



**KNOX GRAMMAR SCHOOL**  
MATHEMATICS DEPARTMENT

**2001**  
TRIAL HSC EXAMINATION

# Mathematics Extension 2

Total marks (120)

- **General Instructions**
  - Reading time – 5 minutes
  - Working time – 3 hours
  - Write using blue or black pen
  - Board-approved calculators may be used
  - A table of standard integrals is provided on page 10
  - All necessary working should be shown in every question
- 
- Attempt Questions 1–8
  - All questions are of equal value
  - Use a **SEPARATE** writing booklet for each question

NAME: \_\_\_\_\_

TEACHER: \_\_\_\_\_

**Total marks (120)**

**Attempt questions 1 – 8**

All questions are of equal value

Answer each question in a SEPARATE writing booklet. Extra writing booklets are available.

**Question 1 (15 marks)** Use a SEPARATE writing booklet

Marks

(a) Find:

(i)  $\int \frac{x}{\sqrt{9-4x^2}} dx$

2

(ii)  $\int \frac{x^2}{x+1} dx$

2

(iii)  $\int_0^{\ln 2} xe^x dx$

3

(b) (i) Find real numbers  $A$ ,  $B$  and  $C$  such that  $\frac{2}{(t+1)(t^2+1)} = \frac{A}{t+1} + \frac{Bt+C}{t^2+1}$ .

3

(ii) Hence, find  $\int_0^1 \frac{2}{(t+1)(t^2+1)} dt$ .

3

(iii) By using the substitution  $t = \tan\left(\frac{x}{2}\right)$  evaluate  $\int_0^{\frac{\pi}{2}} \frac{\sin x}{1 + \sin x - \cos x} dx$ .

2

**Question 2 (15 marks)** Use a SEPARATE writing booklet

Marks

(a) Suppose  $z = 2 + 2i$  and  $w = -1 + \sqrt{3}i$ .

2

(i) Express  $z$  and  $w$  in modulus – argument form.

(ii) Find  $\left| \frac{z}{w} \right|^4$ .

1

(iii) Find the principal argument of  $\left(\frac{z}{w}\right)^4$ .

2

(b) Sketch separately the following loci in an Argand plane and state the cartesian equations in each case given that:

(i)  $|z - 3i| = |z - 4|$

2

(ii)  $\operatorname{Re}\left(\frac{z-2}{2}\right) = 0$

2

(iii)  $\arg(z+2) = -\frac{\pi}{6}$

2

(c) (i) Show that if  $z = x + iy$  then  $|z|^2 = z\bar{z}$ .

1

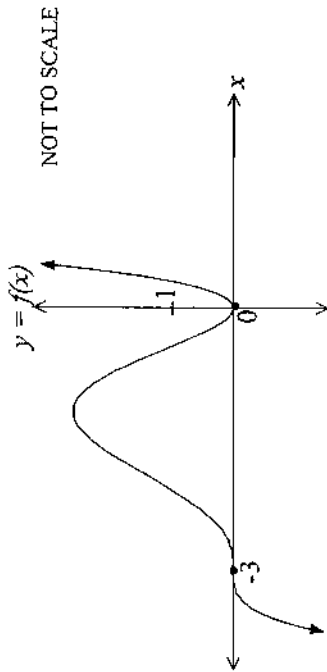
(ii) Using the result of (c)(i), or otherwise, prove that for any two complex numbers  $z$  and  $w$  that:

$$|z+w|^2 + |z-w|^2 = 2|z|^2 + 2|w|^2$$

2

(iii) Interpret this result geometrically. A vector diagram may be useful.

1



- (a) Consider the graph of  $y = f(x)$  as shown above.  
**On the answer sheet provided on pages 11 & 12,** use the graph of  $y = f(x)$  to clearly sketch separately the graphs of:

(i)  $y = \frac{1}{f(x)}$

(ii)  $y^2 = f(x)$

(iii)  $y = f'(x)$ .

- (b) Suggest a possible polynomial equation for the graph of  $y = f(x)$  shown in **part (a)** of **Question 3**.

