KINCOPPAL–ROSE BAY
SCHOOL OF THE SACRED HEART

2009
HIGHER SCHOOL CERTIFICATE
TRIAL EXAMINATION

Extension 2 Mathematics

General Instructions

- Reading time – 5 minutes
- Working time – 3 hours
- Write using black or blue pen
- Board-approved calculators may be used
- A table of standard integrals is provided at the back of this paper
- All necessary working should be shown in every question
- Start a new booklet for each question

Total marks – 120

- Attempt Questions 1 – 8
- All questions are of equal value

Total Marks – 120
Attempt Questions 1-8
All questions are of equal value

Answer each question in a SEPARATE writing booklet. Extra writing booklets are available.

Question 1. (15 marks) Start a new page **Marks**

(a) (i) Show that $y = x\sqrt{4-x^2}$ is an odd function. **1**

(ii) Hence without finding the integral evaluate **2**
$\int_{-2}^{2} \left(x\sqrt{4-x^2} - \sqrt{4-x^2}\right)$, giving reasons.

(b) By using the table of standard integrals, find $\int \frac{dx}{\sqrt{4x^2+36}}$ **2**

(c) Use partial fractions to evaluate $\int_0^1 \frac{5\,dt}{(2t+1)(2-t)}$ **3**

(d) Find $\int \operatorname{cosec} x \; dx$ by using the substitution $t = \tan\frac{x}{2}$ **3**

(e) Find $\int \frac{\sqrt{x^2-16}}{x}\,dx$ using the substitution $x = 4\sec\theta$. **4**

End of Question 1

Question 2. (15 marks) Start a new page **Marks**

(a) Express $\dfrac{2-5i}{4-3i}$ in the form $x + iy$ where x and y are real. **2**

(b) Find all pairs of integers for a and b such that $(a-ib)^2 = -21-20i$ **3**

(c) Find the modulus and argument of $(\sin\theta + i\cos\theta)(\cos\theta - i\sin\theta)$ **3**

(d) (i) If $\left|\dfrac{z-1}{z+1}\right| = 2$, where $z = x + iy$, show that the locus of z is **2**

$$\left(x+\frac{5}{3}\right)^2 + y^2 = \frac{16}{9}$$

(ii) Represent this locus on an Argand Diagram and shade the region for which the inequalities $\left|\dfrac{z-1}{z+1}\right| \le 2$ and $0 \le \arg z \le \dfrac{3\pi}{4}$ are both satisfied. **3**

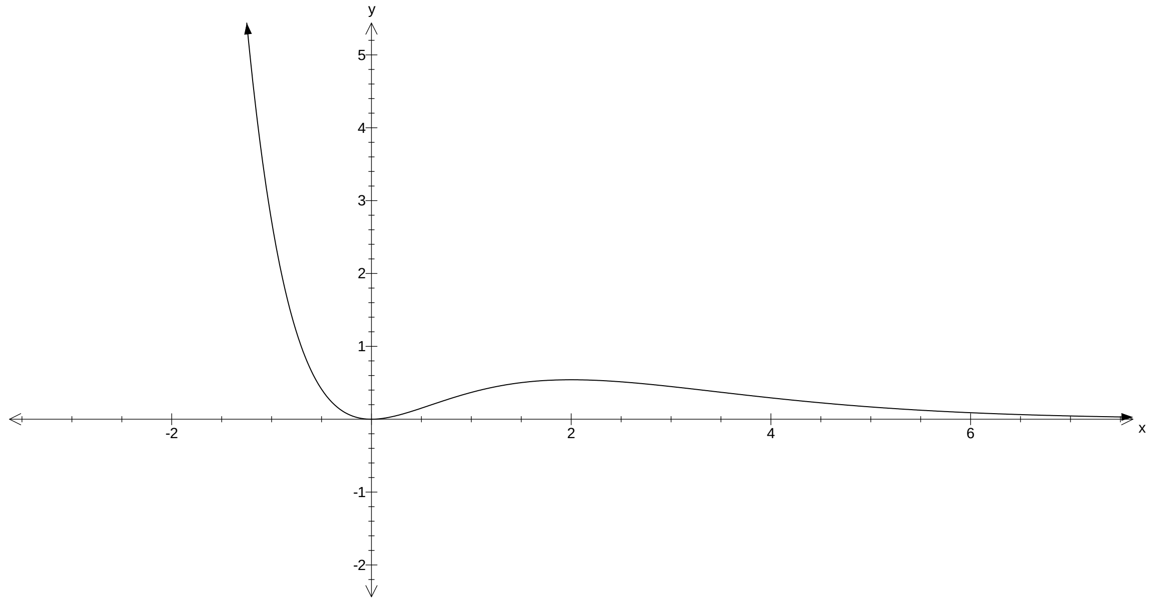
(e) z_1 and z_2 are two complex numbers such that $\dfrac{z_1 + z_2}{z_1 - z_2} = 2i$ **2**

