Set By: IB

KNOX GRAMMAR SCHOOL
MATHEMATICS DEPARTMENT

2002
TRIAL HSC EXAMINATION

Mathematics Extension 2

General Instructions

- Reading time – 5 minutes
- Working time – 3 hours
- Write using blue or black pen
- Board-approved calculators may be used
- A table of standard integrals is provided
- All necessary working should be shown in every question

Total marks (**120**)

- Attempt Questions 1–8
- All questions are of equal value
- Use a **SEPARATE** Writing Booklet for each question

NAME:______________________________ **TEACHER:**____________________________

Total marks (120)
Attempt questions 1 – 8
All questions are of equal value

Answer each question in a SEPARATE writing booklet. Extra writing booklets are available.

Question 1 (15 marks) Use a SEPARATE writing booklet **Marks**

(a) Find $\int \frac{x^2}{1+x}\,dx$. **2**

(b) Find $\int x\sin 2x\,dx$. **2**

(c) Use the substitution $t = \tan\left(\frac{x}{2}\right)$ to evaluate $\int_0^{\frac{\pi}{2}} \frac{1}{2+\sin x}\,dx$. **4**

(d) Find:

(i) $\int \frac{\sin^{-1} x}{\sqrt{1-x^2}}\,dx$. **2**

(ii) $\int \frac{1}{x\ln x}\,dx$. **1**

(e) Evaluate $\int_0^2 \frac{dx}{(x+1)(x^2+4)}$. **4**

Question 2 (15 marks) Use a SEPARATE writing booklet **Marks**

(a) If $z = 3 - 2i$, find in the form $x + iy$ form:

(i) z^2 **1**

(ii) $\frac{1}{z}$ **1**

(iii) $z.\overline{z}$ **1**

(b) On an Argand diagram, sketch the locus of the points z such that $|z + 1| = |z - i|$. **2**

(c) Let $z = i + \sqrt{3}$.

(i) Write z in modulus – argument form. **1**

(ii) Hence, evaluate $z^7 + 64z$. **2**

(d) Shade the region containing all the points satisfying $2 \le |z - 1| \le 4$ ***and*** $0 \le \arg(z - 1) \le \frac{\pi}{6}$. **3**

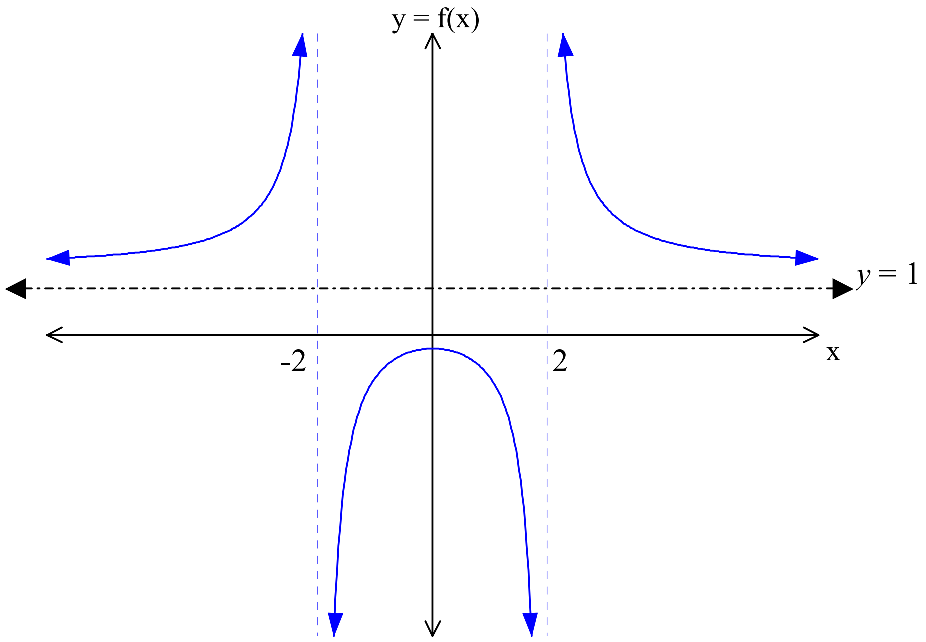
(e) The complex number $z = x + iy$, where x and y are real, satisfies the parametric equation $z = 1 + 2i + t(3 - 4i)$ where t is a real parameter.