(c) (i) Show that  $x = 1$  is a zero of  $x^3 + 3x^2 - 4$ .

- (ii) Sketch the curve with the equation  $y = x^3 + 3x^2 - 4$ , giving the coordinates of any maximum or minimum points and the intercepts made on each axis.

- (iii) Use your results in (c)(i) above to sketch the curves:

( $\alpha$ )  $y = |x^3 + 3x^2 - 4|$

( $\beta$ )  $y = \ln|x^3 + 3x^2 - 4|$

- (iv) Hence, or otherwise, determine the value of  $m$ , where  $m$  is a constant such that the equation  $2 \ln|x + 2| + \ln|x - 1| = m$ .

- (a) (i) Draw a sketch graph of the hyperbola  $\frac{y^2}{b^2} - \frac{x^2}{a^2} = 1$  and shade clearly the region bounded by the lines  $x = \pm a$  and the upper and lower branches of this hyperbola. **1**

- (ii) Show  $\frac{d}{d\theta} \ln(\sec\theta + \tan\theta) = \sec\theta$ . **1**

- (iii) Explain why the area,  $A$ , of the shaded region drawn in (a)(i) above can be by:

$$A = \int_0^a \frac{4b}{a} \sqrt{a^2 + x^2} \, dx.$$

- (iv) By using the substitution  $x = a \tan\theta$  in (a)(iii), show that  $A = 4ab \int_0^{\frac{\pi}{4}} \sec^3\theta \, d\theta$ . **2**

- (v) Show that the integral stated in (a)(iv) simplifies to  $2ab(\sqrt{2} + \ln(\sqrt{2} + 1))$ . **3**

**(Hint:** Write  $\sec^3\theta$  in the form  $\sec\theta \cdot \sec^2\theta$  and then use integration by parts)

- (vi) Use the **method of cylindrical shells** to show that the volume (in cubic units) of the solid generated by revolving this area about the  $y$ -axis is given by:

$$V = \frac{4\pi b a^2}{3} (2\sqrt{2} - 1).$$

- (b) A solid has a base, which is the **standard ellipse**  $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$  with major axis of length  $2a$  units and minor axis of length  $2b$  units ( $a > b$ ). In the vertical plane, the cross-sections of the solid are always isosceles triangles with perpendicular height  $h$  and whose base is parallel to the major axis. **3**

Use the **method of slicing** to find the volume of the solid.

**Question 5 (15 marks)**

Use a SEPARATE writing booklet

**Marks**

- (a) The point  $T$  with coordinates  $(at^2, 2at)$ ,  $t \neq 0$ ,  $a > 0$ , lies on the parabola with equation  $y^2 = 4ax$ . The tangent to the parabola at  $T$  meets the axis of the parabola at  $R$ . The normal at  $T$  meets the axis of the parabola at  $Q$  and the parabola again at  $P$ . The coordinates of  $P$  are  $(ap^2, 2ap)$ .
- 1 Represent this information on a clear and well-labelled diagram.
  - 1 Derive the equations of the tangent and normal to the parabola at  $T$ .
  - 1 Show that the length of  $RQ$  is  $2a(1+t^2)$  units.
  - 2 Show that the values of  $t$  for which  $R$  will lie on the directrix of the parabola satisfy  $t^2 = 1$ .
  - 2 Show that if  $t \neq p$ , then  $p = -\left(t + \frac{2}{t}\right)$ .
  - 1 Find  $TP$ , in terms of  $a$  and simplify your expression as far as possible.
  - 1 Hence, or otherwise, prove that the area of  $\Delta TPR$  is  $16a^2$  square units. (You may assume  $R$  lies on the directrix)
- (b) The equation of a rectangular hyperbola in cartesian form is given by  $xy = c^2$  where  $c > 0$ .
- 1 Verify that the point  $P\left(\frac{c}{cp}, \frac{c}{p}\right)$  lies on  $xy = c^2$ , where  $p$  is a non-zero real number.
  - 2  $Q$  has coordinates  $\left(cq, \frac{c}{q}\right)$  where  $q$  is a non-zero real number. Show that the equation of the chord  $PQ$  is given by  $x + pqy = c(p + q)$ .
  - 2 Find the equation of the locus of the midpoint of the chord  $PQ$  if it is known that the chord must always pass through the point  $(0, 2)$ .

