THE KING'S SCHOOL

2005

Higher School Certificate

Trial Examination

Mathematics Extension 2

General Instructions

- Reading time – 5 minutes
- Working time – 3 hours
- Write using black or blue pen
- Board-approved calculators may be used
- A table of standard integrals is provided
- All necessary working should be shown in every question

Total marks – 120

- Attempt Questions 1-8
- All questions are of equal value

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Attempt Questions 1-8
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Answer each question in a SEPARATE writing booklet. Extra writing booklets are available.

Marks

Question 1 (15 marks) Use a SEPARATE writing booklet.

(a) (i) Express $\dfrac{2}{1-x^2}$ in partial fractions. **2**

(ii) Show that $\displaystyle\int_0^{\frac{1}{4}} \frac{2}{1-x^2}\,dx = \ln\left(\frac{5}{3}\right)$ **2**

(iii) Evaluate $\displaystyle\int_0^{\frac{1}{2}} \frac{2x}{1-x^4}\,dx$ **2**

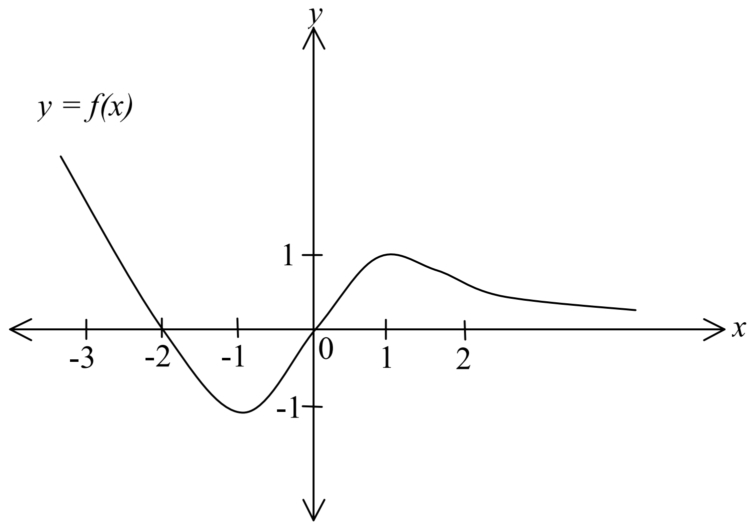
(b) Evaluate $\displaystyle\int_0^{\frac{\pi}{4}} \frac{2}{1+\sin 2x+\cos 2x}\,dx$ **3**

(c) Use completion of square to prove that

$\displaystyle\int_0^{1} \frac{4}{4x^2+4x+5}\,dx = \tan^{-1}\left(\frac{4}{7}\right)$ **3**

Question 1 is continued on the next page

(d)



On separate diagrams, sketch the graphs of:

(i) $y = \ln f(x)$ **2**

(ii) $y = e^{\ln f(x)}$ **1**

End of Question 1

Question 2 (15 marks) Use a SEPARATE writing booklet. **Marks**

(a) (i) Use integration by parts to show that

$$\int_0^1 (x-1)\, f'(x)\, dx = f(0) - \int_0^1 f(x)\, dx$$ **2**

(ii) Hence, or otherwise, evaluate $\displaystyle\int_0^1 \frac{x-1}{(x+1)^2}\, dx$ **2**