(i) Show that the Cartesian equation of the locus of the point P which represents z in an Argand diagram is given by $4x + 3y = 10$. **2**

(ii) Hence find the minimum value of $|z|$. **2**

Question 3 (15 marks) Use a SEPARATE writing booklet **Marks**

(a) The diagram below shows the graph of the function $y = f(x)$ where $f(x) = \dfrac{1 + x^2}{x^2 - 4}$.



(i) On separate number planes, sketch the following graphs showing any intercepts on the coordinate axes and the equations of any asymptotes:

(A) $y = [f(x)]^2$ **2**

(B) $y = \sqrt{f(x)}$ **2**

(C) $y = \dfrac{1}{f(x)}$ **2**

(D) $y = f'(x)$ **2**

(E) $|\,y\,| = f(x)$. **2**

(b) For the curve defined by $3x^2 + y^2 - 2xy - 8x + 2 = 0$,

(i) Show that $\dfrac{dy}{dx} = \dfrac{3x - y - 4}{x - y}$. **2**

(ii) Find the coordinates of the points on the curve where the tangent to the curve is parallel to the line $y = 2x$. **3**

Question 4 (15 marks) Use a SEPARATE writing booklet **Marks**

(a) Find the range of values of x for which the limiting sum exists for the series **3**

$$1+\left(\frac{2x-3}{x+1}\right)+\left(\frac{2x-3}{x+1}\right)^2+\left(\frac{2x-3}{x+1}\right)^3+\ldots\ldots\ldots$$

(b) (i) Sketch the graph of the ellipse $\dfrac{x^2}{2}+y^2=1$, showing the coordinates of the foci and the equations of the directrices. **2**

(ii) The line $y=mx+3$ meets the ellipse in two distinct points $P(x_1,y_1)$ and $Q(x_2,y_2)$. Explain why x_1 and x_2 are the roots of the quadratic equation $(2m^2+1)x^2+12mx+16=0$. **1**

(iii) Find the values of m for which the equation $(2m^2+1)x^2+12mx+16=0$ has real and distinct roots. **1**

(iv) Write down the equations of the tangents to the ellipse from the point (0, 3). **2**

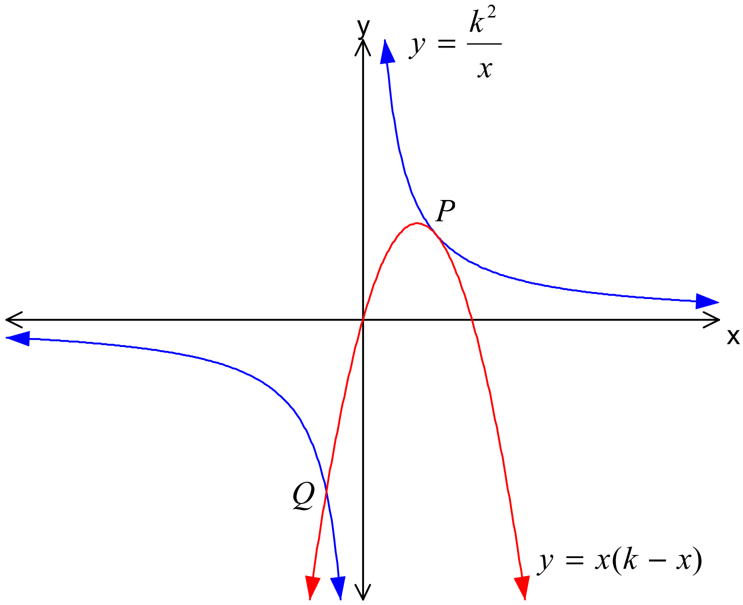
(c) The points $P\left(cp,\dfrac{c}{p}\right)$ and $Q\left(cq,\dfrac{c}{q}\right)$ are points on the rectangular hyperbola $xy=c^2$. Tangents to the rectangular hyperbola at P and Q intersect at the point $R(X,Y)$.