**Question 6 (15 marks)**

Use a SEPARATE writing booklet

**Marks**

- (a) A particle of mass  $m$  units is projected vertically upward from the ground with initial speed  $u$ . The air resistance at any instance is proportional to the velocity  $v$  at that instant. For this question you may assume  $R = kmv$  where  $k$  is a constant.
- 1 With the aid of a suitable diagram show that  $\frac{dv}{dt} = -(g + kv)$ ?
  - 3 Show at any time  $t$ , that  $t = \frac{1}{k} \ln \left| \frac{g + kv}{g + ku} \right|$  seconds.
  - 1 Prove that the particle reaches its highest point in time  $T$  seconds when:
 
$$T = \frac{1}{k} \ln \left( \frac{ku}{g} + 1 \right)$$
  - 1 The highest point reached by the particle is at  $H$  metres above the ground.
    - 3 Prove that  $x = \frac{1}{k^2} (g + kv) (1 - e^{-kv}) - \frac{gu}{k}$ .
    - 2 Prove that  $H = \frac{1}{k} (u - gT)$ .
- (b) Let  $I_n = \int_0^{\frac{\pi}{2}} \sin^n \theta \, d\theta$  where  $n$  is a positive integer such that  $n \geq 2$ .
- 3 By replacing  $\sin^n \theta$  with  $\sin^{n-1} \theta \cdot \sin \theta$ , and using integration by parts or otherwise, show that  $I_n = \frac{n-1}{n} I_{n-2}$ .
  - 2 Hence, or otherwise, evaluate  $I_{10}$ .

Question 7 (15 marks) Use a SEPARATE writing booklet

Marks

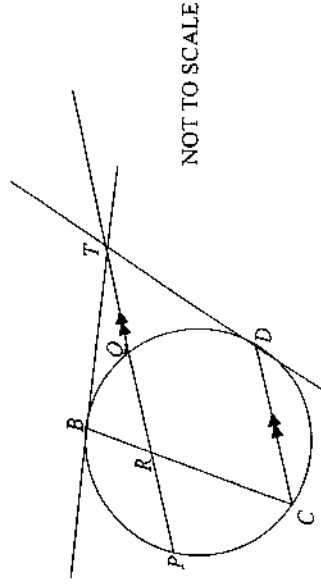
- (a) Let  $\alpha$ ,  $\beta$ , and  $\gamma$  be the non-zero roots of the equation  $x^3 + rx + s = 0$ .
- (i) Find in terms of  $r$ , the simplified value of  $\alpha^2 + \beta^2 + \gamma^2$ . 2
- (ii) Find in terms of  $r$  and  $s$ , the simplified value of  $\frac{1}{\alpha^2} + \frac{1}{\beta^2} + \frac{1}{\gamma^2}$ . 2
- (iii) Find in terms of  $r$  and  $s$ , the cubic equations (in general form) whose roots are
- (A)  $\frac{1}{\alpha}, \frac{1}{\beta}$ , and  $\frac{1}{\gamma}$ ; 2
- (B)  $\alpha + \beta - \gamma, \beta + \gamma - \alpha$  and  $\gamma + \alpha - \beta$  3

- (b) Suppose  $x^3 + rx + s = 0$  (with  $r$  and  $s$  being non-zero and real) has a double root. 2
- Show that  $x = -\frac{3s}{2r}$ .
- (c) Find all the roots of  $p(x) = x^4 - 8x^3 + 39x^2 - 122x + 170$  given that  $3 - i$  is one of the roots. 4

Question 8 (15 marks)

Use a SEPARATE writing booklet

Marks



- (a) In the diagram,  $PQ$  and  $CD$  are parallel chords of a circle. The tangent at  $D$  meets  $PQ$  produced externally at  $T$ .  $B$  is the point of contact of the other tangent from the circle.  $BC$  meets  $PQ$  internally at  $R$ .

