Student Number: ________________

KAMBALA

AUGUST 2007
YEAR 12
HSC ASSESSMENT#4
HIGHER SCHOOL CERTIFICATE
TRIAL EXAMINATION

Mathematics Extension 2

General Instructions

- Reading time – 5 minutes.
- Working time – 3 hours.
- Write using blue or black pen.
- Board-approved calculators may be used.
- A table of standard integrals is provided.
- All necessary working should be shown in every question.

Total marks – 120

- Attempt Questions 1-8.
- All questions are of equal value.

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Answer each question in a SEPARATE writing booklet. Extra writing booklets are available.

Marks

QUESTION 1 (15 marks) Use a SEPARATE writing booklet.

(a) Find $\int \frac{x^2+6}{x^2+4}\,dx$ **2**

(b) (i) Find real numbers A, B and C such that **3**

$$\frac{9}{(2x-1)(x+1)^2} \equiv \frac{A}{2x-1} + \frac{B}{x+1} + \frac{C}{(x+1)^2}$$

(ii) Hence evaluate $\int_1^2 \frac{9}{(2x-1)(x+1)^2}\,dx$ **2**

(c) Use integration by parts to find the exact value of $\int_0^{\frac{\pi}{4}} x\sec^2 x\,dx$. **4**

(d) Show that $\int \frac{x^n}{e^x}\,dx = n\int \frac{x^{n-1}}{e^x}\,dx - \frac{x^n}{e^x}$. **4**

Hence find $\int \frac{x^3}{e^x}\,dx$.

QUESTION 2 (15 marks) Use a SEPARATE writing booklet. **Marks**

(a) (i) Express $z = 1 + \sqrt{3}\,i$ in modulus-argument form. **2**

(ii) Show that $z^7 - 64z = 0$ **3**

(b) (i) Solve the equation $z^4 = 1$. **1**

(ii) Hence find all solutions of the equation $z^4 = (z-1)^4$. **3**

(c) In an Argand diagram the points P, Q and R represent the complex numbers z, w and $z - w$ respectively. O is the origin and $z - w = iz$.

(i) Describe the geometric properties of ΔPOR, giving full reasons for your answer. **2**

(ii) Find the size of $\angle QPR$. **1**

(d) Find the Cartesian equation of the locus represented by $|z|^2 = z + \bar{z}$. **3**

Sketch this locus on an Argand diagram and describe it geometrically.

Marks

QUESTION 3 (15 marks) Use a SEPARATE writing booklet.

(a) Given the function $f(x) = x\sqrt{4-x^2}$:

(i) State the natural domain and show that $f(x)$ is an odd function. **2**

(ii) Show that on the curve $y = f(x)$, stationary points occur at $x = \pm\sqrt{2}$. **3**
Find the co-ordinates of the stationary points and determine their nature.

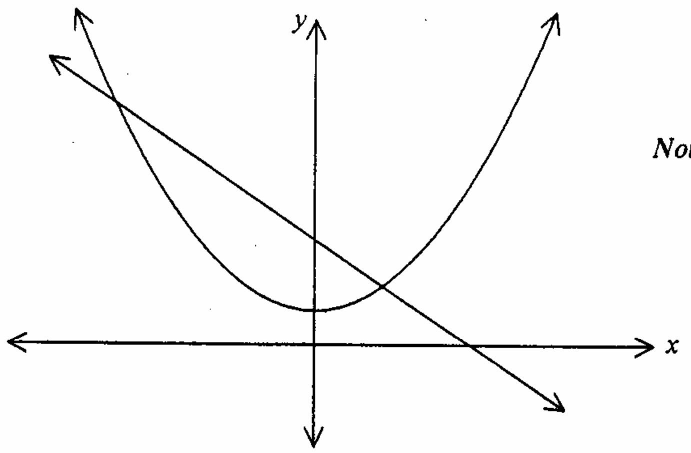
(iii) Draw a neat sketch of the curve $y = f(x)$, indicating the above features, and given that there is a point of inflexion at the origin. **2**

(iv) On separate diagrams, sketch the curves

1. $y^2 = x^2(4-x^2)$ **2**

2. $y = \dfrac{1}{f(x)}$ **2**

(b) The area bounded by the curve $y = x^2 + 1$ and the line $y = 3 - x$ is rotated about the x-axis to form a solid.



Not to scale

(i) By considering slices perpendicular to the x-axis show that the area of one slice is given by $A = \pi(8 - 6x - x^2 - x^4)$. **2**

(ii) Hence find the volume of the solid formed. **2**

Marks

QUESTION 4 (15 marks) Use a SEPARATE writing booklet.