(i) Show that the tangent to the rectangular hyperbola at any point $T\left(ct,\dfrac{c}{t}\right)$ has equation $x+t^2y-2ct=0$. **1**

(ii) Find the coordinates R. **2**

(iii) If P and Q are variable points on the rectangular hyperbola which move so that $p^2+q^2=2$, show that the equation of the locus of R is given by $xy+y^2=2c^2$. **3**

Question 5 (15 marks) Use a SEPARATE writing booklet **Marks**



(a) In the diagram above, the curves $y = \frac{k^2}{x}$ and $y = x(k - x)$, where $k > 0$, touch at the point P and intersect at the point Q.

(i) Explain why the equation $x^3 - kx^2 + k^2 = 0$ has real roots α, α, and β for some $\alpha \neq \beta$? **1**

(ii) Find the exact values of k, α and β. **4**

(b) Given that the polynomial $P(x) = x^4 + x^2 + 6x + 4$ has a rational zero of multiplicity two, find all the zeros of $P(x)$ over the set of complex numbers. **3**

(c) Consider the polynomial $P(x) = x^4 - 4x^3 + 11x^2 - 14x + 10$.

(i) $P(x)$ has complex roots of the form $a + bi$ and $a - 2bi$ (where a and b are real). **3**

Find the values of a and b.

(ii) Hence, or otherwise, express $P(x)$ as a product of two quadratic factors. **1**

(d) Find all the pairs of values of the integers a and b for which the polynomial $P(x) = (ax + b)^2 - x$ is exactly divisible by both $(x - 1)$ and $(x - 4)$. **3**

Question 6 (15 marks) Use a SEPARATE writing booklet **Marks**

(a) The area bounded by $y = e^{2x}$, the x-axis, the y-axis, and the line $x = 2$, is rotated about the y-axis. **4**

Use the method of cylindrical shells to find the volume of the solid formed.

(b) The area bounded by the parabola $y = 2x^2$ and the line $y = 8$ is rotated about the line $y = 10$. **3**

Use the slice (washer) method to find the volume of the solid formed.

(c) The horizontal base of a solid is the circle $x^2 + y^2 = 16$. Vertical cross-sections perpendicular to the x-axis are regions bounded by a parabola and its latus rectum, with the latus rectum lying on the base and the vertex of the parabola vertically above the latus rectum.

The latus rectum is defined to be the chord parallel to the directrix and passing through the focus of the parabola.

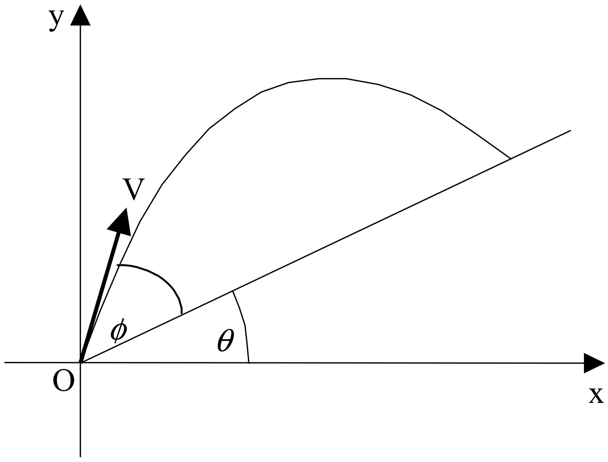
(i) Prove that the area bounded by a parabola of focal length a units and its latus rectum is $\frac{8a^2}{3}$ unit^2. **3**

(ii) Hence find the volume of the solid. **2**

(d) If $P(x) = x^m\left(b^n - c^n\right) + b^m\left(c^n - x^n\right) + c^m\left(x^n - b^n\right)$ where m and n are positive integers, show that $x^2 - (b + c)x + bc$ is a factor of $P(x)$. **3**

Question 7 appears on the next page…

Question 7 (15 marks) Use a SEPARATE writing booklet **Marks**



A projectile is given an initial velocity V at angle ϕ above the surface of an incline, which is in turn inclined at an angle θ above the horizontal.

