THE KING'S SCHOOL

2003
Higher School Certificate
Trial Examination

Mathematics Extension 2

General Instructions

- Reading time – 5 minutes
- Working time – 3 hours
- Write using black or blue pen
- Board-approved calculators may be used
- A table of standard integrals is provided at the back of this paper
- All necessary working should be shown in every question

Total marks – 120

- Attempt Questions 1-8
- All questions are of equal value

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Answer each question in a SEPARATE writing booklet. Extra writing booklets are available.

Marks

Question 1 (15 marks) Use a SEPARATE writing booklet.

(a) Find

(i) $\int \frac{1+x+x^2}{1+x^2}\,dx$ **2**

(ii) $\int \frac{x^2}{1+x^2}\,dx$ **2**

(b) Use integration by parts to evaluate

$\int_0^1 2x\tan^{-1}x\,dx$ **3**

(c) Find $\int_0^1 \frac{x-3}{(x^2+1)(3x+1)}\,dx$, giving your answer in simplest exact form. **4**

(d) $u_n = \int_0^1 \frac{x^n}{1+x^2}\,dx$, $n \geq 0$

(i) Show that $u_{n+2} + u_n = \frac{1}{n+1}$ **2**

(ii) Hence, evaluate $\int_0^1 \frac{x^3}{1+x^2}\,dx$ **2**

End of Question 1

Marks

Question 2 (15 marks) Use a SEPARATE writing booklet.

(a) $u = 2 + ai$, $v = a + 2i$, where a is a real number.

Find in the form $x + iy$,

(i) uv **2**

(ii) $(uv)^{-1}$ **1**

(b) (i) Express $z = -2\sqrt{3} + 2i$ in modulus-argument form **2**

(ii) Hence, find z^3 in the form $x + iy$ **2**

(c) Sketch the region in the complex plane where

$|z - i| \leq |z + 1|$ **3**

(d) Consider the equation $(a + ib)^2 = 1 + 2i$, a, b real

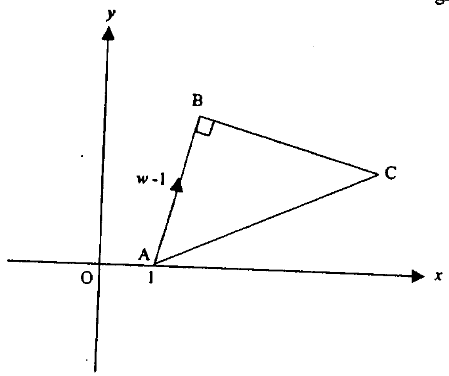
(i) Show that $a^2 + b^2 = \sqrt{1^2 + 2^2}$ **1**

(ii) Hence, or otherwise, find the value of a^2 **2**

Question 2 continues on next page

Question 2 (continued) **Marks**

(e) In the complex plane, A is the point (1,0) and the complex number $\overrightarrow{AB}$ is $w-1$. ΔABC is isosceles and right-angled at B. O is the origin.



Find, in terms of w, the complex numbers

(i) $\overrightarrow{CB}$ **1**

(ii) $\overrightarrow{OC}$ **1**

End of Question 2

Question 3 (15 marks) Use a SEPARATE writing booklet. **Marks**

(a) (i) Sketch on the same axes the graphs of

$y = |x-1|$ and $y = 2x - x^2$ **2**

(ii) Use (i) to show on separate diagrams, the graphs of

(α) $y = \dfrac{|x-1|}{2x-x^2}$, showing any asymptotes **3**

(β) $y = \dfrac{2x-x^2}{|x-1|}$, showing any asymptotes **3**

(b) Consider the function $f(x) = \tan^{-1} x - \dfrac{x}{1+x^2}$

(i) Show that f is an odd function. **1**

(ii) Find $f'(x)$ **2**

(iii) Show that $f(x) > 0$ if $x > 0$ **2**

(iv) Sketch the graph of $y = \tan^{-1} x - \dfrac{x}{1+x^2}$ **2**

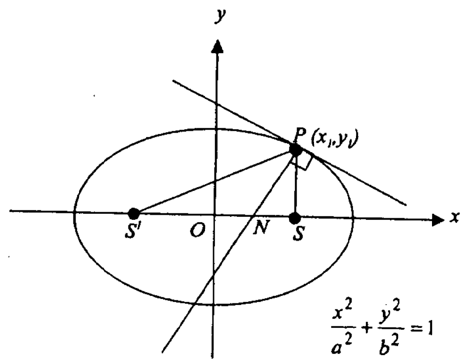
End of Question 3

Question 4 (15 marks) Use a SEPARATE writing booklet. **Marks**

(a) Find the gradient of the tangent to the curve $x^3 + y^2 + xy = 0$ at the point $(-2,4)$ **3**

(b) $P(x_1, y_1)$ is a point in the first quadrant on the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1, \quad a > b > 0$

S and S' are the foci of the ellipse. O is the origin.



(i) Show that the equation of the normal at $P(x_1, y_1)$ is $a^2 y_1(x - x_1) = b^2 x_1(y - y_1)$ **2**

(ii) The normal at P meets the major axis at N.

Prove that the x coordinate at N is $e^2 x_1$, where e is the eccentricity of the ellipse. **2**

(iii) Deduce that N lies between O and S. **2**

(iv) Show that $NS = eSP$ and $NS' = eS'P$ **3**

(v) Using the sine rule in ΔPSN and $\Delta PS'N$, or otherwise, prove that PN bisects $\angle SPS'$ **3**

End of Question 4

Question 5 (15 marks) Use a SEPARATE writing booklet. **Marks**

(a) Four married couples are to be seated at a circular table.

