VIRILE AGITUR

Student Number

2008

TRIAL HIGHER SCHOOL CERTIFICATE EXAMINATION

Knox Grammar School

Mathematics Extension 2

General Instructions

- Reading time – 5 minutes
- Working time – 3 hours
- Write using blue or black pen only
- Board approved calculators may be used
- A table of standard integrals is provided at the back of this paper
- All necessary working should be shown in every question

Total marks – 120

- Attempt Questions 1-8
- All questions are of equal value

Subject Teachers
Mr I Bradford
Mr M Vuletich

This paper MUST NOT be removed from the examination room

Number of Students in Course: 40

Total marks – 120
Attempt Questions 1-8
All questions are of equal value

Answer each question in a SEPARATE writing booklet. Extra writing booklets are available.

Question 1 (15 marks) Use a SEPARATE writing booklet. **Marks**

(a) Evaluate $\int_0^1 xe^{-x^2}\,dx$. **2**

(b) Using the substitution $u = e^x$, or otherwise, find $\int \frac{e^x\,dx}{\sqrt{1-e^{2x}}}$. **2**

(c) Find $\int \frac{4x^3 - 2x^2 + 1}{2x-1}\,dx$. **3**

(d) (i) Find constants a, b and c such that

$$\frac{x^2+2x}{(x^2+4)(x-2)} = \frac{ax+b}{x^2+4} + \frac{c}{x-2}.$$ **2**

(ii) Hence, find $\int \frac{x^2+2x}{(x^2+4)(x-2)}\,dx$. **2**

(e) Show, using integration by parts, that **4**

$$\int_0^{\frac{\pi}{3}} x\sec^2 x\,dx = \frac{\pi\sqrt{3}}{3} - \ln 2.$$

Marks

Question 2 (15 marks) Use a SEPARATE writing booklet.

(a) (i) Express $\sqrt{3}-i$ in modulus-argument form. **4**

(ii) Hence evaluate $\left(\sqrt{3}-i\right)^6$.

(b) (i) Simplify $(2i)^3$. **2**

(ii) Hence find all complex numbers z such that $z^3 = 8i$. **2**
Express your answers in the form $x+iy$.

(c) Sketch the region where the inequalities $|z-3+i| \le 5$ and $|z+1| \le |z-1|$ both hold. **3**

(d) Let $w = \dfrac{3+4i}{5}$ and $z = \dfrac{5+12i}{13}$, so that $|w| = |z| = 1$.

(i) Find wz and $w\bar{z}$ in the form $x+iy$. **2**

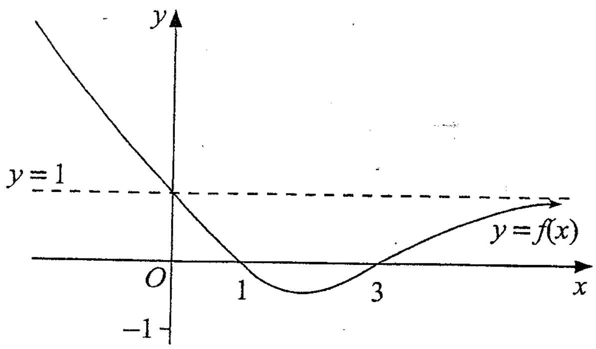
(ii) Hence find two distinct ways of writing 65^2 as the sum of a^2+b^2, where a and b are integers and $0 < a < b$. **2**

Marks

Question 3 (15 marks) Use a SEPARATE writing booklet.

(a) Sketch, without using calculus, the curve $y = \dfrac{4x^2}{x^2 - 9}$ showing all asymptotes. **3**

(b)



The diagram shows the graph of the $y = f(x)$. The graph has a horizontal asymptote at $y = 1$.

Draw separate one-third page sketches of the graphs of the following:

