Total marks (120)
Attempt questions 1 – 8
All questions are of equal value

Knox Grammar 2004 trial
Math Ext. 2

Answer each question in a SEPARATE writing booklet. Extra writing booklets are available.

Question 1 (15 marks) Use a SEPARATE writing booklet. **Marks**

(a) Evaluate $\displaystyle\int_0^{1.5} \frac{2}{\sqrt{9-x^2}}\,dx$. **2**

(b) Find $\displaystyle\int \frac{1}{\sqrt{x^2-4x+5}}\,dx$, with the aid of the Table of Standard Integrals. **2**

(c) Find $\displaystyle\int \sin^2 x\cos^3 x\,dx$. **3**

(d) Using the substitution $x = 3\sec\theta$, evaluate $\displaystyle\int_3^6 \frac{1}{x^2\sqrt{x^2-9}}\,dx$. **4**

(e) (i) Find constants A, B, C such that $\dfrac{x^2+2}{x^2-x-2} \equiv A + \dfrac{Bx+C}{x^2-x-2}$. **1**

(ii) Hence find $\displaystyle\int \frac{x^2+2}{x^2-x-2}\,dx$. **3**

Question 2 (15 marks) Use a SEPARATE writing booklet. **Marks**

(a) Simplify $\dfrac{1-i^3}{1-i}$. **2**

(b) Let $z = \dfrac{8-i}{2+i}$.

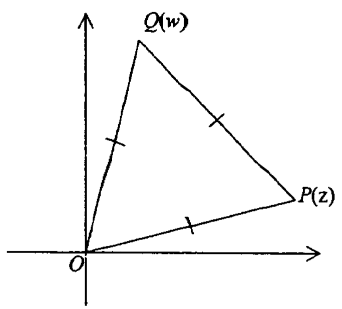
(i) Express z in the form $a+bi$ where a and b are real numbers. **2**

(ii) Hence, or otherwise, find $|z|$ and $\arg z$ (to 3 significant figures in the domain $-\pi < \theta \le \pi$). **3**

(c) Sketch the region in the complex number plane where the inequalities $|z+1-2i| \le 2$ and $\text{Re}(z) \le 0$ hold simultaneously. **2**

(d) Factorise $x^4 + 7x^2 - 18$ into the product of linear factors over the complex field. **2**

(e)



In the Argand diagram, OPQ is an equilateral triangle. P represents the complex number z and Q represents the complex number w.

(i) Explain why $w = z \operatorname{cis}\dfrac{\pi}{3}$. **2**

(ii) Show that $w^3 + z^3 = 0$. **2**

Question 3 (15 marks) Use a SEPARATE writing booklet. **Marks**

(a) Let $f(x) = 2(x-1)(x-3)$.

Draw separate sketches of the following functions (at least one-third of a page), showing clearly the important features, including any intercepts on the axes, turning points, asymptotes, etc.

(i) $y = f(x)$ **1**

(ii) $y = \dfrac{1}{f(x)}$ **2**

(iii) $y = 2 - f(x)$ **2**

(iv) $y = \sqrt{f(x)}$ **2**

(v) $y = \log_e f(x)$ **2**

(b) Let $I_n = \displaystyle\int_0^1 x^n e^{-x}\,dx$.

(i) Evaluate I_0. **1**

(ii) Prove that $I_n = n I_{n-1} - \dfrac{1}{e}$ for $n \ge 1$. **3**

(iii) Hence evaluate $\displaystyle\int_0^1 x^3 e^{-x}\,dx$. **2**

Question 4 (15 marks) Use a SEPARATE writing booklet. **Marks**

(a) Find $\sqrt{9-12i}$. **3**

(b) $(2+i)$ is a zero of the polynomial $P(z) = z^3 - z^2 + az + b$, where a and b are real numbers. **4**

Find the other two zeros, and the values of a and b.

(c) α, β, γ are the roots of the equation $x^3 - 6x^2 + 12x - 35 = 0$.

(i) Form a cubic equation whose roots are $\alpha - 2, \beta - 2, \gamma - 2$. **2**

(ii) Hence, or otherwise, solve the equation $x^3 - 6x^2 + 12x - 35 = 0$ over the complex field. **2**

(d) The roots of the equation $z^2 + 5z - 2i = 0$ are α and β. Without solving this equation, form the cubic equation whose roots are α, β and $(\alpha + \beta)$. **4**

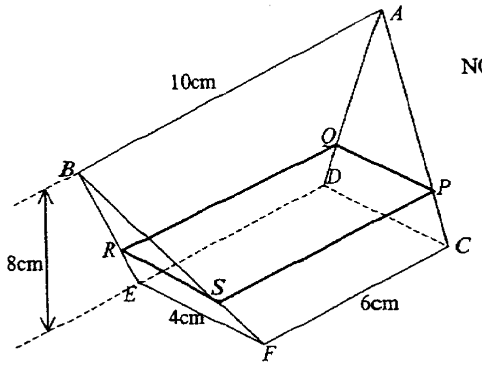
Question 5 (15 marks) Use a SEPARATE writing booklet. **Marks**

(a) Consider the hyperbola $\frac{x^2}{4} - \frac{y^2}{16} = 1$.

(i) Find its eccentricity. **1**

(ii) State the equations of the asymptotes. **1**

(b)



NOT TO SCALE

The diagram shows a wedge with the edge AB parallel to the horizontal rectangular base $CDEF$, and the plane $ABED$ is vertical. AB is 8 cm vertically above DE. $PQRS$ is a rectangular cross-section h cm above the base.