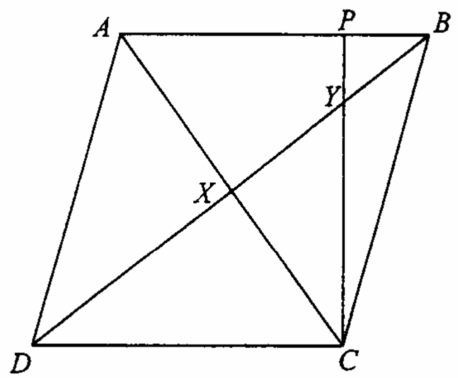
(a) A monic polynomial $P(x)$ of degree 4 with real coefficients has zeros $1+i$ and $-1+i$. **3**

Find $P(x)$, expressing your answer in expanded form.

(b) (i) Show that $\cos(A-B)-\cos(A+B)=2\sin A\sin B$ **2**

(ii) Hence evaluate $\int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \sin 5x \sin x \, dx$. **2**

(c)



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$ABCD$ is a rhombus whose diagonals intersect at X. The perpendicular CP from C to AB cuts BD at Y.

Prove that B, P, X, C are concyclic points. **3**

(d) Consider the hyperbola $\frac{y^2}{9}-\frac{x^2}{4}=1$. The area bounded by the lines $x=\pm 2$ and the hyperbola is rotated about the y-axis to form a solid.

(i) Using the method of cylindrical shells show that the volume of the solid so formed is given by $V=6\pi\int_0^2 x\sqrt{4+x^2}\,dx$. **3**

(ii) Hence find the volume of the solid. **2**

Marks

QUESTION 5 (15 marks) Use a SEPARATE writing booklet.

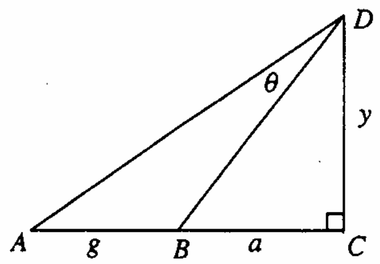
(a) If $(x-1)^2$ is a factor of $Q(x) = Ax^n + Bx^{n-2} + 6$, show that $A = 3n-6$ and $B = -3n$. **3**

(b) The tangent to the hyperbola $\frac{x^2}{16} - \frac{y^2}{9} = 1$ at $P\left(4\sqrt{2}, 3\right)$ meets the asymptotes of the hyperbola at A and B.

(i) Show that P is the midpoint of AB. **4**

(ii) Find the length of AB in exact form. **2**

(c) In the diagram, A, B and C are collinear and DC is perpendicular to AC. AB, BC and DC are g, a and y units long respectively and $\angle ADB = \theta$.



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(i) Show that $\tan\theta = \dfrac{gy}{a(g+a)+y^2}$. **3**

(ii) Find $\dfrac{d\theta}{dy}$ and hence show that the maximum value of θ is when $y = \sqrt{a(g+a)}$. **3**

Marks

QUESTION 6 (15 marks) Use a SEPARATE writing booklet.

(a) If α, β, γ are the roots of the equation $x^3 + 2x^2 - 2x + 3 = 0$ form an equation whose roots are $\alpha^2, \beta^2, \gamma^2$. **2**

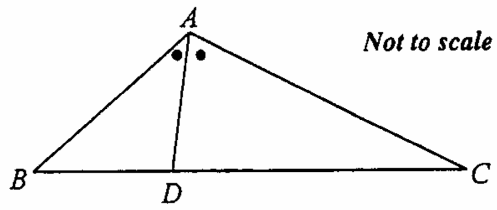
(b) A particle moves in a straight line and its position x at any time t is given by

$$x = \sqrt{3}\cos 3t - \sin 3t$$

(i) Show that the motion is simple harmonic. **2**

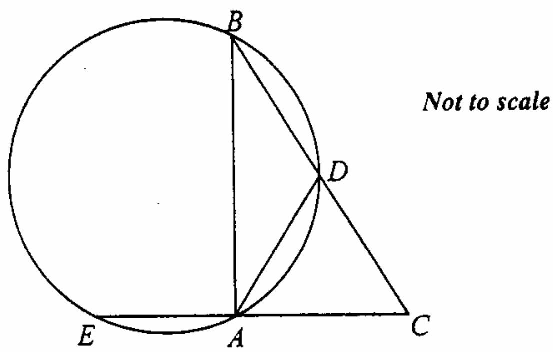
(ii) Determine the period and amplitude of the motion. **3**

(c) (i) In ΔABC, AD bisects $\angle BAC$.