(i) $y = |f(x)|$ **2**

(ii) $y = \dfrac{1}{f(x)}$ **2**

(iii) $y = \ln f(x)$. **2**

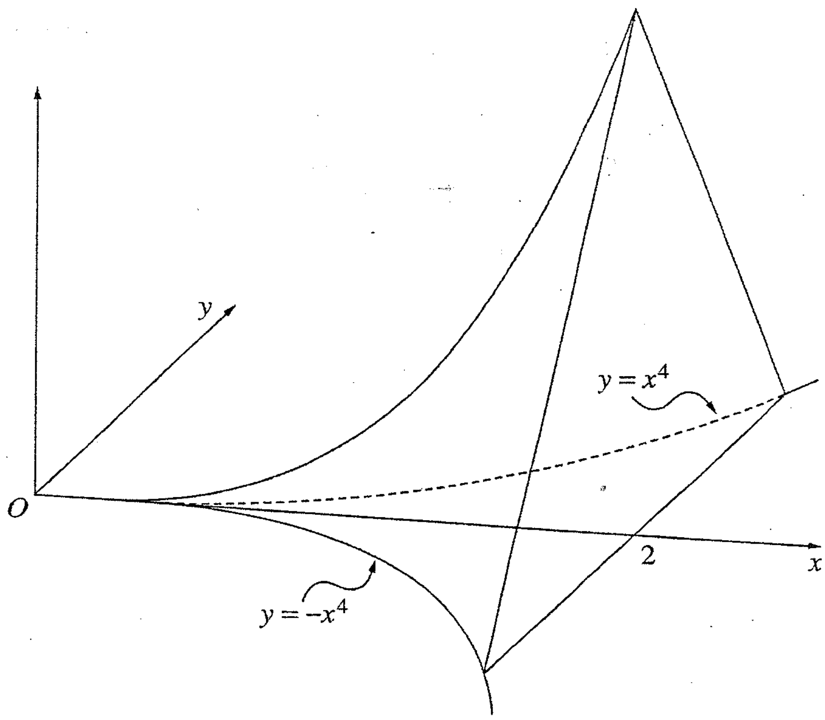
(c) Find the equation, in general form, of the tangent to the curve defined by $x^2 - xy + y^3 = 5$ at the point $(2, -1)$. **2**

Question 3 continues on page 6

Marks

Question 3 (continued)

(c) The base of a solid is the region in the xy plane enclosed by the curves $y = x^4$, $y = -x^4$ and the line $x = 2$. Each cross-section perpendicular to the x-axis is an equilateral triangle.



(i) Show that the area of the triangular cross-section at $x = h$ is $\sqrt{3}h^8$. **2**

(ii) Hence find the volume of the solid. **2**

End of Question 3

Marks

Question 4 (15 marks) Use a SEPARATE writing booklet.

(a) The ellipse E has Cartesian equation $\frac{x^2}{4}+\frac{y^2}{3}=1$.

(i) Write down its eccentricity, the coordinates of its foci S and S' and the equation of each directrix, where S lies on the positive side of the x-axis. **3**

(ii) Sketch E clearly labeling all essential features. **2**

(iii) If P lies on E, then prove that the sum of the distances PS and PS' is independent of P. **2**

(b) $P\left(p,\frac{1}{p}\right)$ and $Q\left(q,\frac{1}{q}\right)$ are two variable points on the rectangular hyperbola $xy=1$. **2**

If M is the midpoint of the chord PQ and OM is perpendicular to PQ, express q in terms of p.

(c) (i) Suppose the polynomial $P(x)$ has a a double root a $x=\alpha$. **2**

Prove that $P'(x)$ also has a root at $x=\alpha$.

(ii) The polynomial $A(x)=x^4+ax^2+bx+36$ has a double root at $x=2$. **2**

Find the values of a and b.

(iii) Factorise the polynomial $A(x)$ of part (ii) over the real numbers. **2**

Question 5 (15 marks) Use a SEPARATE writing booklet. **Marks**

(a) A solid is formed by rotating the region bounded by the curve $y = x(x-1)^2$, the line $y = 0$ and between $x = 0$ and $x = 1$. **3**

Use the method of cylindrical shells to find the exact volume of this solid.

(b) The region between the curve $y = \sin x$ and the line $y = 1$, from $x = 0$ to $x = \frac{\pi}{2}$, is rotated around the line $y = 1$. **4**

Using a slicing technique find the exact volume formed.

(c) A particle is moving in the positive direction along a straight line in a medium that exerts a resistance to motion proportional to the cube of the velocity.
No other forces act on the particle, that is, $\ddot{x} = -kv^3$, where k is a positive constant.

At time $t = 0$, the particle is at the origin and has velocity U. At time $t = T$, the particle is at $x = D$ and has velocity V.

(i) Using the identity $\ddot{x} = \frac{dv}{dt}$ show that **3**

$$\frac{1}{V^2} - \frac{1}{U^2} = 2kT.$$

(ii) Using the identity $\ddot{x} = v\frac{dv}{dx}$, show that **3**

$$\frac{1}{V} - \frac{1}{U} = kD.$$

(iii) Hence show that $\frac{D}{T} = \frac{2UV}{U+V}$. **2**

Marks

Question 6 (15 marks) Use a SEPARATE writing booklet.

(a) (i) Graph $y = \ln x$ and draw the tangent to the graph at $x = 1$. **1**

(ii) By considering the appropriate area under the tangent, deduce that **2**

$$\int_1^{\frac{3}{2}} \ln x \, dx \le \frac{1}{8}.$$

(b) A mass of 2 kg, on the end of a string 0.6 metres long, is rotating as a conical pendulum, with angular velocity 3π radians per second. The acceleration due to gravity is 10 m/s^2.

Let θ be the angle that the string makes with the vertical.

