Alternative Solution to JRAHS Ext. 2 trial, 2011 Q8b

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January 4, 2012

Q8b Prove that $\tan^{-1} 3 + \tan^{-1} 5 + \tan^{-1}(\frac{4}{7}) = \pi$.

James Ruse Agricultural High School provided the following solution:

$0 < \tan^{-1} 5 < \frac{\pi}{2}$

$0 < \tan^{-1} 3 < \frac{\pi}{2}$

$\therefore 0 < \tan^{-1} 3 + \tan^{-1} 5 < \pi$

$\tan(\tan^{-1} 3 + \tan^{-1} 5) = \frac{3+5}{1-15} = -\frac{4}{7}$

$\therefore \tan^{-1} 3 + \tan^{-1} 5 = \tan^{-1}(-\frac{4}{7}) + n\pi$

But as shown above

$0 < \tan^{-1} 3 + \tan^{-1} 5 < \pi$

$\therefore n = 1$

$\tan^{-1} 3 + \tan^{-1} 5 = \tan^{-1}(-\frac{4}{7}) + \pi$

$\tan^{-1} 3 + \tan^{-1} 5 = -\tan^{-1}(\frac{4}{7}) + \pi$

$\therefore \tan^{-1} 3 + \tan^{-1} 5 + \tan^{-1}(\frac{4}{7}) = \pi$

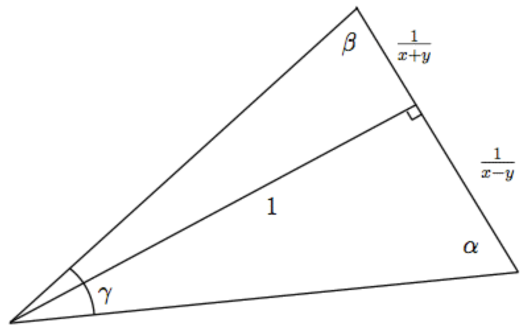
This can however be generalised to the following:

Show that for $x > \sqrt{y^2+1},\ \tan^{-1}(x-y) + \tan^{-1}(x+y) + \tan^{-1}\frac{2x}{x^2-y^2-1} = \pi$

and James Ruse's result follows by letting $x = 4,\ y = 1$

Alternative Solution

Suppose $x > \sqrt{y^2+1}$



Then

$\tan\alpha = \frac{1}{1/(x-y)} = x-y$

$\tan\beta = \frac{1}{1/(x+y)} = x+y$

$$\begin{aligned}\tan\gamma &= \tan\left(\tan^{-1}\tfrac{1/(x-y)}{1} + \tan^{-1}\tfrac{1/(x+y)}{1}\right)\\ &= \frac{\frac{1}{x-y}+\frac{1}{x+y}}{1-\frac{1}{x-y}\cdot\frac{1}{x+y}}\\ &= \tfrac{2x}{x^2-y^2-1}\end{aligned}$$

$\therefore \alpha+\beta+\gamma = \tan^{-1}(x-y)+\tan^{-1}(x+y)+\tan^{-1}\frac{2x}{x^2-y^2-1} = \pi$ (Angle sum of triangle)

Letting $x=4,\ y=1$

$\tan^{-1}3+\tan^{-1}5+\tan^{-1}\frac{4}{7} = \pi$