(b) Let $z = x + iy$, x, y real, where $\arg z = \dfrac{3\pi}{5}$

(i) Sketch the locus of z **1**

(ii) Find $\arg(-z)$ **1**

(c) Sketch the region in the complex plane where $|z - i| \le |z + 1|$ **2**

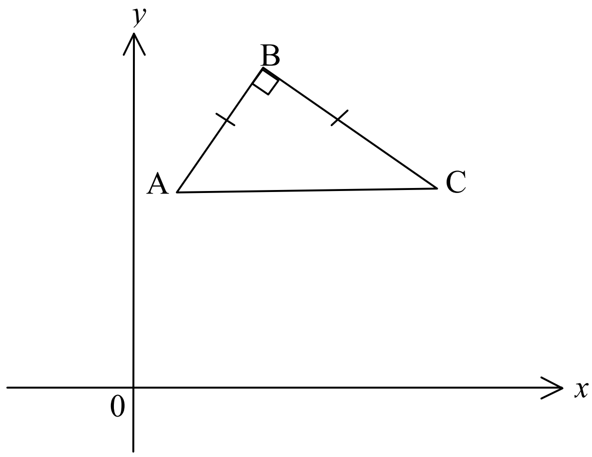
(d) $z = x + iy$, x, y real, is a complex number such that
$(z + \bar{z})^2 + (z - \bar{z})^2 = 4$

(i) Find the cartesian locus of z **2**

(ii) Sketch the locus of z in the complex plane showing any features necessary to indicate your diagram clearly. **2**

Question 2 is continued on the next page

(e)



In the Argand diagram, ΔABC is right-angled at B and isosceles.

A, B, C represent the complex numbers a, b, c respectively.

(i) Find the complex number $\overrightarrow{BA}$ in terms of a and b. **1**

(ii) Prove that $c = ai + b(1 - i)$ **2**

End of Question 2

Question 3 (15 marks) Use a SEPARATE writing booklet. **Marks**

(a) (i) Sketch the parabola $y = \dfrac{1+x^2}{2}$ and use it to sketch the curve $y = \dfrac{2}{1+x^2}$ on the same diagram. **2**

(ii) Hence, or otherwise, find the range of the function $y = \dfrac{2}{1+x^2} - 1$ **1**

(b) Consider the function $y = \cos^{-1}\left(\dfrac{1-x^2}{1+x^2}\right)$

(i) By using (a), or otherwise, find the range of the function. **2**

(ii) Show that $\dfrac{d}{dx}\cos^{-1}\left(\dfrac{1-x^2}{1+x^2}\right) = \dfrac{2x}{(1+x^2)\sqrt{x^2}}$ and

give the simplest expressions for the derivative if

(α) $x > 0$ and (β) $x < 0$ **3**

(iii) Sketch the curve $y = \cos^{-1}\left(\dfrac{1-x^2}{1+x^2}\right)$ **2**

(iv) The region bounded by $y = \cos^{-1}\left(\dfrac{1-x^2}{1+x^2}\right)$ and the line $y = \dfrac{\pi}{2}$ is revolved about the y axis.

Show that the volume of the solid of revolution is given by

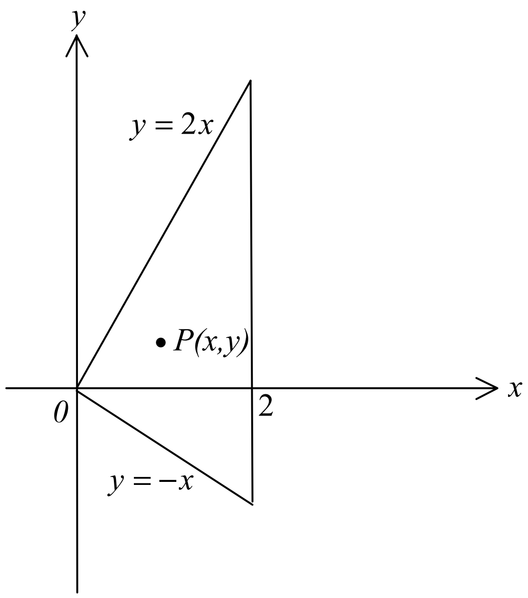
$$V = \pi \int_0^{\frac{\pi}{2}} \frac{1-\cos y}{1+\cos y}\, dy$$ **2**

(v) Find the volume V. **3**

End of Question 3

Question 4 (15 marks) Use a SEPARATE writing booklet. **Marks**

(a)



The base of a solid is the triangular region bounded by the lines $y = 2x$, $y = -x$ and $x = 2$.

