I.S.

KNOX GRAMMAR SCHOOL

TRIAL HIGHER SCHOOL CERTIFICATE

1999

MATHEMATICS

4 UNIT

YEAR 12

Time allowed: Three hours
(Plus 5 minutes reading time)

INSTRUCTIONS

ALL questions should be attempted
ALL questions are of equal value.
ALL necessary working should be shown in every question.
Full marks may not be awarded if work is careless or badly arranged.
Approved calculators may be used.
Each question to be started in a new booklet.

Name: ______________________ Class: ________

Question 1 *(Start a new Booklet)* **Marks**

(a) $u = 4 - i, v = 9 + 12i$. Evaluate in the form $a + bi$: **5**

(i) uv (ii) $\frac{u}{v}$ (iii) $\sqrt{v}$

(b) On separate Argand diagrams, plot the locus of the complex number z, where: **3**

(i) $|z + 2i| = 2$ (ii) $\arg(z - 1) = \frac{2\pi}{3}$

(c) If z is a complex number and $\frac{z+2}{z-i}$ is purely imaginary, find the equation of the locus of z, and describe the locus fully geometrically. **4**

(d) If z is a complex number, $|z| = 1$ and $\arg z = \theta$ for $0 < \theta < \frac{\pi}{2}$, prove that **3**

$\arg(1 - z^2) = \theta - \frac{\pi}{2}$.

Question 2 *(Start a new Booklet)*

(a) $(3 - i)$ is a zero of $P(z) = z^3 - 10z^2 + az + b$, where a, b are real. Find the other two roots, and the values of a, b. **4**

(b) Factorise $z^6 - 1$ into linear factors over the set of complex numbers. **3**

(c) $x^3 + mx + n = 0$ has a double root. Prove that $4m^3 + 27n^2 = 0$. **3**

(d) If α, β, γ are the roots of the equation $x^3 - 5x^2 + 2x + 1 = 0$, form the cubic equations whose roots are: **5**

(i) $\frac{1}{\alpha\beta}, \frac{1}{\alpha\gamma}, \frac{1}{\beta\gamma}$ (ii) $\alpha^2, \beta^2, \gamma^2$

Marks

Question 3 *(Start a new Booklet)*

(a) Find the following indefinite integrals: **6**

(i) $\int \frac{1}{\sqrt{9-x^2}}\,dx$ (ii) $\int \frac{x}{\sqrt{9-x^2}}\,dx$ (iii) $\int \frac{x^2}{\sqrt{9-x^2}}\,dx$

(b) (i) Find $\int \frac{1}{x^2+8x}\,dx$, using partial fractions. **6**

(ii) Hence find $\int \frac{x+1}{x^2+8x}\,dx$.

(c) Find $\int \operatorname{cosec}^6 x\,dx$. **3**

Question 4 *(Start a new Booklet)*

(a) The area bounded by $y=e^{2x}$, the x-axis, the y-axis, and the line $x=2$, is rotated about the y-axis. **5**
Use the method of cylindrical shells to find the volume of the solid formed.

(b) The area bounded by the parabola $y=2x^2$ and the line $y=8$ is rotated about the line $y=10$. **4**
Use the slice (washer) method to find the volume of the solid formed.

(c) The horizontal base of a solid is the circle $x^2+y^2=16$. **6**
Vertical cross-sections perpendicular to the x-axis are regions bounded by a parabola and its latus rectum, with the latus rectum lying on the base and the vertex of the parabola vertically above the latus rectum.

(i) Prove that the area bounded by a parabola of focal length a units and its latus rectum is $\frac{8a^2}{3}$ unit^2.

(ii) Hence find the volume of the solid.

Question 5 *(Start a new Booklet)* **Marks**

(a) For the ellipse $\frac{x^2}{9}+\frac{y^2}{4}=1$, find: 7

(i) the eccentricity
(ii) the coordinates of the foci
(iii) the equations of the directrices
(iv) the equation of the tangent at the point $\left(2,-\frac{2\sqrt{5}}{3}\right)$ on the curve.

(b) The tangent to the hyperbola $xy=c^2$ at a point P on the hyperbola, cuts the x-axis at Q. If the origin is O, prove that the area of ΔOPQ is independent of the position of P. 3

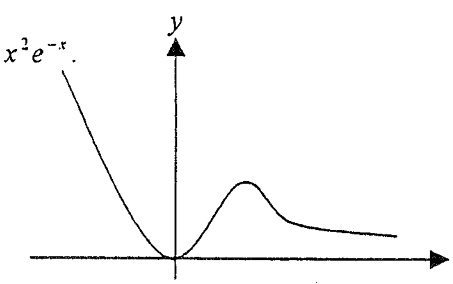
(c) A point $P(x,y)$ moves so that its distance from the point (2,5) is twice its distance from the line $3x-4y-12=0$. 5

(i) What shape is the locus of P?
(ii) Find the equation of the locus of P, expressing your answer in the form $ax^2+bxy+cy^2+dx+ey+f=0$.

