THE KING'S SCHOOL

2007
Higher School Certificate
Trial Examination

Mathematics Extension 2

General Instructions

- Reading time – 5 minutes
- Working time – 3 hours
- Write using black or blue pen
- Board-approved calculators may be used
- A table of standard integrals is provided at the back of this paper
- All necessary working should be shown in every question
- Answer each question in a separate booklet

Total marks – 120

- Attempt Questions 1-8
- All questions are of equal value

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Answer each question in a SEPARATE writing booklet. Extra writing booklets are available.

Question 1 (15 marks) Use a SEPARATE writing booklet. **Marks**

(a) Use the substitution $t = \tan\frac{x}{2}$ to evaluate $\displaystyle\int_0^{\frac{\pi}{2}} \frac{dx}{3 + \cos x}$ **3**

(b) (i) Evaluate $\displaystyle\int_0^{\frac{\sqrt{3}}{2}} \frac{x}{\sqrt{1 - x^2}}\, dx$ **3**

(ii) Evaluate $\displaystyle\int_0^{\frac{\sqrt{3}}{2}} \cos^{-1} x \, dx$ **2**

(c) (i) Find A if $\dfrac{2x + A}{x^2 + 1} - \dfrac{2}{x - 2} \equiv \dfrac{x - 12}{(x^2 + 1)(x - 2)}$ **1**

(ii) Evaluate $\displaystyle\int_0^1 \frac{x - 12}{(x^2 + 1)(x - 2)}\, dx$ **3**

(d) Find $\displaystyle\int \tan^3 x \, dx$ **3**

End of Question 1

Question 2 (15 marks) Use a SEPARATE writing booklet. **Marks**

(a) Let $z = a + i$ and $w = 1 + ai$, a real. Find

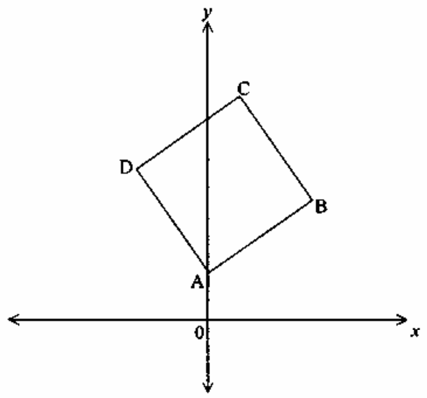
(i) $\left|\frac{z}{w}\right|$ 1

(ii) $\arg zw$ 2

(b) The point $P(x, y)$ represents the complex number z in the Argand diagram.

Sketch the locus of P if $Im(1 - i)z \geq 1$ 3

(c)



The diagram shows the square ABCD in the complex plane.

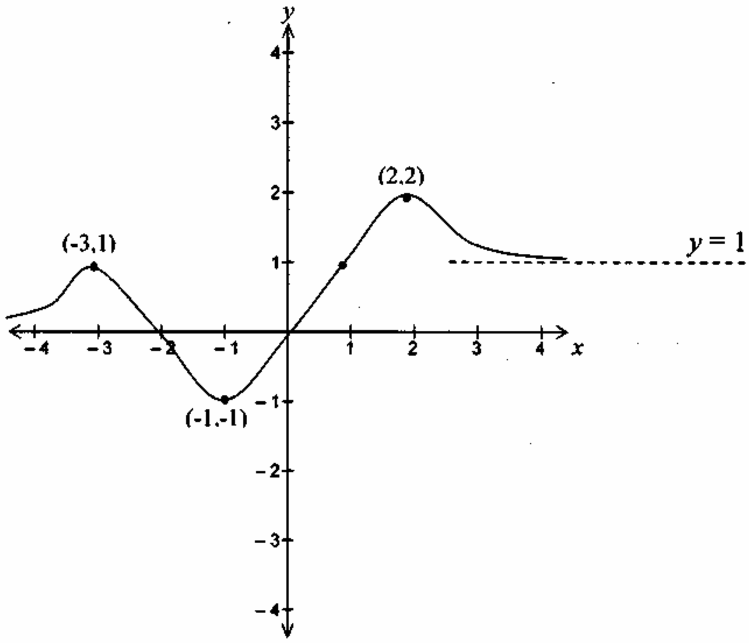
A represents the complex number i and B represents the complex number z.

