Teacher's Name ____________________

Student Number

Knox Grammar School

2014
Trial Higher School Certificate Examination

Mathematics Extension 2

General Instructions
- Reading time – 5 minutes
- Working time - 3 hours
- Write using blue or black pen only
- Board approved calculators may be used
- A table of standard integrals is provided at the back of this paper
- All necessary working should be shown in every question

Subject Teachers
Mr I Bradford
Mr D Sedgman

Setter

Mr M Vuletich

This paper MUST NOT be removed from the examination room

Number of Students in Course: 34

Total Marks – 100

Section I 10 Marks
- Answer Questions 1 to 10
- Use the Multiple Choice Answer Sheet

Section II 90 Marks
- Answer Questions 11 to 16
- All questions are worth 15 marks
- Answer each question in a separate Writing Booklet.

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Section I

10 Marks
Attempt Questions 1–10
Allow about 15 minutes for this section

Use the multiple-choice answer sheet for Questions 1-10

1. Which of these ellipses has foci $(0, \pm 3)$?

A) $8x^2 + y^2 = 8$ B) $5x^2 + 4y^2 = 20$

C) $16x^2 + 25y^2 = 400$ D) $25x^2 + 16y^2 = 400$

2. Find $\int \cot x \, dx$

A) $-\operatorname{cosec}^2 x + c$ B) $-\ln(\operatorname{cosec} x) + c$ C) $\frac{1}{2}\cot^2 x + c$ D) $\ln(\sec x) + c$

3. The speed of a particle moving in a horizontal circle with radius 6 cm is $48\pi \text{ cm s}^{-1}$.

How many revolutions per second does this particle make?

A) 4 B) 240 C) 480 D) 480π

4. The polynomial equation $x^3 - 2x^2 + 3 = 0$ has roots α, β and γ.

What is the value of $\alpha^3 + \beta^3 + \gamma^3$?

A) -2 B) -1 C) -8 D) 8

5. Evaluate $\int_1^e \frac{dx}{x\sqrt{1+(\ln x)^2}}$.

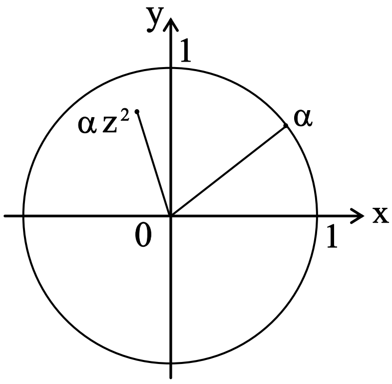
A) $-\frac{\pi}{4} + \tan^{-1} e$ B) $\ln\left(\frac{e+\sqrt{e^2+1}}{1+\sqrt{2}}\right)$ C) $\frac{\pi}{4}$ D) $\ln(1+\sqrt{2})$

6. What is the simplest expression of

$$\text{Arg}\left(\frac{1}{1+i}\right) + \text{Arg}\left(\frac{1}{(1+i)^2}\right) + \text{Arg}\left(\frac{1}{(1+i)^3}\right) + \ldots + \text{Arg}\left(\frac{1}{(1+i)^{20}}\right) ?$$

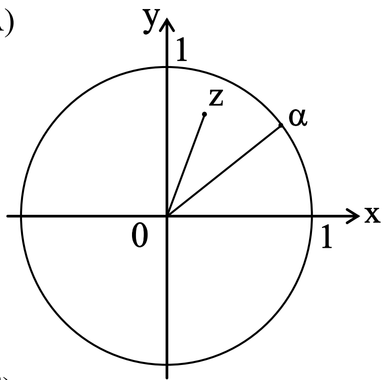
A) $-\frac{\pi}{2}$ B) $-\frac{\pi}{4}$ C) $\frac{\pi}{4}$ D) $\frac{\pi}{2}$

7. The Argand diagram below shows the complex numbers α and αz^2.

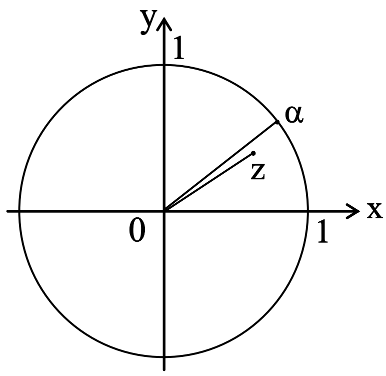


Which of the following best represents the positions of z and α?