(i) Show that the area of the cross-section $PQRS$ is $\left(6+\frac{h}{2}\right)\left(4-\frac{h}{2}\right)$ cm². **2**

(ii) Hence find the volume of the wedge. **2**

(c) Consider the function $y = \frac{x^2 - 3x}{x+1}$.

(i) Find the equations of the two asymptotes. **2**

(ii) Find the coordinates of the stationary points and determine their nature. **5**

(iii) Sketch the graph of the function. **1**

(iv) For what values of k does the equation $\frac{x^2 - 3x}{x+1} = k$ have two real roots? **1**

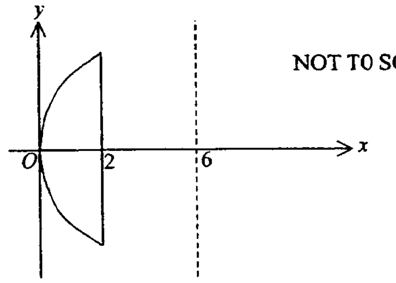
Question 6 (15 marks) Use a SEPARATE writing booklet. **Marks**

(a) (i) Show that $f(x) = x\sqrt{4-x^2}$ is an odd function. **1**

(ii) Hence, without finding any primitives, evaluate **2**

$$\int_{-2}^{2}\left(x\sqrt{4-x^2}-\sqrt{4-x^2}\right)dx,$$ giving reasons.

(b)

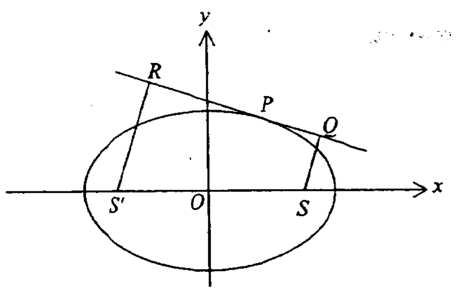


NOT TO SCALE

The region bounded by the parabola $y^2 = 4x$ and the line $x = 2$ is rotated about the line $x = 6$. **5**

Using the method of cylindrical shells, find the volume of the solid formed.

(c)



(i) Prove that the equation of the tangent to the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ at the point $P(a\cos\theta, b\sin\theta)$ is $(b\cos\theta)x+(a\sin\theta)y-ab=0$. **3**

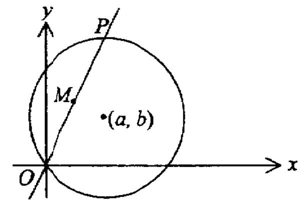
(ii) Q and R are the feet of the perpendiculars to the tangent from the foci S and S' respectively. **4**

Prove that $SQ \times S'R = b^2$.

Question 7 (15 marks) Use a SEPARATE writing booklet. **Marks**

(a) Find the general solution of the inequality $\cos\theta \geq \frac{1}{2}$. **2**

(b)



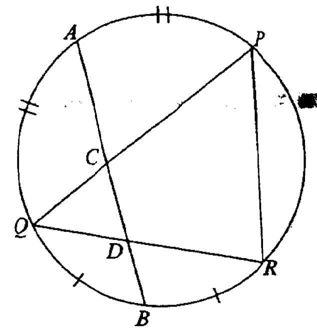
The diagram shows the graph of the circle $(x-a)^2+(y-b)^2 = a^2+b^2$, which passes through the origin O. The line $y = mx$ cuts the circle at O and P.

(i) Show that the x coordinate of P is $\frac{2(a+bm)}{1+m^2}$. **2**

(ii) Hence write down the coordinates of M, the midpoint of OP. **2**

(iii) Hence show that the locus of M, as the gradient of OP varies, is a circle, and state its centre and radius. **4**

(c)



A circle is drawn through the vertices of the triangle PQR. A is the midpoint of the arc PQ and B is the midpoint of the arc QR. The chord AB intersects PQ at C and QR at D.

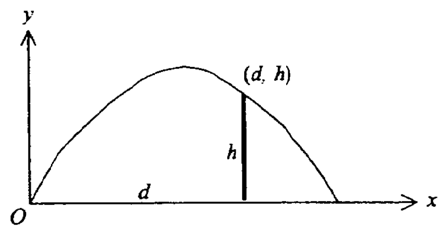
Copy or trace the diagram into your Writing Booklet.

(i) Explain why $\angle QPB = \angle BPR$. **1**

(ii) Prove that $QC = QD$. **4**

Question 8 (15 marks) Use a SEPARATE writing booklet. **Marks**

(a)



A stone is projected from a point on the ground, and it just clears a fence d metres away. The height of the fence is h metres. The angle of projection to the horizontal is θ and the speed of projection is V m/s.
The displacement equations, measured from the point of projection, are:

$$x = V\cos\theta\, t \quad \text{and} \quad y = V\sin\theta\, t - \frac{1}{2}gt^2 .$$

(i) Show that $V^2 = \dfrac{gd^2\sec^2\theta}{2(d\tan\theta - h)}$. **2**

(ii) Show that the maximum height reached is $\dfrac{d^2\tan^2\theta}{4(d\tan\theta - h)}$ **3**

(iii) Show that the stone will just clear the fence at its highest point if $\tan\theta = \dfrac{2h}{d}$. **3**

(b) (i) Prove by mathematical induction that $\left(\sqrt{3}-1\right)^n = p_n + q_n\sqrt{3}$, where n is a positive integer and p_n and q_n are unique integers. **5**

(ii) Hence show that $p_n^2 - 3q_n^2 = (-2)^n$. **2**

End of paper