(i) Find the complex number $\overrightarrow{AD}$ 2

(ii) Hence or otherwise find the complex number represented by the point C. 1

Question 2 continues next page

(d) The diagram shows the graph of $y = f(x)$



The lines $y = 1$ and the x axis are asymptotes.

Draw separate sketches of the graphs of:

(i) $y = \dfrac{1}{f(x)}$ **2**

(ii) $y = \ln f(x)$ **2**

(iii) $y = f(x + |x|)$ **2**

End of Question 2

Question 3 (15 marks) Use a SEPARATE writing booklet. **Marks**

(a) (i) $S(c, o)$ and $S^1(-c, o)$, where $c > 0$, are the foci of the hyperbola $x^2 - y^2 = 2$

Find S and sketch the hyperbola showing its foci, directrices, asymptotes and any intercepts made with the coordinate axes. **3**

(ii) $P(\sqrt{2}\sec\theta, \sqrt{2}\tan\theta)$ is a point in the first quadrant on the hyperbola $x^2 - y^2 = 2$ in (i).

A circle with centre P and radius PS is drawn.

(α) Find the length of the radius in simplest form. **2**

(β) The line S^1P cuts the circle at Q and R where Q is between S^1 and P.

It can be shown that $Q = \left(\dfrac{2}{\sqrt{2}\sec\theta + 1}, \dfrac{2\sqrt{2}\tan\theta}{\sqrt{2}\sec\theta + 1} \right)$

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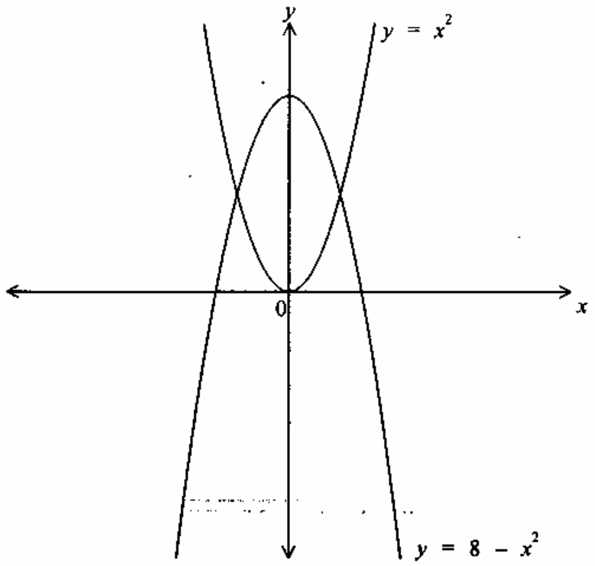
Prove that QS is parallel to the normal to the hyperbola at P. **3**

(γ) Explain why RS is parallel to the tangent to the hyperbola at P. **2**

Question 3 continues next page

(b)



The diagram shows the two parabolae $y = x^2$ and $y = 8 - x^2$

A solid is formed using the region enclosed between the two parabolae as its base.

Cross-sections parallel to the y axis and perpendicular to the xy plane are semi-circles where the diameters are in the base of the solid.

Prove that the volume of this solid is $\frac{256\pi}{15}$ cubic units. **5**

End of Question 3

Question 4 **(15 marks)** Use a SEPARATE writing booklet. **Marks**

(a) A particular curve passes through the origin and its derivative is given by $\frac{dy}{dx} = \sqrt{4y^2 + 1}$

(i) Prove that $\frac{d^2y}{dx^2} = 4y$ **2**

(ii) Use the table of standard integrals to find x as a function of y. **2**

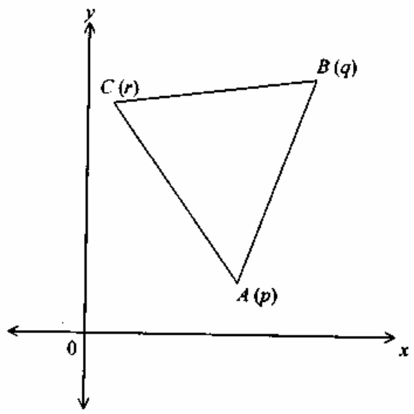
(b) (i) Sketch the region where $0 \le y \le x - x^2$ **1**