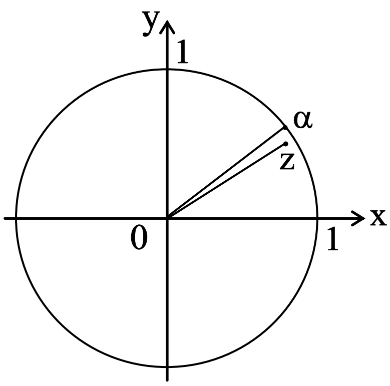
A)



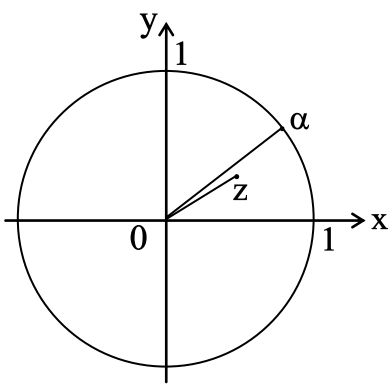
B)



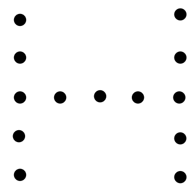
C)



D)



8. The diagram shows a shape made by 13 points.



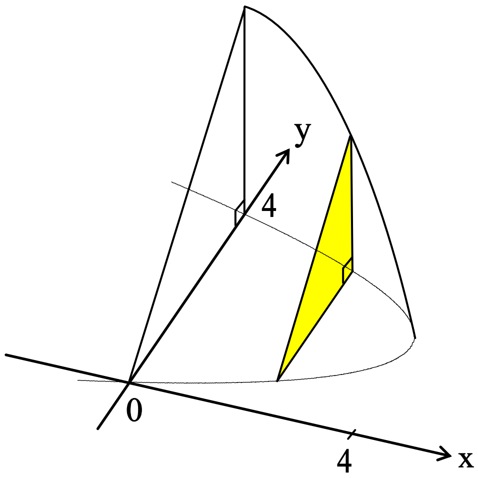
How many triangles can be made with these points as vertices?

A) ${}^{13}C_3 - 3\,{}^{5}C_3 - 3$ B) ${}^{13}C_3 - 2\,{}^{5}C_3 - 4$

C) ${}^{13}C_3 - 3\,{}^{5}C_3 - 4$ D) ${}^{13}C_3 - 3\,{}^{5}C_3 - 5$

9. The base of a solid is the region bounded by the parabola $x = 4y - y^2$ and the y axis.

Vertical cross sections are right angled isosceles triangles perpendicular to the x axis as shown.

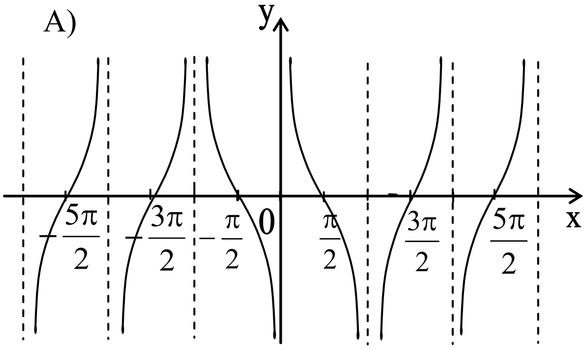


Which integral represents the volume of this solid?

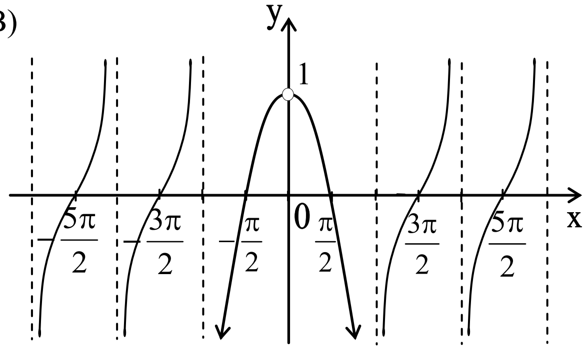
A) $\int_0^4 2\sqrt{4-x}\,dx$ B) $\int_0^4 \pi(4-x)\,dx$ C) $\int_0^4 (8-2x)\,dx$ D) $\int_0^4 (16-4x)\,dx$