Question 6 *(Start a new Booklet)*

(a) For $0\le\theta\le2\pi$, solve $4\cos^2\theta<3$. 3

(b) The diagram shows the graph of $y=x^2e^{-x}$. 7



(i) Find the coordinates of the relative maximum value (give the y value to 1 d.p.)

(ii) On the separate answer sheet for this question draw the graphs of:

(α) $y=x^4e^{-2x}$ (β) $y=\frac{e^x}{x^2}$ (γ) $y=x^2e^{-|x|}$

(c) Find the six complex roots of $z^6+1=0$, in the form $x+yi$ where x,y are real. Hence express z^6+1 as the product of three real quadratic factors. 5

Question 7 *(Start a new Booklet)* **Mark**

(a) For a, b, c, d positive, prove that: **9**

(i) $a^2 + b^2 \geq 2ab$

(ii) $a^4 + b^4 + c^4 + d^4 \geq 4abcd$

(iii) $a^2 + b^2 + c^2 \geq ab + ac + bc$

(iv) $ab + ac + bc \leq 12$, if $a + b + c = 6$.

(b) Prove by mathematical induction that, for $n = 1, 2, 3, \ldots\ldots$, **6**

$\tan\theta + 2\tan 2\theta + 4\tan 4\theta + \ldots\ldots\ldots\ldots + 2^{n-1}\tan 2^{n-1}\theta = \cot\theta - 2^n \cot 2^n\theta$.

Question 8 *(Start a new Booklet)*

(a) A particle moves in a straight line. Prove that its acceleration at any instant can be expressed as: **2**

(i) $v\dfrac{dv}{dx}$

(ii) $\dfrac{d}{dx}\left(\frac{1}{2}v^2\right)$,

where x denotes its position and v its velocity at any instant.

(b) A particle of mass m kg is projected upwards with initial speed U m/s. The particle is subject to gravity and a resistive force of magnitude mkv Newtons, where v m/s is the speed of the particle at any instant and k is a constant, $k>0$. **13**

(i) Explain why $\ddot{x} = -g - kv$, where g is the acceleration due to gravity.

(ii) Prove that the particle will reach a maximum height after T seconds given by $T = \dfrac{1}{k}\log_e\left(\dfrac{g+kU}{g}\right)$.

(iii) Prove that the maximum height reached is $\dfrac{1}{k}(U - gT)$ metres.

End of Examination

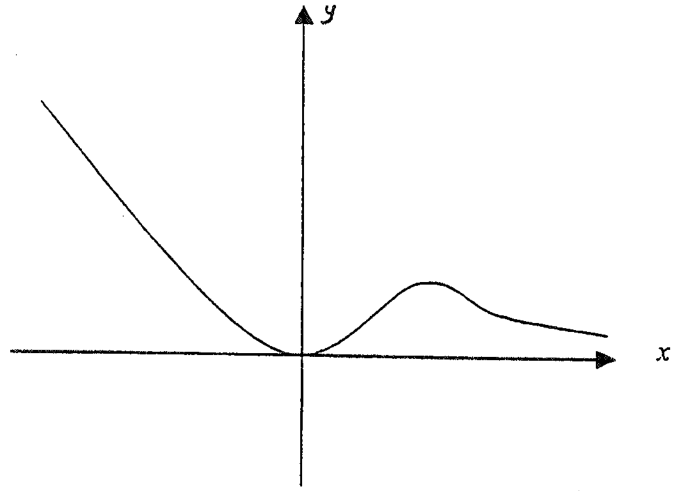
Name...

Answer sheet for Question 6(b)(ii)

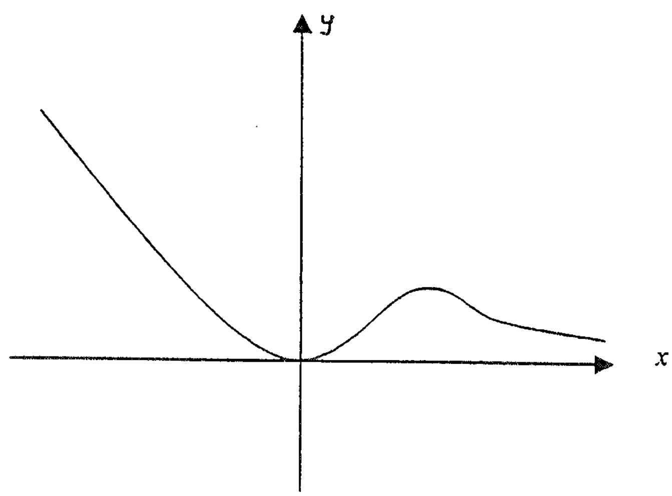
Place this sheet between the first two pages of the Question 6 Booklet

Note: In each diagram, the graph drawn is the graph of $y = x^2 e^{-x}$.

(α) $y = x^4 e^{-2x}$



(β) $y = \dfrac{e^x}{x^2}$



(γ) $y = x^2 e^{-|x|}$