2003
HIGHER SCHOOL CERTIFICATE
TRIAL EXAMINATION

Mathematics Extension 2

General Instructions

- Reading time – 5 minutes
- Working time – 3 hours
- Write using black or blue pen
- Approved scientific calculators and templates may be used
- Attempt all questions
- Start a new booklet for each question
- A standard integral sheet is included on the back of this paper

Total marks – 120

- All questions should be attempted
- All questions are of equal value

Examiner MN June 2003

Total Marks –120
Attempt Questions 1-8
All questions are of equal value

Answer each question in a SEPARATE writing booklet. Extra writing booklets are available.

Marks

Question One (15 marks). Use a SEPARATE writing booklet.

(a) $\int \sin\theta \cos^5\theta \, d\theta$. **2**

(b) (i) Use partial fractions to find the values of A, B and C if **3**

$$\frac{x^2 - x - 21}{(x^2+4)(2x-1)} = \frac{Ax+B}{x^2+4} + \frac{C}{2x-1}.$$

(ii) Hence find $\int \frac{x^2 - x - 21}{(x^2+4)(2x-1)} \, dx$. **2**

(c) Use integration by parts to evaluate $\int_0^{\frac{\pi}{4}} x \sec^2 x \, dx$. **3**

(d) By using the substitution $t = \tan\frac{\theta}{2}$, show that. $\int_0^{\frac{\pi}{3}} \sec\theta \, d\theta = \ln\left(2+\sqrt{3}\right)$ **5**

Question Two (15 marks). Use a SEPARATE writing booklet.

(a) Let $z = -24 + 28i$, and $w = 3 + 5i$.

(i) Find $z + 3w$. **1**

(ii) Find $\arg(z + 3w)$. Give your answer in radians correct to 3 significant figures. **2**

(iii) Express $\frac{z}{w}$ in the form $a + ib$. **2**

(b) Express $1 - i\sqrt{3}$ in modulus-argument form. **2**

(c) (i) Sketch on an Argand diagram the locus defined by $\arg(z + 2i) = \frac{3\pi}{4}$ **2**

(ii) Let $z_1 = 1 + i$ and $z_2 = 2 - i$. Sketch on an Argand diagram the locus defined by $\arg\left(\frac{z - z_2}{z - z_1}\right) = \frac{\pi}{2}$. **2**

(d) In an Argand diagram the point A represents the complex number $z = \cos\frac{\pi}{6} + i\sin\frac{\pi}{6}$.

(i) Let C represent the number w, where $w = 2iz$. Find w in the form $a + ib$. **2**

(ii) The point B completes the rectangle $COAB$, where O is the origin
Let u be the number represented by B. Find u in the form $a + ib$. **1**

(iii) Find the value of $|w - u| \times |z - u|$. **1**

Question Three (15 marks).Use a SEPARATE writing booklet.

(a) Let α, β and γ be the roots of $x^3 + 2x^2 - 2 = 0$.

(i) Find the value of $\alpha^2 + \beta^2 + \gamma^2$. **2**

(ii) Form the equation whose roots are $\alpha - 1, \beta - 1$ and $\gamma - 1$. **2**

(b) Let $P(x)$ be a polynomial.

(i) Prove that if α is a double zero of $P(x)$, then $P'(\alpha) = 0$. **2**

(ii) Hence find the roots of the equation $12x^3 + 44x^2 - 5x - 100 = 0$, given that two of the roots are equal. **2**

(c) Let z_1 and z_2 be complex numbers.