(ii) The region in (i) is revolved about the line $x = -1$

Use the method of cylindrical shells to find the volume of the solid of revolution generated. **4**

(c) (i) Express $\frac{1}{2}(1 + i\sqrt{3})$ in mod-arg form. **1**

(ii)



In the Argand diagram the points A, B, C represent the complex numbers p, q, r, respectively, where $r - p = \frac{1}{2}(1 + i\sqrt{3})(q - p)$

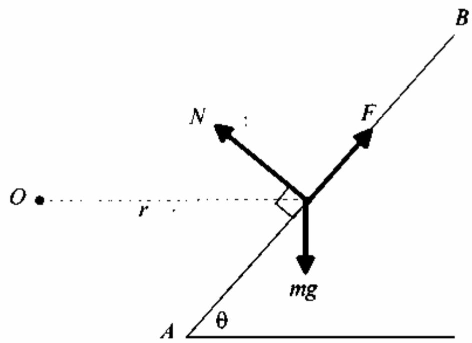
(α) Prove that $p - q = \frac{1}{2}(1 + i\sqrt{3})(r - q)$ **3**

(β) Deduce that $p^2 + q^2 + r^2 = pq + qr + rp$ **2**

End of Question 4

Question 5 (15 marks) Use a SEPARATE writing booklet. **Marks**

(a)



The diagram shows the forces exerted on a car of mass m travelling at speed v on a banked circular track AB of radius r. The track is banked inwards at θ to the horizontal. The road exerts the normal force N at right angles to the road and there is a frictional force F exerted up the track.

(i) By resolving the forces in the direction BA, or otherwise, show that **2**

$$F = mg\sin\theta - \frac{mv^2}{r}\cos\theta$$

(ii) Deduce that $v^2 < gr\tan\theta$ **2**

(iii) Draw a diagram showing the forces on the car if $v^2 > gr\tan\theta$ **1**

(iv) Find an expression for N not involving F. **2**

(b) u, v, w are the roots of $x^3 + Ax + B = 0$

(i) Show that $u^2 + v^2 + w^2 = -2A$ **2**

(ii) Let $y = \frac{v}{w} + \frac{w}{v}$

Prove that $u^3 + 2Au - By = 0$ **2**

(iii) By using another equation involving u^3 show that $u = \frac{B}{A}(y + 1)$ **2**

(iv) Show that the equation with roots $\frac{v}{w} + \frac{w}{v}, \frac{w}{u} + \frac{u}{w}$ and $\frac{u}{v} + \frac{v}{u}$ is

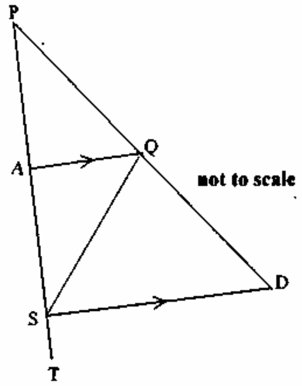
$B^2(x + 1)^3 + A^3(x + 1) + A^3 = 0$ **1**

(v) Evaluate $\frac{v}{w} + \frac{w}{v} + \frac{w}{u} + \frac{u}{w} + \frac{u}{v} + \frac{v}{u}$ **1**

End of Question 5

Question 6 **(15 marks)** Use a SEPARATE writing booklet. **Marks**

(a)



In the diagram PAST and PQD are straight lines and AQ || SD

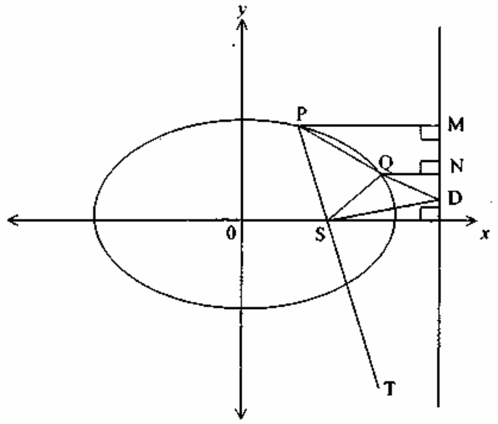
Further, $\dfrac{PS}{QS} = \dfrac{PD}{QD}$

(i) Explain why QS = AS **1**

(ii) Deduce that $\angle DSQ = \angle DST$ **2**

Question 6 continues next page

(b)



The diagram shows a chord PQ of an ellipse meeting a directrix at D. S is the corresponding focus. PM and QN meet this directrix at right angles at M and N, respectively.