Prove that $\dfrac{BD}{DC} = \dfrac{BA}{AC}$ **4**

(ii)



In the diagram $AB = BC$ and AD bisects $\angle BAC$.

Prove that $BD = CE$. **4**

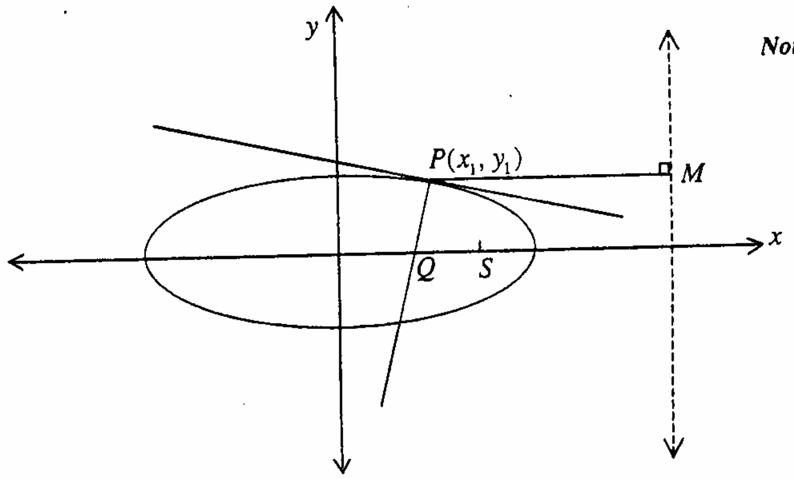
Marks

QUESTION 7 (15 marks) Use a SEPARATE writing booklet.

(a) Use the substitution $t = \tan\frac{x}{2}$ to evaluate $\int_0^{\frac{\pi}{2}} \frac{dx}{2+\cos x}$. **3**

(b) Consider the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ with eccentricity e.

Not to scale



(i) Write down in terms of a and e the foci and equation of the directrices. **1**

(ii) Show that the equation of the normal to the ellipse at the point $P(x_1, y_1)$ is given by $\frac{a^2x}{x_1} - \frac{b^2y}{y_1} = a^2 - b^2$. **2**

(iii) Let Q be the x-intercept of the normal and let M be the foot of the perpendicular from P to the directrix as shown in the diagram. **3**

Prove that $QS = e^2PM$.

Question 7 continues on the next page

Marks

QUESTION 7 continued

(c) (i) State why, for $x < 1$, the sum of n terms of **1**

$$1 + x + x^2 + x^3 + \dots + x^{n-1} = \frac{1 - x^n}{1 - x}$$

(ii) Show that **2**

$$1 + 2x + 3x^2 + \dots + (n-1)x^{n-2} = \frac{(n-1)x^n - nx^{n-1} + 1}{(1-x)^2}$$

(iii) Hence find an expression for $1 + 1 + \frac{3}{4} + \frac{4}{8} + \frac{5}{16} + \dots + \frac{n-1}{2^{n-2}}$ and show that this sum is always less than 4. **3**

Marks

QUESTION 8 (15 marks) Use a SEPARATE writing booklet.

(a) (i) Use De Moivre's theorem to show that $\cos 3\theta = 4\cos^3\theta - 3\cos\theta$. **2**

(ii) Using $\cos 3\theta = \frac{1}{2}$ deduce that $8x^3 - 6x - 1 = 0$ has solutions $x = \cos\theta$. **1**

(iii) Use this result to solve the equation $8x^3 - 6x - 1 = 0$ in terms of $\cos\theta$. **2**

(iv) Hence prove that $\cos\frac{2\pi}{9} + \cos\frac{4\pi}{9} = \cos\frac{\pi}{9}$. **2**

(b) (i) If $f(x)$, $g(x)$ and $h(x)$ are distinct non-negative continuous functions of x in the interval $a \le x \le b$ and $f(x) < g(x) < h(x)$, explain why **1**

$$\int_a^b f(x)\,dx < \int_a^b g(x)\,dx < \int_a^b h(x)\,dx$$

(ii) By considering the interval $0 < x < 1$ as an inequality, use algebra to show that **3**

$$\frac{1}{2}x(1-x)^3 < \frac{x(1-x)^3}{1+x} < x(1-x)^3$$

(iii) Deduce that $\frac{1}{2}\int_0^1 x(1-x)^3\,dx < \int_0^1 \frac{x(1-x)^3}{1+x}\,dx < \int_0^1 x(1-x)^3\,dx$ **1**

(iv) Given that $\int_0^1 \frac{x(1-x)^3}{1+x}\,dx = \frac{67}{12} - 8\ln 2$, deduce that $\frac{83}{120} < \ln 2 < \frac{667}{960}$ **3**

End of Assessment