question 5 (15 Marks) START A NEW PAGE Marks

A toy soldier of mass 500 grams has a parachute attached to it. The toy soldier is released from rest at a position 60 *m* above the ground. During its fall the forces acting on the toy are gravity and, owing to the parachute, a resistance force of magnitude $\frac{1}{80}v^2$ when the velocity of the toy is $v\ m/s$. After $5\ln 2$ seconds the parachute fails and from then on the only force acting on the toy is gravity. The acceleration due to gravity is taken as 10 ms^{-2} and at time *t* seconds the toy has fallen $x\ m$ from its release point and its velocity is $v\ m/s$.

(i) With the aid of a force diagram show that while the parachute is operating, $\ddot{x} = 10 - \frac{1}{40}v^2$. 1

(ii) Show that $v = 20\left(\frac{e^t - 1}{e^t + 1}\right)$. 4

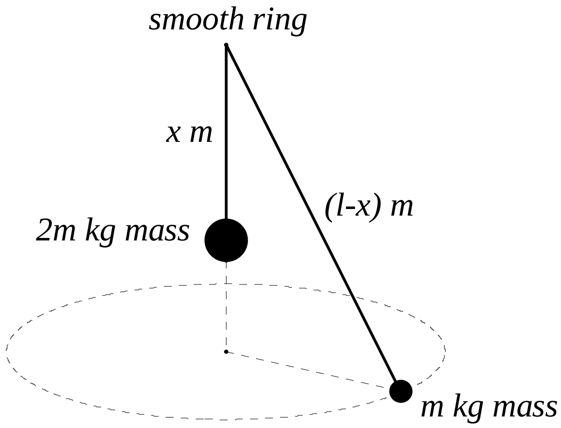
(iii) Show that $x = -20\ln\left[1 - \left(\frac{v}{20}\right)^2\right]$. 4

(iv) Find the exact speed of the toy and the exact distance fallen at the instant the parachute fails. 3

(v) After the parachute fails, find an expression for the acceleration $\ddot{x}$ and hence find the speed of the toy the instant it hits the ground. Give your answer correct to two significant figures. 3

Question 6 (15 Marks) START A NEW PAGE Marks

(a) Two particles of masses $2m$ kg and m kg respectively are connected to a light inextensible string of length l m. The string passes through a fixed smooth ring. While the $2m$ kg mass hangs in equilibrium a distance x m below the ring, the other mass describes a horizontal circle with angular velocity ω rad/sec. (Take the acceleration due to gravity as g m/s^2)

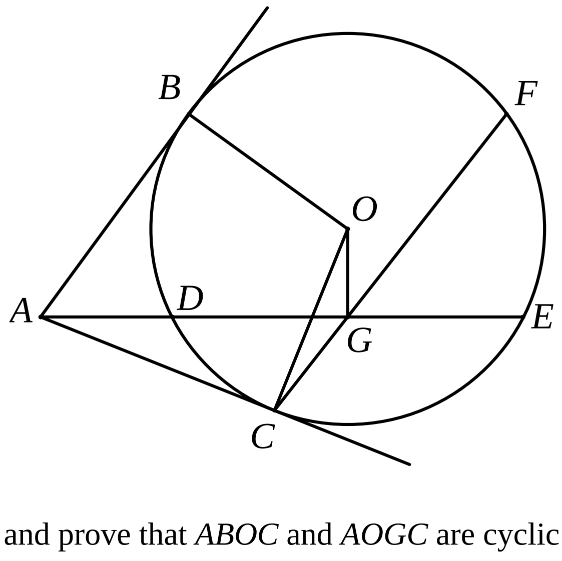


(i) Draw a diagram showing the forces acting on each mass. 2

(ii) Find the angle that the string makes with the vertical. 3

(iii) Prove that the angular velocity of the smaller mass equals $\sqrt{\dfrac{2g}{l-x}}$ rad/sec. 2

(b) In the diagram, *AB* and *AC* are tangents from *A* to the circle with centre *O*, meeting the circle at *B* and *C*. *ADE* is a secant of the circle. *G* is the midpoint of *DE*. *CG* produced meets the circle at *F*.

