

Total marks (120)  
Attempt questions 1 – 8  
All questions are of equal value

Key Grammar 2004 Final  
Math, Ex. 2

Answer each question in a SEPARATE writing booklet. Extra writing booklets are available.

Question 1	(15 marks)	Use a SEPARATE writing booklet.	Marks
(a)	Evaluate $\int_0^{1.5} \frac{2}{\sqrt{9-x^2}} dx$ .		2
(b)	Find $\int \frac{1}{\sqrt{x^2-4x+5}} dx$ , with the aid of the Table of Standard Integrals.		2
(c)	Find $\int \sin^2 x \cos^3 x dx$ .		3
(d)	Using the substitution $x = 3 \sec \theta$ , evaluate $\int_3^6 \frac{1}{x^2 \sqrt{x^2-9}} dx$ .		4
(e)	(i) Find constants $A, B, C$ such that $\frac{x^2+2}{x^2-x-2} \equiv A + \frac{Bx+C}{x^2-x-2}$ .		1
	(ii) Hence find $\int \frac{x^2+2}{x^2-x-2} dx$ .		3

**Question 2** (15 marks) Use a SEPARATE writing booklet.

**Marks**

(a) Simplify  $\frac{1-i^2}{1-i}$ . 2

(b) Let  $z = \frac{8-i}{2+i}$ .

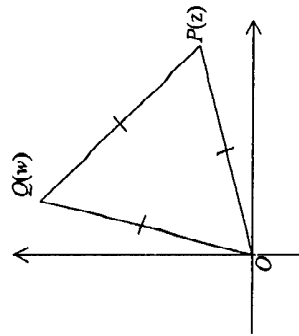
(i) Express  $z$  in the form  $a+bi$  where  $a$  and  $b$  are real numbers. 2

(ii) Hence, or otherwise, find  $|z|$  and  $\arg z$  (to 3 significant figures in the domain  $-\pi < \theta \leq \pi$ ). 3

(c) Sketch the region in the complex number plane where the inequalities  $|z+1-2i| \leq 2$  and  $\operatorname{Re}(z) \leq 0$  hold simultaneously. 2

(d) Factorise  $x^4 + 7x^2 - 18$  into the product of linear factors over the complex field. 2

(e)



In the Argand diagram,  $OPQ$  is an equilateral triangle.  $P$  represents the complex number  $z$  and  $Q$  represents the complex number  $w$ .

(i) Explain why  $w = z \operatorname{cis} \frac{\pi}{3}$ . 2

(ii) Show that  $w^3 + z^3 = 0$ . 2

**Question 3** (15 marks) Use a SEPARATE writing booklet.

**Marks**

(a) Let  $f(x) = 2(x-1)(x-3)$ .

Draw separate sketches of the following functions (at least one-third of a page), showing clearly the important features, including any intercepts on the axes, turning points, asymptotes, etc.

(i)  $y = f(x)$  1

(ii)  $y = \frac{1}{f(x)}$  2

(iii)  $y = 2 - f(x)$  2

(iv)  $y = \sqrt{f(x)}$  2

(v)  $y = \log_e f(x)$  2

(b) Let  $I_n = \int_0^1 x^n e^{-x} dx$ .

(i) Evaluate  $I_0$ . 1

(ii) Prove that  $I_n = nI_{n-1} - \frac{1}{e}$  for  $n \geq 1$ . 3

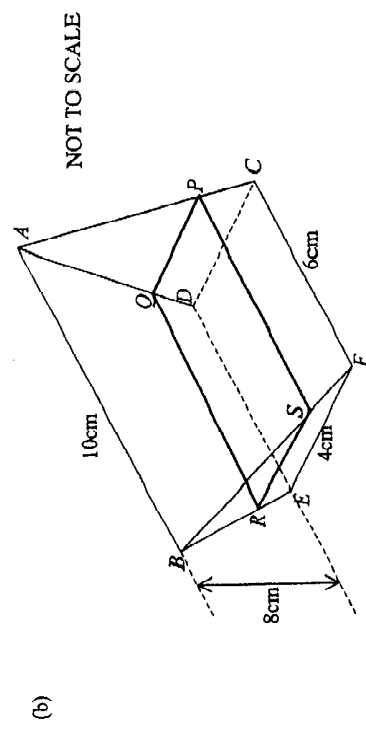
(iii) Hence evaluate  $\int_0^1 x^3 e^{-x} dx$ . 2