(i) How many arrangements are possible if the men and women are to be separated? **2**

(ii) For the arrangements in (i), find the probability that no woman is sitting next to her husband. **2**

(b) The equation $x^3 + ax^2 + bx + c = 0$ has one root the sum of the other two roots.

Prove that $a^3 - 4ab + 8c = 0$ **4**

(c) (i) By considering the circle $x^2 + y^2 = a^2$, or otherwise, find

$$\int_0^a \sqrt{a^2 - x^2}\, dx$$ **2**

(ii) The ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1, \quad a > b > 0$, is revolved about the line $y = a$.

By considering slices perpendicular to the line $y = a$, find the volume of the solid of revolution generated. **5**

End of Question 5

Marks

Question 6 (15 marks) Use a SEPARATE writing booklet.

(a) A particle of mass m kg falls vertically from rest from point O in a medium whose resistance is mkv, where k is a positive constant and v is its velocity in m/s. After t seconds the particle has fallen x metres.

g m/s^2 is the acceleration due to gravity.

(i) Show that $\frac{dv}{dt} = g - kv$ **1**

(ii) Find the terminal velocity, V m/s, of the particle. **1**

(iii) Use integration to prove that $v = \frac{g}{k}\left(1 - e^{-kt}\right)$ **3**

(iv) Find the distance the particle has fallen when its velocity is one half of its terminal velocity. **4**

(b) α, β are the two complex roots of the equation $x^3 + 5x + 1 = 0$

(i) Explain why α, β are complex conjugates. **1**

(ii) Show that the real root is $\frac{-1}{|\alpha|^2}$ **2**

(iii) Show that $\alpha\beta$ is a root of the equation $x^3 - 5x^2 - 1 = 0$ **3**

End of Question 6

Marks

Question 7 (15 marks) Use a SEPARATE writing booklet.

(a) By mathematical induction it is easy to show that

$$1^2 - 2^2 + 3^2 - \ \dots\ - (2n)^2 = -n(2n+1)$$

If, further, it is known that

$$1^2 + 2^2 + 3^2 + \ \dots\ + (2n)^2 = \frac{n}{3}(2n+1)(4n+1),$$

deduce that

$$1^2 + 2^2 + 3^2 + \ \dots\ + n^2 = \frac{n}{6}(n+1)(2n+1)$$

(Do not use induction) **3**

(b) (i) Using the substitution $t = \tan\frac{x}{2}$, or otherwise, evaluate

$$\int_0^{\frac{\pi}{2}} \frac{dx}{1 + \sin x}$$ **3**

(ii) Let $F(x)$ be a primitive function of $f(x)$.

Using this, or otherwise, show that

$$\int_0^{2a} f(x)\,dx = \int_0^a f(x) + f(2a - x)\,dx$$ **2**

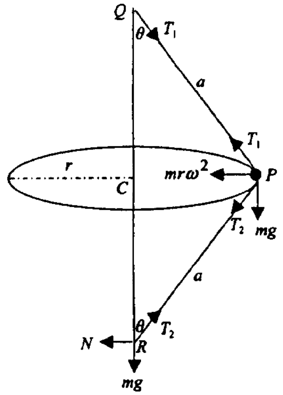
(iii) Deduce $\int_0^{\pi} \frac{x}{1 + \sin x}\,dx$ **3**

Question 7 continues next page

Marks

Question 7 (continued)

(c) A mass m at P is freely joined to two equal light rods PQ and PR of length a. The end Q of PQ is pivoted to a fixed point Q and the end R of PR is freely joined to a ring of mass m which slides on a smooth vertical pole. If P rotates in a horizontal circle with uniform angular velocity ω, show the angle of inclination of the rods PQ and PR to the vertical is $\tan^{-1}\left(\frac{rw^2}{3g}\right)$. T_1, T_2 are tensions in the rods, N is the normal reaction of QR on the ring R. **4**



End of Question 7

Marks

Question 8 (15 marks) Use a SEPARATE writing booklet.

(a) The roots of $z^n = 1$, n a positive integer, are

$$z_k = \cos\frac{2\pi k}{n} + i\sin\frac{2\pi k}{n}, \quad k = 1, 2, \dots, n$$

(i) Show that $z_k^n = z_1^{kp}$, p a positive integer **2**

(ii) If z_k is such that $z_k, z_k^2, z_k^3, \dots, z_k^n$ generates all the roots of $z^n = 1$, then z_k is called a primitive root of $z^n = 1$

(α) Show that z_1 is a primitive root of $z^n = 1$ **1**

(β) Show that z_5 is a primitive root of $z^6 = 1$ **2**

(γ) Suppose the highest common factor of n and k is h, ie, $n = ph$ and $k = qh$, p, q integers.

Show that for z_k to be a primitive root of $z^n = 1$, then $h = 1$ **2**

(b) (i) Show that $\sum_{k=0}^{n-1}(1-x)^k = \frac{1-(1-x)^n}{x}$, $x \neq 0$ **2**

(ii) Deduce that $\sum_{k=0}^{n-1}(1-x)^k = \sum_{k=1}^{n}(-1)^{k-1}\binom{n}{k}x^{k-1}$ **2**

(iii) Explain or show why $\int \sum_{k=0}^{n-1}(1-x)^k\,dx = \sum_{k=0}^{n-1}\int(1-x)^k\,dx$ **1**

(iv) Deduce that $\sum_{k=1}^{n}(-1)^{k-1}\binom{n}{k}\frac{1}{k} = 1 + \frac{1}{2} + \frac{1}{3} + \dots + \frac{1}{n}$ **3**