On an Argand diagram show vectors representing $z_1, z_2, z_1 + z_2$ *and* $z_1 - z_2$

End of Question 2

Question 3. (15 marks) Start a new page **Marks**

(a)



The graph of $y = x^2 e^{-x}$ is sketched above. There is a stationary point at (0,0) and $\left(2, \frac{4}{e^2}\right)$

On separate diagrams, draw a neat sketch showing the main features of each of the following

(i) $y = f(x) + 1$ **1**

(ii) $y = f(|x|)$ **1**

(iii) $y = \{f(x)\}^2$ **2**

(iv) $y = \dfrac{1}{f(x)}$ **2**

(v) $y^2 = f(x)$ **2**

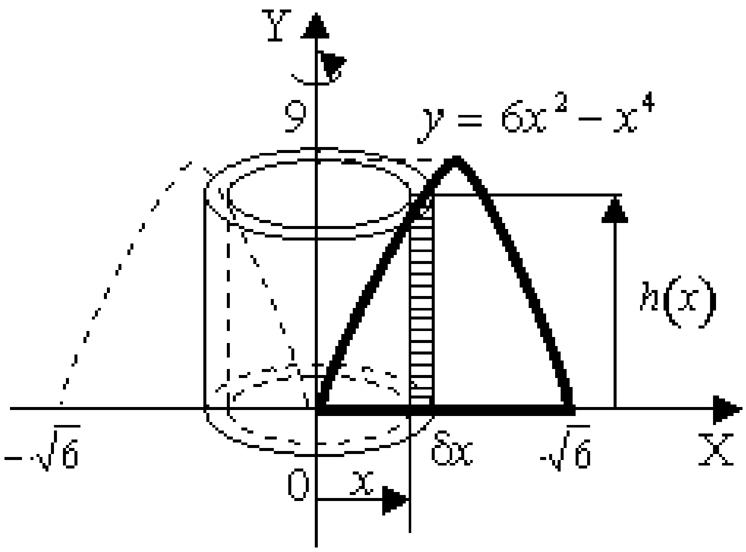
(vi) $y = \cos^{-1}(f(x))$ **2**

(b) If $x^m y^n = k$, where k is a constant, show that $\dfrac{dy}{dx} = -\dfrac{my}{nx}$ **2**

Question 3 continues on page 5

Question 3 continued **Marks**

(c) Using the method of cylindrical shells find the volume of the solid of revolution generated when the area enclosed by the curve $y = 6x^2 - x^4$ the x-axis and $0 \le x \le \sqrt{6}$ is rotated about the y- axis. **3**



End of Question 3

Question 4. (15 marks) Start a new page **Marks**

(a) If α, β, γ are the roots of the equation $x^3 - 4x^2 + 2x + 5 = 0$. Evaluate:

(i) $\alpha^2 + \beta^2 + \gamma^2$ **1**

(ii) $\alpha^3 + \beta^3 + \gamma^3$ **2**

(b) $P(x)$ is a monic polynomial of degree 4 with integer coefficients and constant term 4. **3**
One zero is $\sqrt{2}$, another zero is rational and the sum of the zeros is positive.
Factorise $P(x)$ fully over **R**.