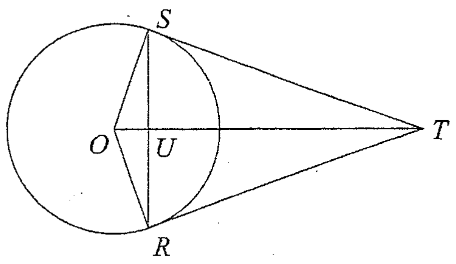
(i) Draw a diagram showing all forces acting on the mass. **1**

(ii) By resolving all forces show that the tension in the string is $10.8\pi^2$ **3**

(iii) Hence, or otherwise, find θ correct to the nearest degree. **1**

(c) Solve for x: $\tan^{-1} 5x - \tan^{-1} 3x = \tan^{-1} \frac{1}{4}$. **3**

(d)



The points R and S lie on a circle with centre O and radius 1. The tangents to the circle at R and S meet at T. The lines OT and RS meet at U, and are perpendicular. **4**

By considering ΔSOU and ΔTOS, show that

$$OU \times OT = 1.$$

Marks

Question 7 (15 marks) Use a SEPARATE writing booklet.

(a) (i) Let x be a fixed, non-zero number satisfying $x > -1$. **3**

Use the method of mathematical induction to prove that

$$(1+x)^n > 1+nx \text{ for } n = 2, 3, \ldots.$$

(ii) Deduce that $\left(1-\frac{1}{2n}\right)^n > \frac{1}{2}$ for $n = 2, 3, \ldots.$ **1**

(b) (i) Differentiate $\sin^{-1}(u) - \sqrt{1-u^2}$. **2**

(ii) Hence show that **1**

$$\int_0^\alpha \left(\frac{1+u}{1-u}\right)^{\frac{1}{2}} du = \sin^{-1}\alpha + 1 - \sqrt{1-\alpha^2} \text{ for } 0 < \alpha < 1.$$

(c) A ball of mass 2 kilograms is thrown vertically upward from the origin with an initial speed of 8 metres per second. The ball is subject to a downward gravitational force of 20 newtons and air resistance of $(v^2/5)$ newtons in the opposite direction to the velocity, v metres per second.

Hence, until the ball reaches its highest point, the equation of motion is:

$$\ddot{y} = -\frac{v^2}{10} - 10 \text{ where } y \text{ metres is its height.}$$

(i) Using the fact that $\ddot{y} = v\frac{dv}{dy}$, show that, while the ball is rising, **3**

$$v^2 = 164e^{-\frac{y}{5}} - 100$$

(ii) Hence find the exact maximum height reached. **1**

(iii) Using the fact that $\ddot{y} = \frac{dv}{dt}$, find how long the ball takes to reach this maximum height. **2**

(iv) How fast is the ball travelling when it returns to the origin? **2**

Marks

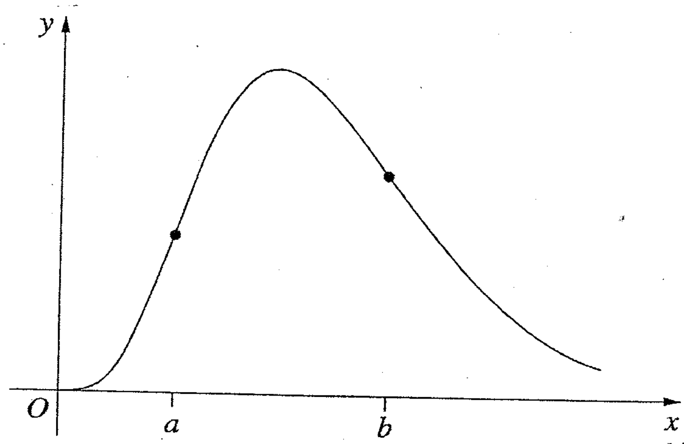
Question 8 (15 marks) Use a SEPARATE writing booklet.

(a) (i) Show that $\left(1-x^2\right)^{\frac{n-3}{2}}-\left(1-x^2\right)^{\frac{n-1}{2}}=x^2\left(1-x^2\right)^{\frac{n-3}{2}}$. **1**

(ii) Let $I_n=\int_0^1\left(1-x^2\right)^{\frac{n-1}{2}}dx$ where $n=0,1,2,\ldots$. **3**

Show that $nI_n=(n-1)I_{n-2}$ for $n=2,3,4,\ldots$.

(b) For $x>0$, let $f(x)=x^n e^{-x}$, where n is an integer and $n\geq 2$. **4**



The two points of inflexion of $f(x)$ occur at $x=a$ and $x=b$, where $0<a<b$.

Find a and b in terms of n.

(c) A straight line is drawn through a fixed point $P(a,b)$ in the first quadrant on a number plane. The line cuts the positive part of the x-axis at A and the positive part of the y-axis at B.

(i) If $\angle OAB=\theta$, prove that the length of AB is given by $AB=a\sec\theta+b\operatorname{cosec}\theta$. **2**

(ii) Show that the length of AB will be a minimum if $\cot\theta=\left(\frac{a}{b}\right)^{\frac{1}{3}}$. **3**

(iii) Show that the minimum length of AB is $\left(a^{\frac{2}{3}}+b^{\frac{2}{3}}\right)^{\frac{3}{2}}$. **2**

End of paper