(i) Prove that $|z_1|^2 = z_1\overline{z_1}$. **1**

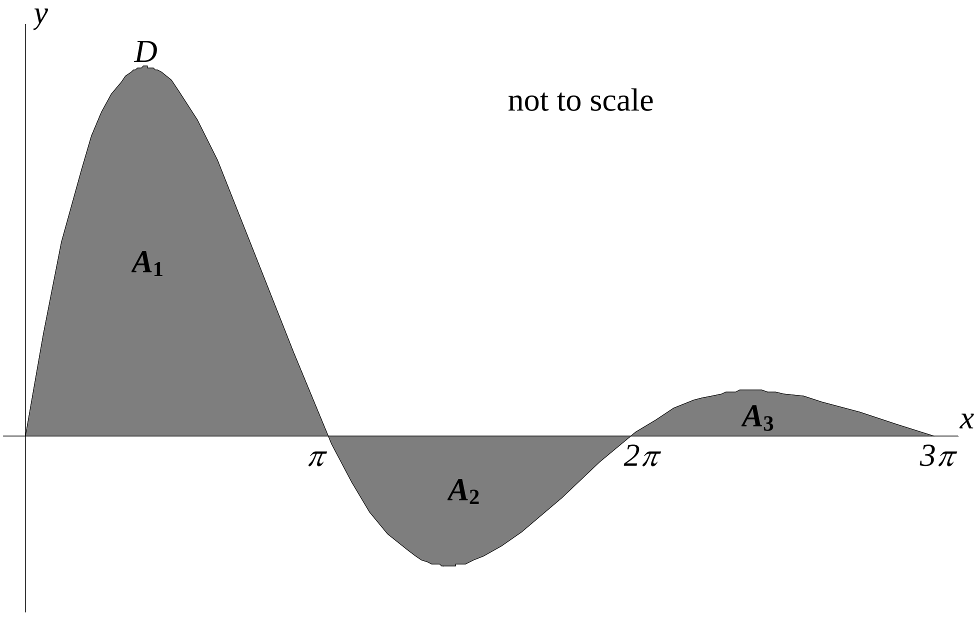
(ii) By using the fact that $\overline{z_1 \times z_2} = \overline{z_1} \times \overline{z_2}$, prove that $\overline{\left(z_1 \times \overline{z_2}\right)} = \overline{z_1} \times z_2$. **1**

(iii) Hence prove that $|z_1 + z_2|^2 = |z_1|^2 + |z_2|^2 + 2\,\mathrm{Re}\left(z_1\overline{z_2}\right)$ **3**

(iv) Hence prove that $|z_1 + z_2| \le |z_1| + |z_2|$ **2**

Question Four (15 marks). Use a SEPARATE writing booklet.

The graph below is $y = e^{-x}\sin x$



(i) Find the coordinates of D, the absolute maximum of $y = e^{-x}\sin x$. **3**

(ii) Prove that the shaded area A_1 is equal to $\dfrac{e^0 + e^{-\pi}}{2}$. **4**

(iii) Prove that the shaded area A_2 is equal to $\dfrac{e^{-\pi} + e^{-2\pi}}{2}$. **2**

(iv) Write down the value of A_3. **1**

(v) Show that $\dfrac{A_2}{A_1} = e^{-\pi}$. **2**

(vi) Given that the shaded areas form a geometric progression, find the limiting sum of such areas as $x \to \infty$. **3**

Question Five (15 marks). Use a SEPARATE writing booklet.

The point $P\left(ct, \frac{c}{t}\right)$ lies on the hyperbola $xy = c^2$.

(i) Sketch the hyperbola and mark on it the point *P* where $t \neq 1$. **1**

(ii) Derive the equation of the tangent at *P*. **2**

(iii) Prove that the equation of the normal at *P* is given by $y = t^2 x + \frac{c}{t} - ct^2$. **2**

(iv) The tangent at *P* meets the line $y = x$ at *T*. Find the coordinates of *T*. **3**

(v) The normal at *P* meets the line $y = x$ at *N*. Find the coordinates of *N*. **3**

(vi) Prove that $OT \times ON = 4c^2$ **4**

Question Six (15 marks). Use a SEPARATE writing booklet.

The ellipse *E* has equation $\frac{x^2}{4} + \frac{y^2}{3} = 1$. *P* is a point on *E*.

(i) Calculate the eccentricity **1**

(ii) Write down the coordinates of the foci *S* and S' . **2**

(iii) Write down the equation of each directrix. **1**

(iv) Sketch *E* showing all important features. **1**

(v) Prove that the sum of the distances $SP + S'P$ is independent of *P*. **3**

(vi) Derive the equation of the normal at *P*. **3**

(vii) Prove that the normal at *P* bisects $\angle SPS'$. **4**

Question Seven (15 marks). Use a SEPARATE writing booklet.

(a) The curve $f(x) = x + 6x^3$ is defined over the domain $\frac{-1}{\sqrt{2}} \le x \le \frac{1}{\sqrt{2}}$.