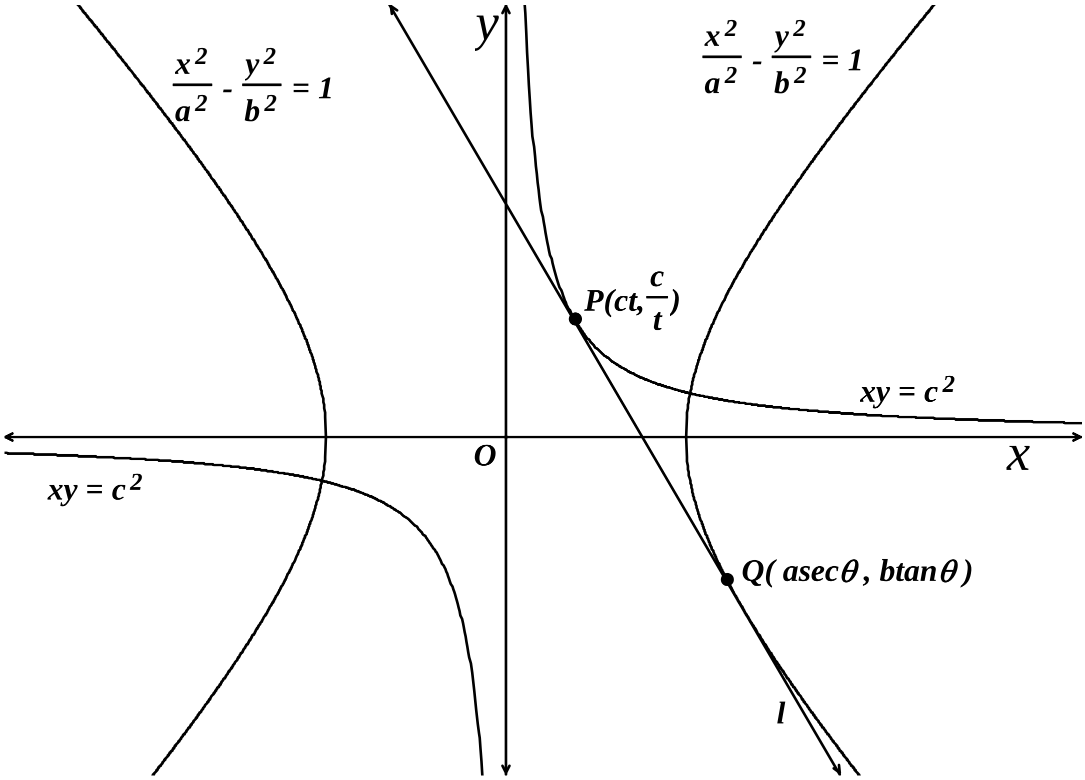


(i) Copy the diagram onto your answer sheet and prove that *ABOC* and *AOGC* are cyclic quadrilaterals. 2

(ii) Explain why $\angle OGF = \angle OAC$. 1

(iii) Prove that $BF \parallel ADE$. 5

Question 7 (15 Marks) START A NEW PAGE Marks



The line l is a common tangent to the hyperbola $xy = c^2$ and $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ with points of contact P and Q respectively.

(i) Considering l as a tangent to $xy = c^2$ at $P\left(ct, \frac{c}{t}\right)$, prove that l has the equation $x + t^2 y = 2ct$. **3**

(ii) Considering l as a tangent to $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ at $Q(a\sec\theta, b\tan\theta)$, prove that l has the equation $\frac{x\sec\theta}{a} - \frac{y\tan\theta}{b} = 1$. **3**

(iii) Deduce that $\frac{\sec\theta}{a} = \frac{-\tan\theta}{bt^2} = \frac{1}{2ct}$. **1**

(iv) Write the coordinates of Q in terms of t, a, b and c, and show that $b^2t^4 + 4c^2t^2 - a^2 = 0$. Deduce that there are exactly two such common tangents to the hyperbolas. **3**

(v) Copy the diagram onto your answer sheet, and using the symmetry of the graph, draw in the second common tangent with points of contact R on $xy = c^2$ and S on $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$. Write down the coordinates of R and S in terms of t, a, b and c. **2**

(vi) Show that if $PQRS$ is a rhombus, then $b^2 = a^2$. **3**

Question 8 (15 Marks) Marks

(a) Six lines are drawn on a plane such that no two lines are parallel and no three lines are concurrent (i.e. pass through the same point).

(i) Show that there are 15 points of intersection. 1

(ii) If three of these points are chosen at random, show that the probability that they all lie on one of the given lines is $\frac{12}{91}$. 2

(iii) Find the probability that if three of these points are chosen at random then none of them lie on the same line. 2

(b) (i) Prove that: 2

$$(\alpha)\ \frac{{}^1C_0}{x} - \frac{{}^1C_1}{x+1} = \frac{1!}{x(x+1)}.$$

$$(\beta)\ \frac{{}^2C_0}{x} - \frac{{}^2C_1}{x+1} + \frac{{}^2C_2}{x+2} = \frac{2!}{x(x+1)(x+2)}.$$

(ii) Given $T(k,x) = \dfrac{k!}{x(x+1)(x+2)\ldots(x+k)}$, prove that $T(k,x) - T(k,x+1) = T(k+1,x)$. 3

(iii) Hence prove, using Mathematical Induction or otherwise, that for $n \geq 1$: 5

$$\frac{{}^nC_0}{x} - \frac{{}^nC_1}{x+1} + \frac{{}^nC_2}{x+2} - \frac{{}^nC_3}{x+3} + \ldots + (-1)^n \frac{{}^nC_n}{x+n} = \frac{n!}{x(x+1)(x+2)(x+3)\ldots(x+n)}.$$

[You may use the result : ${}^{k+1}C_r = {}^kC_r + {}^kC_{r-1}$]

THIS IS THE END OF THE EXAMINATION PAPER