(i) Show that $\dfrac{PS}{QS} = \dfrac{PM}{QN}$ **1**

(ii) Deduce that $\dfrac{PS}{QS} = \dfrac{PD}{QD}$ **1**

(iii) Deduce that $\angle DSQ = \angle DST$ **1**

(iv) Deduce that if the tangent at P meets the directrix at R then $\angle PSR = 90^\circ$ **2**

(c) (i) w is a complex root of $x^3 - 1 = 0$

(α) Explain why $\overline{w}$ is the other complex root and deduce that $1 + w + \overline{w} = 0$ **2**

(β) Show that $\overline{w} = w^2$ **1**

(ii) $A(x)$ and $B(x)$ are two polynomials with complex coefficients such that $A(x^3) + xB(x^3) \equiv (x^2 + x + 1)Q(x)$, where $Q(x)$ is a polynomial with complex coefficients.

(α) Prove that $A(1) = 0$ and $B(1) = 0$ **3**

(β) Deduce that $A(x^3) + xB(x^3)$ is divisible by $x^3 - 1$ **1**

End of Question 6

Question 7 (15 marks) Use a SEPARATE writing booklet. **Marks**

(a) You are given the identity $\cos(A + B) + \cos(A - B) \equiv 2\cos A \cos B$

(i) Evaluate $\displaystyle\int_0^{\frac{\pi}{4}} \cos 5x \cos 3x \quad dx$ **2**

(ii) Find the general solutions of the equation $\cos 5x + \cos 3x + 2\cos x = 0$ **3**

(b) A particle of unit mass moves on the x axis against a resistance numerically equal to $v^2 + v^3$, where v is its velocity. Initially the particle is travelling with velocity u, where $u > 0$.

(i) Prove that when the velocity is $\frac{u}{2}$ the distance X travelled by the particle is given by $X = \ln\left(\frac{2 + u}{1 + u}\right)$ **4**

(ii) Prove that if T is the time taken to travel the distance X then $u(T + X) = 1$ **4**

(iii) Thomas examined the motion of the particle more thoroughly. Thomas alleged that if the particle started at the origin then the velocity v, displacement x and time t were related by the equation

$$v = \frac{u}{ux + ut + 1}$$ **2**

By finding a suitable derivative, show that Thomas is correct.

End of Question 7

Question 8 (15 marks) Use a SEPARATE writing booklet. **Marks**

(a) Let $u_n = \int_0^1 x^{2007}(1-x)^n \, dx, \qquad n = 0, 1, 2, \ldots$

(i) By considering $u_n - u_{n-1}$ show that $u_n < u_{n-1}$ **2**

(ii) Use integration by parts to show that $u_n = \frac{n}{2008+n} u_{n-1}$, $n \geq 1$ **3**

(iii) Deduce that $u_n = \frac{2007!\, n!}{(2008+n)!}$ **2**

(b) An extraordinary identity, due to the Swiss mathematician Leonard Euler (1707-83), states

$$e^{i\theta} = \cos\theta + i\sin\theta \quad \text{for all real values of } \theta$$

(i) Show that $\cos\theta = \frac{1}{2}(e^{i\theta} + e^{-i\theta})$ and $\sin\theta = \frac{1}{2i}(e^{i\theta} - e^{-i\theta})$ **2**

(ii) Deduce that

$$1 + 2\cos\theta + 2\cos 2\theta + \ldots + 2\cos n\theta = \frac{\sin\left(n+\frac{1}{2}\right)\theta}{\sin\frac{1}{2}\theta}, \quad n \geq 1$$ **3**

(iii) Hence or otherwise find $\lim_{\theta \to 0} \frac{\sin\left(n+\frac{1}{2}\right)\theta}{\sin\frac{1}{2}\theta}$ **1**

(iv) Use (ii) to show that

$$1 + (2\cos\theta)^2 + (2\cos 2\theta)^2 + \ldots + (2\cos n\theta)^2 = 2n + \frac{\sin(2n+1)\theta}{\sin\theta}, \quad n \geq 1$$ **2**

End of Examination