(i) Show that this function has no turning points. **2**

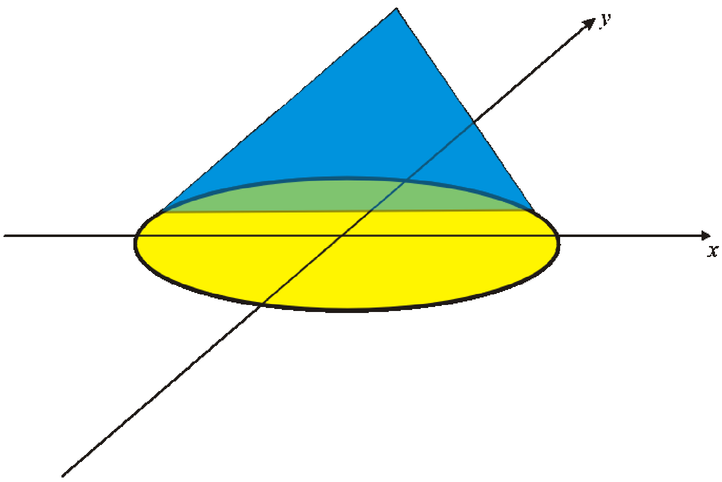
(ii) Show that $f(x) = x + 6x^3$ is an odd function. **2**

(iii) Draw a neat sketch of the function, about one third of a page in size. **1**

(iv) On the same diagram, sketch the solid formed when $y = f(x)$ is rotated about the y-axis. **1**

(v) Use the method of cylindrical shells to find the exact volume of this solid. **4**

(b) A solid is constructed on a circular base of radius 6cm. Parallel cross-sections are right-angled isosceles triangles with the hypotenuse in the base of the solid. Find the volume of the solid. **5**

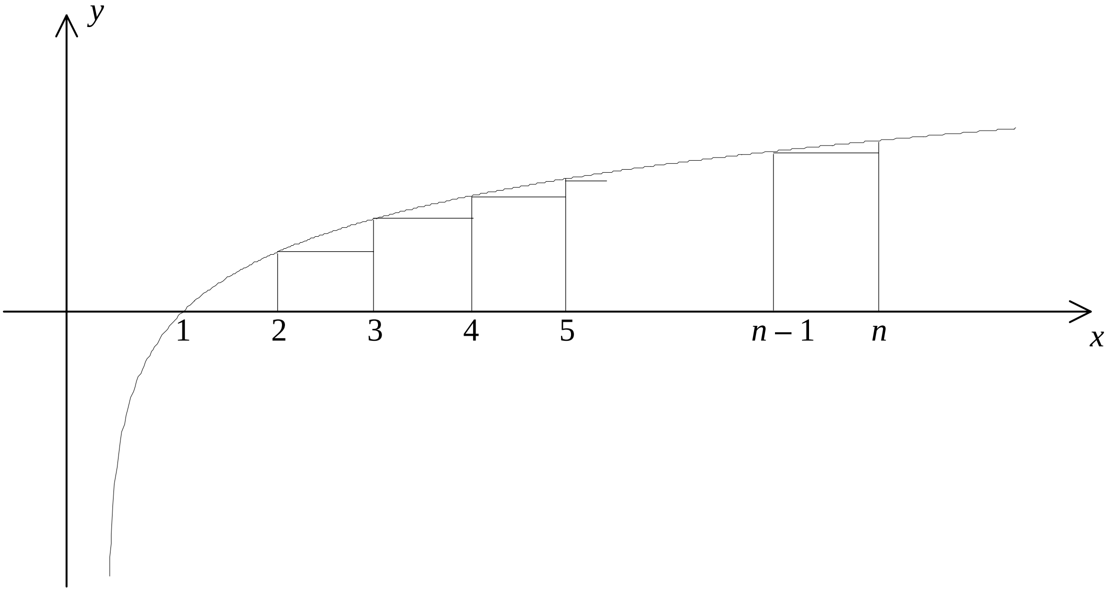


Question Eight (15 marks). Use a SEPARATE writing booklet.

(a) (i) Prove that $\cot\frac{\alpha}{2} - \cot\alpha = \operatorname{cosec}\alpha$ **3**

(ii) Hence find a simplified expression for $\sum_{r=1}^{n} \operatorname{cosec}\left(2^r \alpha\right)$. **3**

(b)



The graph above is of the curve $y = \ln x$.

(i) Find $\int_1^n \ln x \, dx$ **3**

(ii) Prove that the sum of the areas of the rectangle is given by $\ln(n-1)!$. **2**

(iii) What can you say about your answer in (ii) compared to your answer in (i)? **1**

(iv) Prove that for any integer $n > 1$, $\ln\left(\frac{n^n}{(n-1)!}\right) > n - 1$. **3**