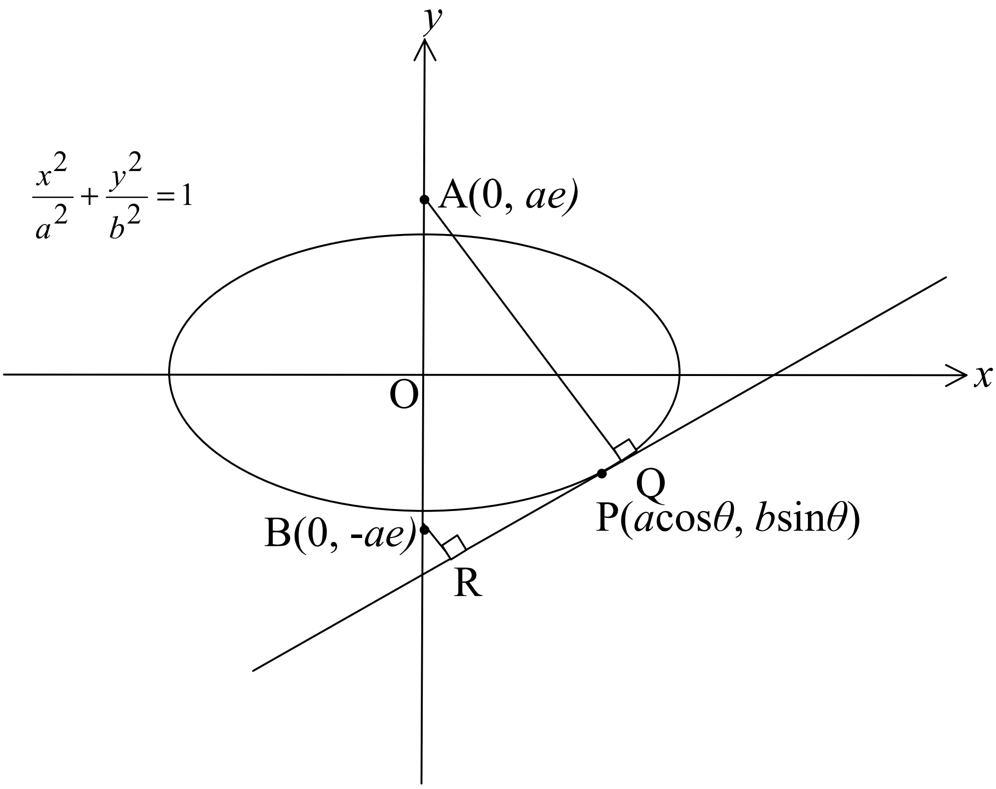
At each point $P(x, y)$ in the base the height of the solid is $4x^2 + x$

Find the volume of the solid. **4**

(b) If $xy^2 + 1 = x^2$, $y \neq 0$, show that $\dfrac{dy}{dx} = \dfrac{1}{y} - \dfrac{y}{2x}$ **2**

Question 4 is continued on the next page

(c)



P($a\cos\theta$, $b\sin\theta$) is a point on the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, $a > b > 0$,

where e is the eccentricity of the ellipse.

From A(0, ae) and B(0, $-ae$) perpendiculars are drawn to meet the tangent at P($a\cos\theta$, $b\sin\theta$) at Q and R, respectively.

(i) Prove that the equation of the tangent at P is

$\frac{\cos\theta}{a}x+\frac{\sin\theta}{b}y=1$ **3**

(ii) Hence, or otherwise, show that the line $x\cos\alpha + y\sin\alpha = k$ is a tangent to the ellipse if $a^2\cos^2\alpha + b^2\sin^2\alpha = k^2$ **2**

(iii) Hence, or otherwise, prove that $AQ^2 + BR^2 = 2a^2$ **4**

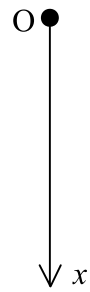
End of Question 4

Question 5 (15 marks) Use a SEPARATE writing booklet. **Marks**

(a) A particle of mass m moving with speed v experiences air resistance mkv^2, where k is a positive constant. g is the constant acceleration due to gravity.