10. Which of the following shows the graph of y = $x \cot x$?

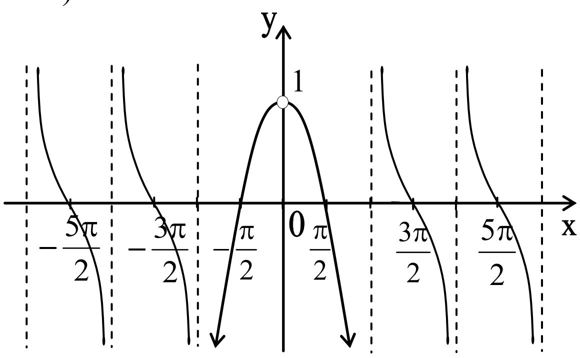
A)



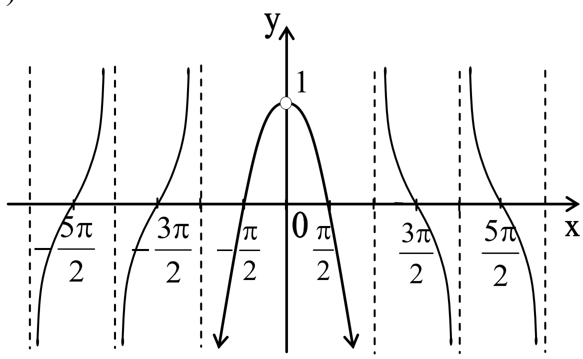
B)



C)



D)



End of Section I

Section II

90 Marks
Attempt Questions 11-16
Allow about 2 hours and 45 minutes for this section

Answer each question in a separate writing booklet. Extra writing booklets are available.

All necessary working should be shown in every question.

Question 11 **(15 marks)** Use a SEPARATE writing booklet **Marks**

(a) The complex number z is given by $z = -\sqrt{3} + i$.

(i) Express z in modulus argument form. **2**

(ii) Hence show that $z^7 + 64z = 0$ **2**

(b) Find values *A*, *B* and *C* such that:

$$\frac{8(1-x)}{\left(2-x^2\right)\left(2-2x+x^2\right)} = \frac{A-Bx}{\left(2-2x+x^2\right)} - \frac{Cx}{\left(2-x^2\right)}$$ **3**

(c) Factorise $z^2 + 4iz + 5$ over the complex field. **1**

(d) Using the substitution $x = 2\sin\theta$, show that **4**

$$\int_{-1}^{\sqrt{3}} \frac{x^2}{\sqrt{4-x^2}}\,dx = \pi - \sqrt{3}$$

(e) Sketch the region in the Argand diagram defined by $z\overline{z} + 2(z + \overline{z}) \le 0$ **3**

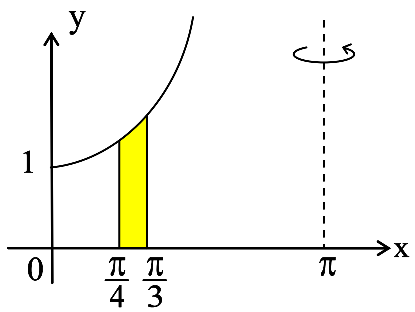
End of Question 11

Question 12 (15 marks) Use a SEPARATE writing booklet

(a) Use the substitution $t = \tan\frac{x}{2}$ to evaluate $\int \frac{dx}{1+\sin x+\cos x}$. **3**

(b) Consider the equation $z^3 + mz^2 + nz + 6 = 0$, where m and n are real. **3**
It is known that $1-i$ is a root of the equation. Find the values of m and n.

(c) The area bounded by the curve $y = \sec^2 x$, the x-axis, **4**
$x = \frac{\pi}{4}$ and $x = \frac{\pi}{3}$ is rotated about the line $x = \pi$ to form a solid.

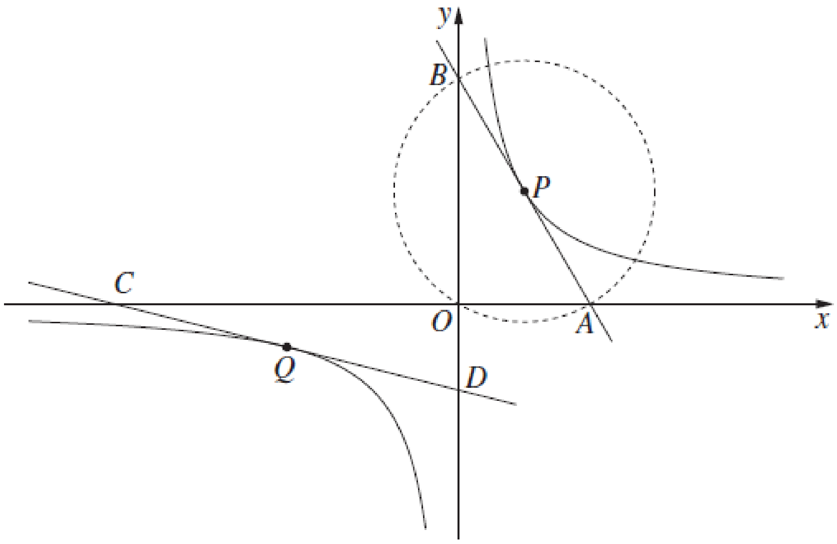


Use the method of cylindrical shells to find the volume of the solid.

