

Answer each question in a SEPARATE writing booklet. Extra writing booklets are available.

Marks

Question 1 (15 marks) Use a SEPARATE writing booklet.

(a) (i) Find a, b, c if $\frac{1}{x(x+1)^2} = \frac{a}{x} + \frac{b}{x+1} + \frac{c}{(x+1)^2}$

2

(ii) Find $\int \frac{dx}{x(x+1)^2}$

2

(b) Evaluate $\int_0^5 \frac{x}{\sqrt{4-x^2}} dx$

3

(c) (i) Simplify $\frac{1}{\sqrt{2}} \left(\frac{1}{\sqrt{2+x}} + \frac{1}{\sqrt{2-x}} \right)$

1

(ii) Show that $\frac{1}{\sqrt{2}} \int_{-1}^0 \left(\frac{1}{\sqrt{2+x}} + \frac{1}{\sqrt{2-x}} \right) dx = \sqrt{2} \ln(\sqrt{2}+1)$

2

(iii) Use completion of square to evaluate $\int_0^1 \frac{2}{1-t^2+2t} dt$

3

(d) Use the substitution $t = \tan \frac{x}{2}$ to evaluate $\int_0^{\frac{\pi}{2}} \frac{dx}{\cos x + \sin x}$

2

End of Question 1

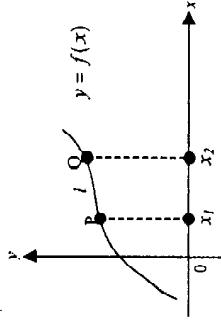
Question 2 (15 marks) Use a SEPARATE writing booklet.

Marks

(a) (i) If $x^2 + y^2 = 1$, show that $\frac{dy}{dx} = \frac{-x}{y}, y \neq 0$

1

(ii)



In the diagram, the length, l , of the arc PQ is given by $l = \int_{x_1}^{x_2} \sqrt{1 + \left(\frac{dy}{dx}\right)^2} dx$

Use this result to prove that the length of the arc of the circle $x^2 + y^2 = 1$ between the points $(0, 1)$ and $\left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$ is $\frac{\pi}{6}$

3

(b) (i) $f(x), f'(x)$ and $f''(x)$ exist for $a \leq x \leq b$

Show that

(i) $\int_a^b f'(x) dx = \int_a^b f''(a+b-x) dx$

2

(ii) $\int_a^b x f''(x) dx = bf'(b) - f(b) - (af'(a) - f(a))$

3

(c) Let $z = 1-i$ and $w = -3+3i$

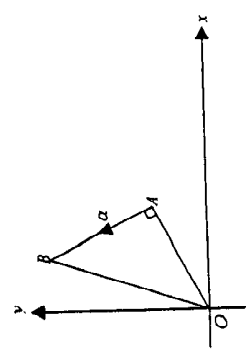
(i) Find $\frac{1}{z}$ in the form $x+iy$

1

(ii) Find $\arg(z+w)$

2

Question 2 continues next page



(d) In the Argand diagram, $\triangle OAB$ is isosceles and right angled at A . \vec{AB} represents the complex number α

- (i) What complex number corresponds to the vertex A ? 1
- (ii) What complex number corresponds to the vertex B ? 1
- (iii) Show that the area of $\triangle OAB$ is $\frac{1}{2}\alpha\bar{\alpha}$ 1

End of Question 2

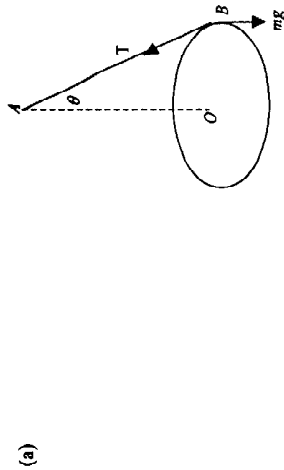
- (a) Consider $(x+iy)^2 = i$, x, y real
 - (i) Show that $|x+iy| = 1$ 1
 - (ii) Solve the equation $(x+iy)^2 = i$ 3
- (b) z is any complex number such that $|z-1| = 1$
 - (i) Sketch the locus of z in the Argand diagram. 1
 - (ii) Hence, or otherwise, show that $|z| + |z-2| \geq 2$ 2
 - (iii) If z were not on the locus in (i) would the result in (ii) still be true? Give a reason for your answer. 1

- (iv) If $0 < \arg z < \frac{\pi}{2}$, find the value of $\arg\left(\frac{z-2}{z}\right)$ 2
- (c) α, β, γ are the roots of $x^3 + x - 1 = 0$.