- (a) Find  $\sqrt{9-12i}$ . 3
- (b)  $(2+i)$  is a zero of the polynomial  $P(z) = z^2 - z^2 + az + b$ , where  $a$  and  $b$  are real numbers. 4

- Find the other two zeros, and the values of  $a$  and  $b$ .
- (c)  $\alpha, \beta, \gamma$  are the roots of the equation  $x^3 - 6x^2 + 12x - 35 = 0$ . 3
- (i) Form a cubic equation whose roots are  $\alpha - 2, \beta - 2, \gamma - 2$ . 2
- (ii) Hence, or otherwise, solve the equation  $x^3 - 6x^2 + 12x - 35 = 0$  over the complex field. 2

- (d) The roots of the equation  $z^2 + 5z - 2i = 0$  are  $\alpha$  and  $\beta$ . Without solving this equation, form the cubic equation whose roots are  $\alpha, \beta$  and  $(\alpha + \beta)$ . 4

- (a) Consider the hyperbola  $\frac{x^2}{4} - \frac{y^2}{16} = 1$ . 1
- (i) Find its eccentricity. 1
- (ii) State the equations of the asymptotes. 1



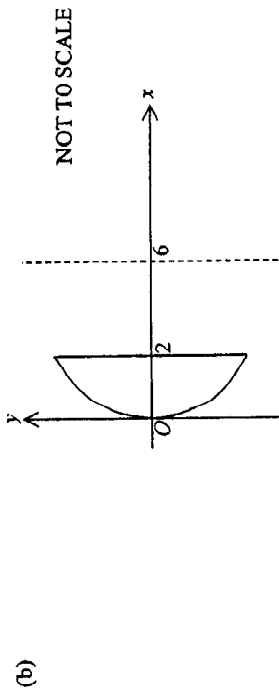
The diagram shows a wedge with the edge  $AB$  parallel to the horizontal rectangular base  $CDEF$ , and the plane  $ABED$  is vertical.  $AB$  is 8 cm vertically above  $DE$ .  $PQRS$  is a rectangular cross-section  $h$  cm above the base.

- (i) Show that the area of the cross-section  $PQRS$  is  $\left(6 + \frac{h}{2}\right)\left(4 - \frac{h}{2}\right)$  cm<sup>2</sup>. 2
- (ii) Hence find the volume of the wedge. 2
- (c) Consider the function  $y = \frac{x^2 - 3x}{x + 1}$ . 2
- (i) Find the equations of the two asymptotes. 2
- (ii) Find the coordinates of the stationary points and determine their nature. 5
- (iii) Sketch the graph of the function. 1
- (iv) For what values of  $k$  does the equation  $\frac{x^2 - 3x}{x + 1} = k$  have two real roots? 1

**Question 6** (15 marks) Use a SEPARATE writing booklet.

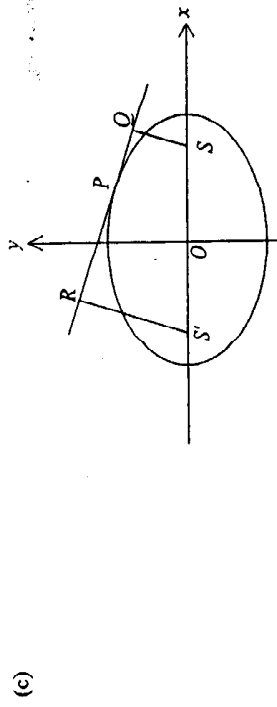
**Marks**

- (a) (i) Show that  $f(x) = x\sqrt{4-x^2}$  is an odd function. **1**  
 (ii) Hence, without finding any primitives, evaluate  $\int_{-2}^2 (x\sqrt{4-x^2} - \sqrt{4-x^2}) dx$ , giving reasons. **2**



The region bounded by the parabola  $y^2 = 4x$  and the line  $x = 2$  is rotated about the line  $x = 6$ . **5**

Using the method of cylindrical shells, find the volume of the solid formed.