(c) (i) Use De Moivre's theorem to show $\cos 3\theta = 4\cos^3\theta - 3\cos\theta$ **3**

(ii) Hence solve $8x^3 - 6x - 1 = 0$ leaving answer in terms of $\cos\theta$ **3**

(d) For a real number *r*, the polynomial $8x^3 - 4x^2 - 42x + 45$ is divisible by $(x - r)^2$. **3**
Find the value of *r*.

End of Question 4

Question 5. (15 marks) Start a new page **Marks**

(a) Evaluate $\int_{1}^{\infty} \frac{1}{x+1} - \frac{1}{x+3}\, dx$ **2**

(b) (i) Use integration by parts to show that a reduction (recurrence) formula **3**

for $\mathrm{I}_n = \int \sin^n x\, dx$ is $\mathrm{I}_n = -\frac{1}{n}\cos x \sin^{n-1} x + \frac{n-1}{n}\mathrm{I}_{n-2}$

(ii) Hence evaluate $\int_{0}^{\frac{\pi}{2}} \sin^4 x\, dx$ **2**

(c) The hyperbola H has equation $xy = 4$.

(i) Sketch the hyperbola and indicate on your diagram the position and coordinates of all points at which H intersects the axes of symmetry. **1**

(ii) Show that the equation of the tangent at $P\left(2t, \frac{2}{t}\right)$ where $t \neq 0$, is $x + t^2 y = 4t$ **2**

(iii) If $s \neq 0$ and $s^2 \neq t^2$, show that the tangents to H at P and $Q\left(2s, \frac{2}{s}\right)$ intersect at **2**

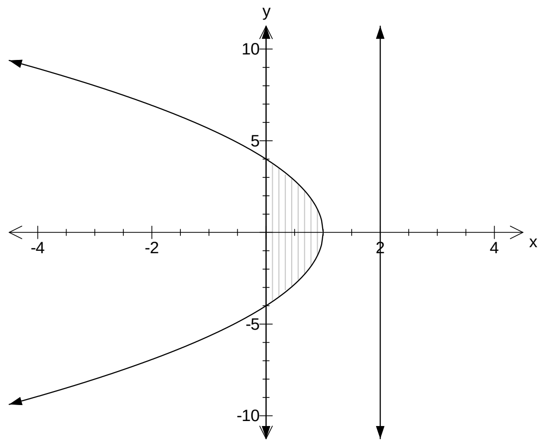
$M\left(\frac{4st}{s+t}, \frac{4}{s+t}\right)$

(iv) Suppose that in (iii) the parameter $s = -\frac{1}{t}$. Show that the locus of M is a straight line through, but excluding the origin. **3**

End of Question 5

Question 6. (15 marks) Start a new page **Marks**

(a)



A solid S is formed by rotating the region bounded by the parabola $y^2 = 16(1-x)$ and the y – axis around the line $x = 2$. **4**
By using the method of slices find the exact volume of S.

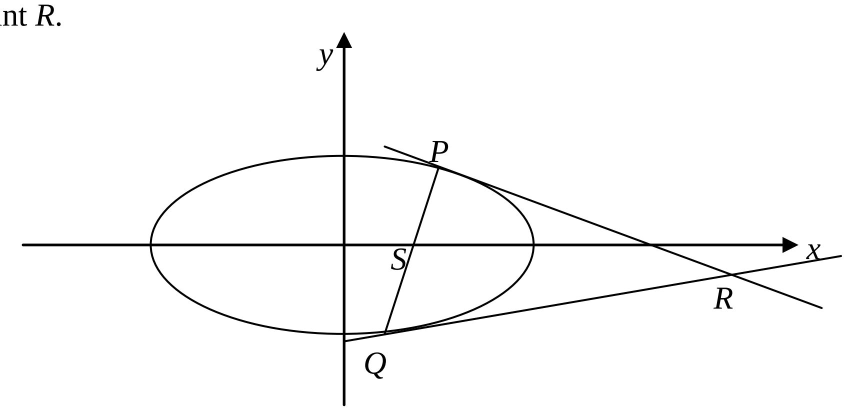
(b) A hyperbola has foci $(\pm 10, 0)$ and asymptotes $y = \pm \frac{4x}{3}$.