Copy or trace this diagram into your writing booklet

- (i) Explain why  $\angle BDT = \angle BRT$ ? 2
- (ii) Show that  $B, T, D$  and  $R$  are concyclic points. 2
- (iii) Prove that  $\angle BRT = \angle DRT$ . 2
- (iv) Show that  $\triangle RCD$  is isosceles. 2
- (v) Show that  $BC$  bisects  $PQ$ . 2
- (b) (i) Show that  $\cos x = \sin\left(x + \frac{\pi}{2}\right)$ . 1
- (ii) Given that  $y = 3 \sin x + 4 \cos x$ , prove by the Principle of Mathematical Induction that  $\frac{d^n y}{dx^n} = 5 \sin\left(x + \alpha + \frac{n\pi}{2}\right)$  where  $\frac{d^n y}{dx^n}$  means the  $n$ th derivative of  $y$  with respect to  $x$  and  $n = 1, 2, 3, \dots$ . 4

You are advised to first express  $y = 3 \sin x + 4 \cos x$  in the form  $R \sin(x + \alpha)$ .

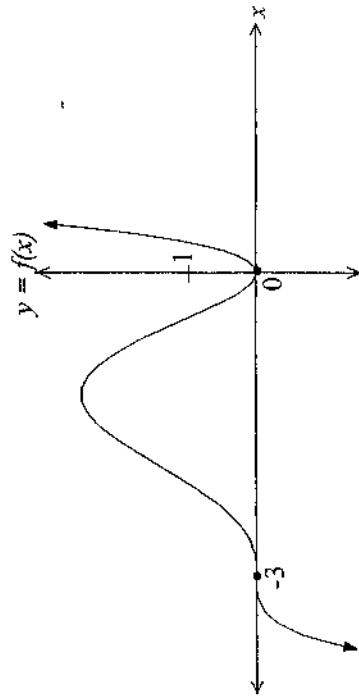
End of Paper

Detach and submit this page with your solutions to Question 3

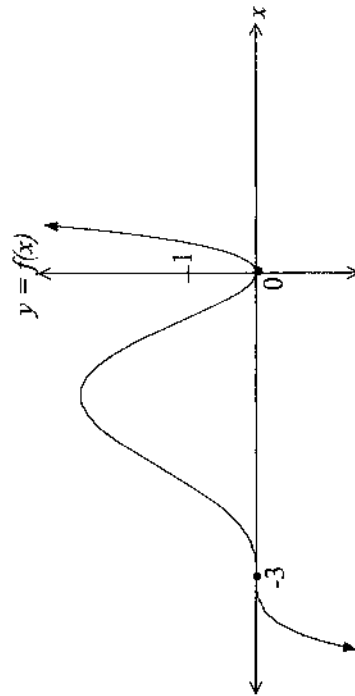
Student Name: \_\_\_\_\_

**Question 3 (a)** In each case use the graph of  $y = f(x)$  to clearly sketch the following:

(i)  $y = \frac{1}{f(x)}$ .

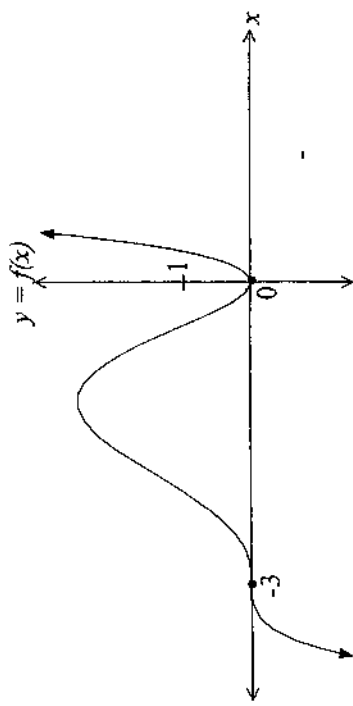


(ii)  $y^2 = f(x)$ .



Please turn over for part (a)(iii).

(iii)  $y = f'(x)$



**Question 3 (b):**

Possible polynomial equation for  $y = f(x)$ : \_\_\_\_\_