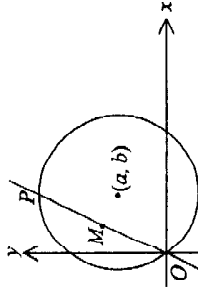


- (i) Prove that the equation of the tangent to the ellipse  $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$  at the point  $P(a \cos \theta, b \sin \theta)$  is  $(b \cos \theta)x + (a \sin \theta)y - ab = 0$ . **3**  
 (ii)  $Q$  and  $R$  are the feet of the perpendiculars to the tangent from the foci  $S$  and  $S'$  respectively. **4**  
 Prove that  $SQ \times S'R = b^2$ .

**Question 7** (15 marks) Use a SEPARATE writing booklet.

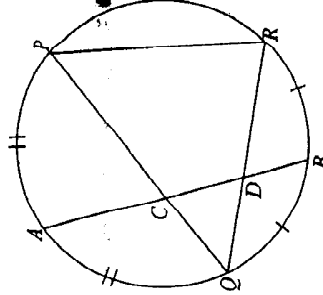
**Marks**

- (a) Find the general solution of the inequality  $\cos \theta \geq \frac{1}{2}$ . **2**  
 (b)



The diagram shows the graph of the circle  $(x-a)^2 + (y-b)^2 = a^2 + b^2$ , which passes through the origin  $O$ . The line  $y = mx$  cuts the circle at  $O$  and  $P$ .

- (i) Show that the  $x$  coordinate of  $P$  is  $\frac{2(a+bm)}{1+m^2}$ . **2**  
 (ii) Hence write down the coordinates of  $M$ , the midpoint of  $OP$ . **2**  
 (iii) Hence show that the locus of  $M$ , as the gradient of  $OP$  varies, is a circle, and state its centre and radius. **4**



A circle is drawn through the vertices of the triangle  $PQR$ .  $A$  is the midpoint of the arc  $PQ$  and  $B$  is the midpoint of the arc  $QR$ . The chord  $AB$  intersects  $PQ$  at  $C$  and  $QR$  at  $D$ .

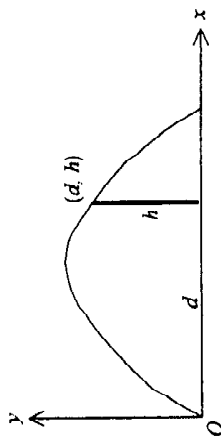
Copy or trace the diagram into your Writing Booklet.

- (i) Explain why  $\angle QPB = \angle BPR$ . **1**  
 (ii) Prove that  $QC = QD$ . **4**

Question 8 (15 marks) Use a SEPARATE writing booklet.

Marks

(a)



A stone is projected from a point on the ground, and it just clears a fence  $d$  metres away. The height of the fence is  $h$  metres. The angle of projection to the horizontal is  $\theta$  and the speed of projection is  $V$  m/s. The displacement equations, measured from the point of projection, are:

$$x = V \cos \theta t \quad \text{and} \quad y = V \sin \theta t - \frac{1}{2} g t^2.$$

- |       |                                                                                                                                                           |   |
|-------|-----------------------------------------------------------------------------------------------------------------------------------------------------------|---|
| (i)   | Show that $V^2 = \frac{2gd \sec^2 \theta}{2(d \tan \theta - h)}$ .                                                                                        | 2 |
| (ii)  | Show that the maximum height reached is $\frac{d^2 \tan^2 \theta}{4(d \tan \theta - h)}$ .                                                                | 3 |
| (iii) | Show that the stone will just clear the fence at its highest point if $\tan \theta = \frac{2h}{d}$ .                                                      | 3 |
| (b)   | (i) Prove by mathematical induction that $(\sqrt{3}-1)^n = p_n + q_n \sqrt{3}$ , where $n$ is a positive integer and $p_n$ and $q_n$ are unique integers. | 5 |
|       | (ii) Hence show that $p_n^2 - 3q_n^2 = (-2)^n$ .                                                                                                          | 2 |

End of paper