(i) Find the eccentricity. **1**

(ii) State the equation of the hyperbola. **1**

(iii) Sketch the hyperbola indicating important features such as vertices, foci, directrices and asymptotes **2**

(c) Let $P(a\cos\theta, b\sin\theta)$ and $Q(a\cos\phi, b\sin\phi)$ be points on the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ at the extremities of a focal chord PQ. The tangents drawn from the extremities intersect at a point R.



(i) Show that the tangent at P is given by $\frac{x}{a}\cos\theta + \frac{y}{b}\sin\theta = 1$. **2**

Question 6 continues on page 9

Question 6 continued **Marks**

(ii) Use simultaneous equations to show that the x coordinate of the point R is given by $x = \dfrac{a(\sin\phi - \sin\theta)}{\cos\theta\sin\phi - \sin\theta\cos\phi}$ **2**

(iii) Use the fact that the gradient of PS = gradient of SQ to show that $\dfrac{\sin\phi - \sin\theta}{\cos\theta\sin\phi - \sin\theta\cos\phi} = \dfrac{1}{e}$ **2**

(iv) Hence or otherwise show that R lies on the directrix of the ellipse. **1**

End of Question 6

Question 7. (15 marks) Start a new page **Marks**

(a) Let α, β, γ be the roots of the equation $x^3 + qx + r = 0$. **2**
Write down the cubic equation whose roots are $\alpha^{-1}, \beta^{-1}, \gamma^{-1}$.

(b) Let ω be a non-real root of $z^7 - 1 = 0$.

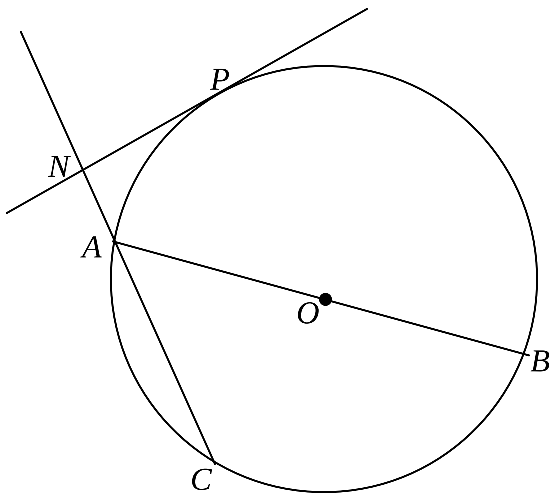
(i) Show that $1 + \omega + \omega^2 + \omega^3 + \omega^4 + \omega^5 + \omega^6 = 0$. **1**

(ii) Show that $(1+\omega)(1+\omega^2)(1+\omega^4) = 1$. **1**

(iii) Simplify $(\omega + \omega^2 + \omega^4)(\omega^6 + \omega^5 + \omega^3)$ **2**

(iv) Sketch on the Argand diagram all seven roots of $z^7 - 1 = 0$ **1**

(c) In a circle centre *O*, a diameter *AB* and a chord *AC* are drawn.
P is the point on the circumference on the side of *AB* opposite to *C*, such that the tangent at *P* is perpendicular to *CA* produced.
The tangent at *P* and the line *CA* produced intersect at the point *N*.



Copy this diagram into your examination booklet.