Question 12 continues on page 8

(d) The points $P\left(cp, \frac{c}{p}\right)$ and $Q\left(cq, \frac{c}{q}\right)$, where $|p| \neq |q|$, lie on the rectangular hyperbola with equation $xy = c^2$.

The tangent to the hyperbola at P intersects the x-axis at A and the y-axis at B. Similarly, the tangent to the hyperbola at Q intersects the x-axis at C and the y-axis at D.



(i) Show that the equation of the tangent at P is $x + p^2y = 2cp$. **2**

(ii) Show that A, B and O are on a circle with centre P. **2**

(iii) Prove that BC is parallel to PQ. **1**

End of Question 12

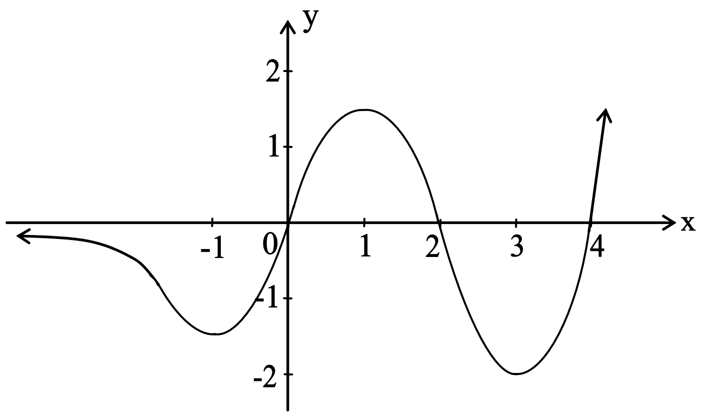
Question 13 (15 marks) Use a SEPARATE writing booklet

(a) (i) Let $I_n = \int_0^1 x^n \sqrt{1-x^3}\,dx$ for $n \geq 2$. Show that: **3**

$$I_n = \frac{2n-4}{2n+5} \times I_{n-3} \text{ for } n \geq 5$$

(ii) Hence find I_8 **2**

(b) The graph of a certain function $y = f(x)$ is shown below.



Sketch the following curves:

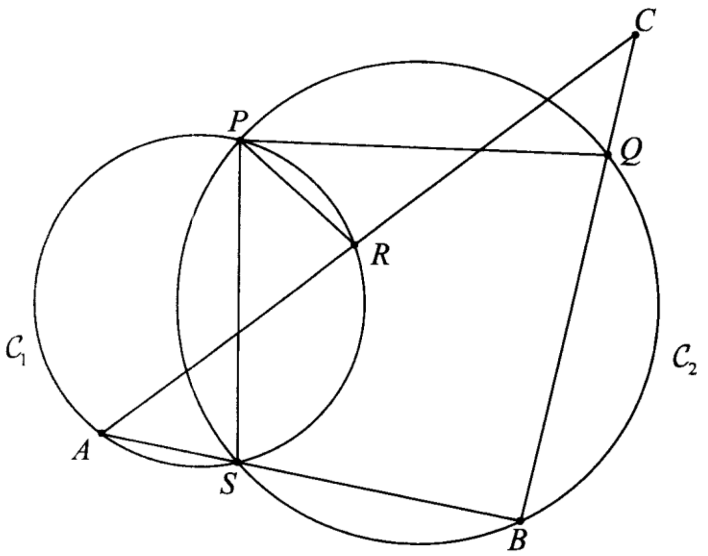
(i) $y = \dfrac{1}{f(x)}$ **2**

(ii) $y = \ln[1 - f(x)]$ **2**

(iii) $y^2 = 1 + f(x)$ **2**

Question 13 continues on page 10

(c) Two circles C_1 and C_2 meet at P and S. Points A and R lie on C_1 and points B and Q lie on C_2. AB passes through S and AR produced meets BQ produced at C, as shown in the diagram.