(i) The particle of mass m falls from rest from a point O.

Taking the positive x axis as vertically downward, show that $\ddot{x} = k(V^2 - v^2)$, where V is the terminal speed.



2

(ii) Another particle of mass m is projected vertically upward from ground level with a speed V^2, where V is the terminal speed as in (i).

Prove that the particle will reach a maximum height of $\frac{1}{2k}\ln(1 + V^2)$ **3**

(iii) Prove that the particle in (ii) will return to the ground with speed U where $U^{-2} = V^{-2} + V^{-4}$ **4**

(b) The ellipse $\frac{x^2}{4} + \frac{y^2}{3} = 1$ is revolved about the line $x = 4$.

(i) Use the method of cylindrical shells to show that the volume of the solid of revolution is given by

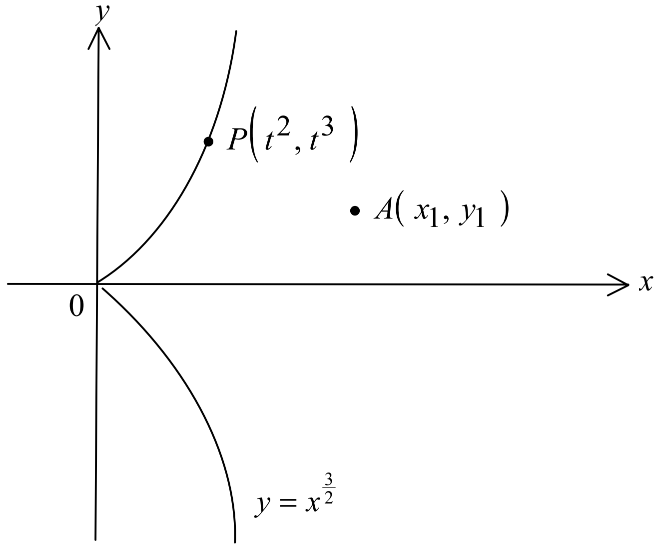
$$V = 8\sqrt{3}\,\pi \int_{-2}^{2} \sqrt{4 - x^2}\, dx - 2\sqrt{3}\,\pi \int_{-2}^{2} x\sqrt{4 - x^2}\, dx$$ **4**

(ii) Prove that the volume $V = 16\sqrt{3}\,\pi^2$ **2**

End of Question 5

Question 6 (15 marks) Use a SEPARATE writing booklet. **Marks**

(a)



$P(t^2, t^3)$ is any point in the curve $y = x^{\frac{3}{2}}$

(i) Show that the equation of the tangent at $P(t^2, t^3)$ is $3tx - 2y - t^3 = 0$ **2**

(ii) $A(x_1, y_1)$ is a point not on the curve $y = x^{\frac{3}{2}}$

Deduce that at most three tangents to the curve pass through A. **1**

(iii) If the tangents with parameters t_1, t_2, t_3 do pass through $A(x_1, y_1)$, show that

(α) $t_1^3 + t_2^3 + t_3^3 = -6y_1$ **2**

(β) $(t_1 t_2)^2 + (t_2 t_3)^2 + (t_3 t_1)^2 = 9x_1^2$ **2**

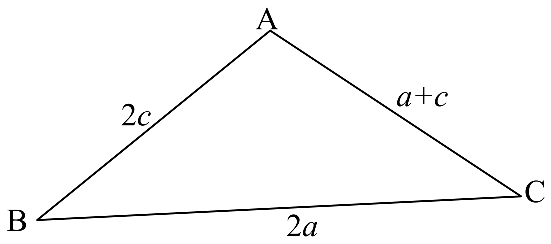
(iv) Find a cubic equation with roots $\frac{1}{t_1}, \frac{1}{t_2}, \frac{1}{t_3}$ **2**