Find the values of

 - (i) $\alpha^2 + \beta^3 + \gamma^3$ 2
 - (ii) $\alpha^2 + \beta^3 + \gamma^3$ 2
- and (iii) write down an equation with roots $\frac{\alpha}{2}, \frac{\beta}{2}, \frac{\gamma}{2}$ 1

End of Question 3



A particle of mass m kg is attached to one end of a light string at B. The other end of the string is fixed at a point A. The particle rotates in a horizontal circle of radius r metres at g rad/s, the centre O of the circle being directly below A.

The forces acting on the particle are the tension in the string T and the gravitational force mg .

Let $\angle BAO = \theta$

- (i) Show that $T \sin \theta = mg^2 r$ 2
- (ii) Prove that $\theta = \tan^{-1}(gr)$ 2
- (iii) Prove that $T = mg\sqrt{1 + g^2 r^2}$ 2

Question 4 continues next page

(b) A is the series $x + x^2 + x^3 + \dots + x^n = \frac{x(1-x^n)}{1-x}$, $|x| < 1$ and

B is the series $1 + 2x + 3x^2 + \dots + nx^{n-1}$, $|x| < 1$

- (i) Deduce that the sum of series B is $\frac{1 - (n+1)x^n + nx^{n+1}}{(1-x)^2}$ 2

[DO NOT USE INDUCTION]

- (ii) Prove the result in (i) by induction, $n \geq 1$ 4

- (iii) The limiting sum of series A is 1. Find the limiting sum of series B. 3

End of Question 4

Question 5 (15 marks) Use a SEPARATE writing booklet.

Marks

(a) Consider the function $f(x) = \frac{1}{2}(\sqrt{x^2-1} - \ln(x + \sqrt{x^2-1}))$

(i) Find the domain of f .

2

(ii) Show that $f'(x) = \sqrt{x^2-1}$

3

(iii) Sketch the function.

1

(b) (i) Sketch the hyperbola $x^2 - y^2 = 1$, showing its foci, directrices and asymptotes.

3

(ii) A particular solid has as its base the region bounded by the hyperbola $x^2 - y^2 = 1$ and the line $x = 2$. Cross-sections perpendicular to this base and the x axis are semi-circles whose diameters are in the base.

Find the volume of this solid.

4

(iii) Show that the area of the base of the solid in (ii) is $2\sqrt{3} - \ln(2 + \sqrt{3})$

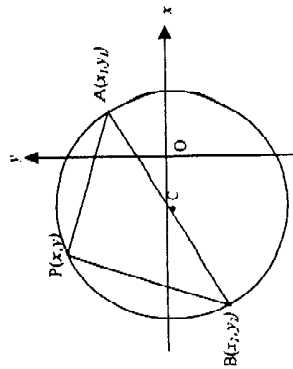
2

End of Question 5

Question 6 (15 marks) Use a SEPARATE writing booklet.

Marks

(a)



$P(x, y)$ is any point on the circle, centre C . $A(x_1, y_1)$ and $B(x_2, y_2)$ are the end points of a diameter of the circle.

Show that the equation of the circle is $(x - x_1)(x - x_2) + (y - y_1)(y - y_2) = 0$

2

(b) $A(a, 0)$ and $A'(-a, 0)$ are the vertices of the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1, a > 0$

$P(a \sec \theta, b \tan \theta)$ is any point on the hyperbola, $P \neq A$ or A' .

The tangent at P meets the tangents at A, A' at Q, R respectively.

(i) Prove that the equation of the tangent at P is $\frac{\sec \theta}{a}x - \frac{\tan \theta}{b}y = 1$

3

(ii) Find the coordinates of Q and R .

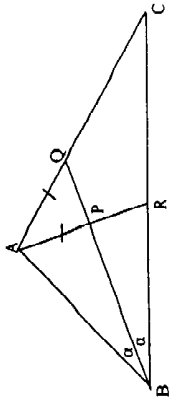
2

(iii) Prove that the circle with QR as a diameter passes through the two foci of the hyperbola.

4

Question 6 continues next page

(c)



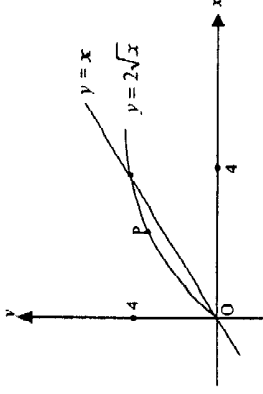
In the diagram, BQ bisects $\angle ABC$ and P is the point on BQ so that $AP = AQ$.