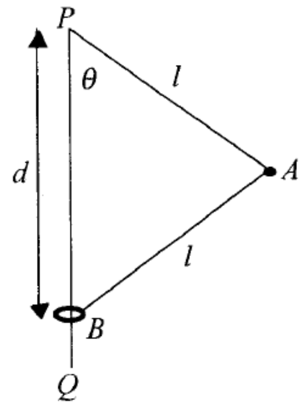
(i) Prove that $\angle PRA = \angle PQB$. **2**

(ii) Prove that the points P, R, Q and C are concyclic. **2**

End of Question 13

Question 14 **(15 marks)** Use a SEPARATE writing booklet

(a)



PQ is a smooth vertical rod. Particle *A* of mass *m* is attached to a point *P* by a string of length l and *A* is also attached by a second string of length l to a smooth ring *B* of mass *M* which is free to slide on the rod *PQ* without friction. *A* is set in motion in a horizontal circle about *PQ* with constant angular velocity ω. *B* is in equilibrium.

T_1 and T_2 are the tensions in the strings *AP* and *AB* respectively when *AP* makes an angle θ with the vertical.

(i) Draw diagrams showing the forces acting on each of *A* and *B*. **1**

(ii) Hence show that $T_1 - T_2 = \dfrac{mg}{\cos\theta}, T_1 + T_2 = ml\omega^2 \text{ and } T_2 = \dfrac{Mg}{\cos\theta}$. **2**

(iii) Deduce that $d = \dfrac{2g}{\omega^2}\left(1 + 2\dfrac{M}{m}\right)$. **2**

Question 14 continues on page 12

Question 14 (continued)

(b) A particle's resistance to motion in a medium is proportional to mv^2 where m is the particle's mass and v is its velocity at time t.

(i) Initially the particle is projected downwards in the medium where the speed of projection is equal to the terminal velocity V_T. **1**

Show that $V_T^2 = \dfrac{g}{k}$ where k is the constant of proportionality.

The particle is now projected vertically <u>upwards</u> in the same medium.

(ii) Show that $x = \dfrac{V_T^2}{2g}\ln\left(\dfrac{2V_T^2}{V_T^2+v^2}\right)$. **2**

(iii) Hence show that $H = \dfrac{V_T^2 \ln 2}{2g}$ where H represents the particle's maximum height above its point of projection. **1**

(iv) Show that during the particle's ascent $v = V_T \tan\left(\dfrac{\pi}{4} - \dfrac{g}{V_T}t\right)$. **2**

(v) Hence show that $\dfrac{2V_T^2}{V_T^2+v^2} = 1 + \sin\left(\dfrac{2g}{V_T}t\right)$. **2**

(vi) If T is the time taken to achieve *half* its maximum height, show that **2**

$T = \dfrac{V_T}{2g}\sin^{-1}\left(\sqrt{2}-1\right)$.

End of Question 14

Question 15 **(15 marks)** Use a SEPARATE writing booklet

(a) (i) If $0 \le a \le b$ show that $\dfrac{a}{1+a} \le \dfrac{b}{1+b}$ **2**

(ii) Hence or otherwise show that $\dfrac{a}{1+a} \le \dfrac{b}{1+b} + \dfrac{c}{1+c}$ where $a \le b+c$ and $a, b, c \ge 0$ **2**

(b) An urn contains 5 balls numbered from 1 to 5. A ball is chosen at random and its number is noted. The ball is then returned to the urn. This is done a total of five times.

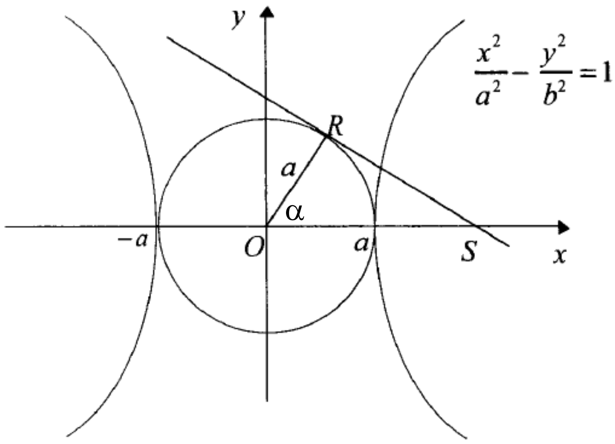
(i) What is the probability that each ball is selected exactly once? **1**

(ii) What is the probability that at least one ball is not selected? **1**

(iii) What is the probability of obtaining 1,1,2,3,4 in any order? **1**

(iv) What is the probability that exactly one of the balls is not selected? **2**

(c)



S is the focus of the hyperbola $\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = 1$, $a \ne b$, which lies on the positive x-axis. R is a point on the auxiliary circle of the hyperbola such that R lies in the first quadrant and SR is tangent to the auxiliary circle. The eccentricity of the hyperbola is e and $\angle ROS = \alpha$.