Question 6 is continued on the next page

(b) (i) Given that $\sin(X+Y)+\sin(X-Y)=2\sin X\cos Y$, show that

$$\sin A+\sin C=2\sin\frac{A+C}{2}\cos\frac{A-C}{2}$$ **1**

(ii) Consider ΔABC where



(α) Use the sine rule to show that $\sin A+\sin C=2\sin B$ **2**

(β) Deduce that $\sin\frac{B}{2}=\frac{1}{2}\cos\frac{A-C}{2}$ **3**

End of Question 6

Question 7 (15 marks) Use a SEPARATE writing booklet. **Marks**

(a) Let $f(n) = (n+1)^3 + (n+2)^3 + \ldots + (2n-1)^3 + (2n)^3$, $n = 1, 2, 3, \ldots$

(i) Show that $f(n+1) - f(n) = (2n+1)^3 + 7(n+1)^3$ **2**

(ii) Show that

$$(2n+1)^3 - \frac{2n+1}{4}(3n+1)(5n+3) = \frac{2n+1}{4}(n+1)^2$$ **1**

(iii) Use mathematical induction for integers $n = 1, 2, 3, \ldots$ to prove that

$$f(n) = (n+1)^3 + (n+2)^3 + \ldots + (2n)^3 = \frac{n^2}{4}(3n+1)(5n+3)$$

4

(iv) Given that $1^3 + 2^3 + \ldots + n^3 = \left[\frac{n}{2}(n+1)\right]^2$, prove that

$(n+1)^3 + (n+2)^3 + \ldots + (2n)^3 = \frac{n^2}{4}(3n+1)(5n+3)$ without induction. **2**

(b) (i) Show that $\dfrac{\binom{n}{k}}{n^k} = \dfrac{\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)\ldots\left(1-\frac{k-1}{n}\right)}{k!}$, $2 \le k \le n$ **2**

(ii) Deduce that $\dfrac{\binom{n+1}{k}}{(n+1)^k} > \dfrac{\binom{n}{k}}{n^k}$, $2 \le k \le n$ **2**

(iii) Deduce that, if n is a positive integer, $\left(1+\frac{1}{n+1}\right)^{n+1} > \left(1+\frac{1}{n}\right)^n$ **2**

End of Question 7

Question 8 (15 marks) Use a SEPARATE writing booklet. **Marks**

(a) Consider the equation

$z^7 - 1 = (z-1)(z^6 + z^5 + z^4 + z^3 + z^2 + z + 1) = 0$

(i) Show that $v = \cos\frac{2\pi}{7} + i\sin\frac{2\pi}{7}$ is a complex root of $z^7 - 1 = 0$ **1**

(ii) Show that the other five complex roots of $z^7 - 1 = 0$ are

$$v^k \text{ for } k = 2, 3, 4, 5, 6$$ **2**

(iii) Show that $\overline{\left(v^{7-k}\right)} = v^k$ for $k = 1, 2, \ldots, 6$

i.e. show that the conjugate of v^{7-k} is v^k **2**

(iv) Deduce that $v + v^2 + v^4$ and $v^3 + v^5 + v^6$ are conjugate complex numbers. **1**

(v) Deduce that $\cos\frac{\pi}{7} - \cos\frac{2\pi}{7} + \cos\frac{3\pi}{7} = \frac{1}{2}$ **3**

Question 8 is continued on the next page

(b) (i) Use a suitable substitution to show that

$$\int_0^{\frac{\pi}{2}} \cos x \sin^{n-1} x \, dx = \frac{1}{n}, \quad n = 1, 2, 3, \ldots$$ **1**

(ii) Show by integration that

$$\int x \sin x \, dx = -x \cos x + \sin x$$ **1**

(iii) Let $t_n = \int_0^{\frac{\pi}{2}} x \sin^n x \, dx, \quad n = 0, 1, 2, \ldots$

Use integration by parts to prove that

$$t_n = \frac{1}{n^2} + \frac{n-1}{n} t_{n-2}, \quad n = 2, \quad 3, \quad 4, \quad \ldots$$ **4**

End of Examination