Prove that BA is a tangent to the circle through the points A, R and C.

4

End of Question 6

(a)



The region bounded by the curve $y = 2\sqrt{x}$ and the line $y = x$ is revolved about the line $y = x$.

Let $P(x, 2\sqrt{x})$ be a point on $y = 2\sqrt{x}$, $0 \leq x \leq 4$.

By considering slices through P perpendicular to the line $y = x$, find the volume of the solid of revolution.

5

(b) A particle of mass M moves in a straight line with velocity v under the action of two propelling forces $\frac{Mu^2}{y}$ and Mk^2v , u, k positive constants.

(i) Show that the acceleration equation of motion is $\frac{u^2 + k^2v^2}{y}$

1

(ii) Show that the distance travelled by the particle in increasing its velocity from $\frac{u}{k}$ to $\frac{2u}{k}$ is $\frac{u}{k^3} \left(1 - \tan^{-1} \frac{1}{3} \right)$

5

Question 7 continues next page

Question 7 (continued)

Marks

(c) (i) $x^2 + Ax + B = 0$ has integer coefficients. If $\alpha + \sqrt{\beta}$ is a root, α, β , rational, $\beta \geq 0$, show that $\alpha - \sqrt{\beta}$ is also a root.

1

(ii) $f(x) = x^4 - 4x^3 - 4x^2 + 16x + 16 = 0$ is known to have only real roots. Further, it is also known that there is at least one double root.

Express $f(x)$ as a product of factors with integer coefficients.

3

End of Question 7

Question 8 (15 marks) Use a SEPARATE writing booklet.

Marks

(a) (i) Given that $e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots$... when expressed as an infinite series, show that $\lim_{x \rightarrow \infty} \left(\frac{x^n}{e^x} \right) = 0$, $n = 0, 1, 2, 3, \dots$

1

(ii) Let $u_n = \int_0^x t^n e^{-t} dt$, $n = 0, 1, 2, 3, \dots$

Show that $u_n = nu_{n-1} - x^n e^{-x}$, $n \geq 1$

2

(iii) Let $f(n) = \lim_{x \rightarrow \infty} \int_0^x t^n e^{-t} dt = \int_0^{\infty} t^n e^{-t} dt$, $n = 0, 1, 2, 3, \dots$

Deduce that $f(n) = n!$

2

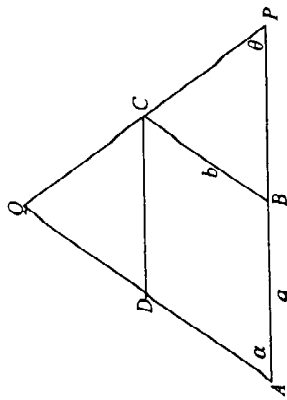
(iv) Evaluate $\int_0^{\infty} t^7 e^{-t^2} dt$

2

Question 8 continues next page

- (b) Show that $\frac{d}{d\theta} \left(\frac{\sin \theta}{\sin(\theta + \alpha)} \right) = \frac{\sin \alpha}{\sin^2(\theta + \alpha)}$, α constant 1

(c)



In the diagram, $ABCD$ is a fixed parallelogram where $AB = a$ and $BC = b$. $\angle DAB = \alpha$.
 A variable line through C meets AB produced at P and AD produced at Q .

Let $\angle BPC = \theta$, $0 < \theta < \pi - \alpha$

- (i) Show that the area of $\triangle MPQ$ is given by

$$A(\theta) = \frac{1}{2} \sin \alpha \left(\frac{a^2 \sin \theta}{\sin(\theta + \alpha)} + \frac{b^2 \sin(\theta + \alpha)}{\sin \theta} + 2ab \right) \quad 2$$

- (ii) Show that as $\theta \rightarrow 0$, $A(\theta) \rightarrow \infty$ 1

- (iii) Prove that the minimum area of $\triangle MPQ$ occurs when $\cot \theta = \frac{a}{b} \operatorname{cosec} \alpha - \cot \alpha$ 3

- (iv) Draw a diagram to show clearly the position of the side PQ for which the area of $\triangle MPQ$ is a minimum. Include on your diagram the parallelogram $ABCD$. 1

End of Examination