Prove that:

(i) $PC = PB$ **3**

(ii) $\angle APC + 2\angle ACP = 90^\circ$ **3**

(iii) $\angle PAB = \angle NPC$ **2**

End of Question 7

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Question 8. (15 marks) Start a new page **Marks**

(a) (i) Sketch $y = \sec x$ in the domain $-2\pi \le x \le 2\pi$ **1**

(ii) Using a suitable domain sketch $y = \sec^{-1} x$. **2**

(b) For all integers $n \ge 1$, let

$$t_n = \frac{1}{n+1} + \frac{1}{n+2} + \frac{1}{n+3} + ... + \frac{1}{2n-1} + \frac{1}{2n}$$

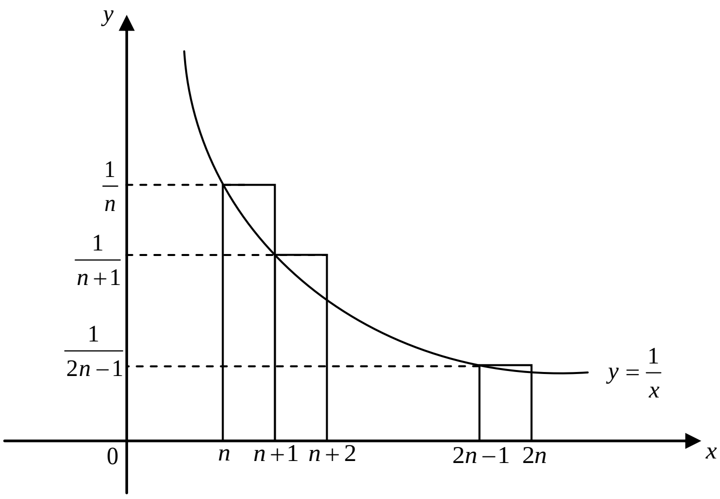
That is: $t_1 = \frac{1}{2}$

$t_2 = \frac{1}{3} + \frac{1}{4}$

$t_3 = \frac{1}{4} + \frac{1}{5} + \frac{1}{6}$

......

(i) Show that $t_n + \frac{1}{2n} = \frac{1}{n} + \frac{1}{n+1} + ... + \frac{1}{2n-1}$ **2**



The diagram above shows the graph of the function $y = \frac{1}{x}$ for $n \le x \le 2n$.

(ii) By using the diagram and the area of upper rectangles, show that $t_n + \frac{1}{2n} > \ln 2$ **3**

[Note that it can similarly be shown that $t_n < \ln 2$]

Questions 8 continued on page 13

Question 8 continued **Marks**

For all integers $n \geq 1$ let

$$s_n = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \ldots + \frac{1}{2n-1} - \frac{1}{2n}$$

That is:

$$s_1 = 1 - \frac{1}{2}$$
$$s_2 = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4}$$
$$s_3 = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6}$$

.....

(iii) Prove by mathematical induction that $s_n = t_n$ **4**

(iv) Hence find, to three decimal places, the value of $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \ldots. + \frac{1}{9999} - \frac{1}{10\,000}$ **3**

End of Test

STANDARD INTEGRALS

$$\int x^n \, dx = \frac{1}{n+1} x^{n+1}, \; n \neq -1; \; x \neq 0, \; if \; n < 0$$

$$\int \frac{1}{x} \, dx = \ln x, \; x > 0$$

$$\int e^{ax} \, dx = \frac{1}{a} e^{ax}, \; a \neq 0$$

$$\int \cos ax \, dx = \frac{1}{a} \sin ax, \; a \neq 0$$

$$\int \sin ax \, dx = -\frac{1}{a} \cos ax, \; a \neq 0$$

$$\int \sec^2 ax \, dx = \frac{1}{a} \tan ax, \; a \neq 0$$

$$\int \sec ax \tan ax \, dx = \frac{1}{a} \sec ax, \; a \neq 0$$

$$\int \frac{1}{a^2 + x^2} \, dx = \frac{1}{a} \tan^{-1} \frac{x}{a}, \; a \neq 0$$

$$\int \frac{1}{\sqrt{a^2 - x^2}} \, dx = \sin^{-1} \frac{x}{a}, \; a \neq 0, \; -a < x < a$$

$$\int \frac{1}{\sqrt{x^2 - a^2}} \, dx = \ln(x + \sqrt{x^2 - a^2}), \; x > a > 0$$

$$\int \frac{1}{\sqrt{x^2 + a^2}} \, dx = \ln\left(x + \sqrt{x^2 + a^2}\right)$$

NOTE: $\ln x = \log_e x, \; x > 0$