(i) Show that R lies on a directrix of the hyperbola. **1**

(ii) Show that SR has equation $y = -\dfrac{1}{\sqrt{e^2-1}}\left(x - ae\right)$. **1**

(iii) If SR meets the hyperbola at the point $\left(a\sec\theta, b\tan\theta\right)$, show that **2**

$e^2(2-e^2)\sec^2\theta - 2e\sec\theta + \left\{e^2 + (e^2-1)^2\right\} = 0$.

(iv) By considering this as a quadratic equation in $\sec\theta$, deduce that SR intersects the hyperbola in two distinct points P and Q, lying on the same branch of the hyperbola if $e^2 < 2$ and lying on opposite branches if $e^2 > 2$. **2**

End of Question 15

Question 16 (15 marks) Use a SEPARATE writing booklet

(a) The nth Fermat number, F_n, is defined by $F_n = 2^{2^n} + 1$ for $n = 0, 1, 2, 3...$, where 2^{2^n} means 2 raised to the power of 2^n. **4**

Prove by mathematical induction, that for all positive integers:

$$F_0 \times F_1 \times F_2 \times ... \times F_{n-1} = F_n - 2$$

(b) A coin is tossed six times. What is the probability that there will be more tails on the first three of the six throws than on the last three throws? **3**

(c) Let $x = \cos\theta + i\sin\theta$ for $0 < \theta < 2\pi$, and let n be a positive integer.

(i) Show that $x^k + \frac{1}{x^k} = 2\cos k\theta$, for any positive integer k. **2**

(ii) Show that **3**

$$\left(x + \frac{1}{x}\right)^{2n} = \left(x^{2n} + \frac{1}{x^{2n}}\right) + \binom{2n}{1}\left(x^{2n-2} + \frac{1}{x^{2n-2}}\right) + \binom{2n}{2}\left(x^{2n-4} + \frac{1}{x^{2n-4}}\right) +$$
$$\ldots + \binom{2n}{n-1}\left(x^2 + \frac{1}{x^2}\right) + \binom{2n}{n}.$$

(iii) Deduce that **1**

$$2^{2n-1}\cos^{2n}\theta = \cos 2n\theta + \binom{2n}{1}\cos(2n-2)\theta + \binom{2n}{2}\cos(2n-4)\theta +$$
$$\ldots + \binom{2n}{n-1}\cos 2\theta + \frac{1}{2}\binom{2n}{n}.$$

(iv) Hence show that $\int_0^{2\pi} \cos^{2n}\theta \, d\theta = \frac{\pi}{2^{2n-1}}\binom{2n}{n}$. **2**

End of Paper

STANDARD INTEGRALS

$$\int x^n\,dx = \frac{1}{n+1}x^{n+1}, \quad n \neq -1; \quad x \neq 0, \text{ if } n < 0$$

$$\int \frac{1}{x}\,dx = \ln x, \quad x > 0$$

$$\int e^{ax}\,dx = \frac{1}{a}e^{ax}, \quad a \neq 0$$

$$\int \cos ax\,dx = \frac{1}{a}\sin ax, \quad a \neq 0$$

$$\int \sin ax\,dx = -\frac{1}{a}\cos ax, \quad a \neq 0$$

$$\int \sec^2 ax\,dx = \frac{1}{a}\tan ax, \quad a \neq 0$$

$$\int \sec ax\,\tan ax\,dx = \frac{1}{a}\sec ax, \quad a \neq 0$$

$$\int \frac{1}{a^2+x^2}\,dx = \frac{1}{a}\tan^{-1}\frac{x}{a}, \quad a \neq 0$$

$$\int \frac{1}{\sqrt{a^2-x^2}}\,dx = \sin^{-1}\frac{x}{a}, \quad a > 0, \quad -a < x < a$$

$$\int \frac{1}{\sqrt{x^2-a^2}}\,dx = \ln\left(x+\sqrt{x^2-a^2}\right), \quad x > a > 0$$

$$\int \frac{1}{\sqrt{x^2+a^2}}\,dx = \ln\left(x+\sqrt{x^2+a^2}\right)$$

NOTE : $\ln x = \log_e x, \quad x > 0$