(a) Show that the Cartesian equation for the projectile's trajectory is given by: **4**

$$y = x\tan(\theta+\phi) - \frac{gx^2}{2V^2\cos^2(\theta+\phi)}.$$

(b) When the projectile has progressed a horizontal distance of X metres, it collides with the incline. Show that the projectile's vertical height relative to the horizontal Ox is given by $X\tan\theta$. **1**

(c) Hence, show that $X = \dfrac{2V^2\cos^2(\theta+\phi)}{g}\left[\tan(\theta+\phi) - \tan\theta\right]$. **2**

(d) Deduce that the distance R travelled up the incline is given by: **1**

$$R = \frac{2V^2\cos^2(\theta+\phi)}{g\cos\theta}\left[\tan(\theta+\phi) - \tan\theta\right].$$

(e) Show that $\dfrac{2V^2\cos^2(\theta+\phi)}{g\cos\theta}\left[\tan(\theta+\phi) - \tan\theta\right] = \dfrac{2V^2\cos(\theta+\phi)}{g\cos^2\theta}\sin\phi$. **3**

(f) Deduce that $\dfrac{dR}{d\phi} = \dfrac{2V^2}{g\cos^2\theta}\cos(\theta+2\phi)$. **2**

(g) Hence, show that the maximum range R measured along the incline is achieved when the angle of trajectory bisects the vertical and the incline. **2**

Question 8 (15 marks) Use a SEPARATE writing booklet **Marks**

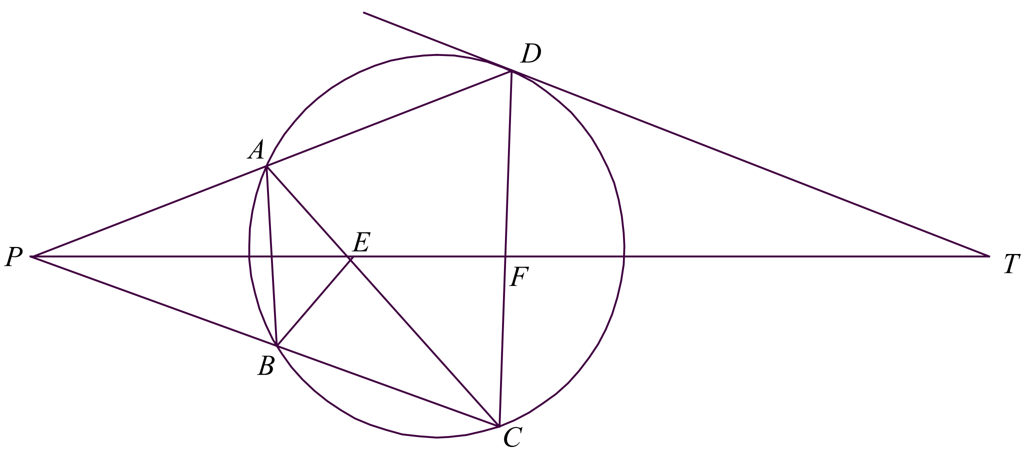
(a) (i) If $I_n = \int_1^e x(\ln x)^n \, dx$, $n = 0, 1, 2, 3, \ldots$ then show that for $n = 1, 2, 3, \ldots$ **3**

$$I_n = \frac{e^2}{2} - \frac{n}{2} I_{n-1}$$

(ii) Hence, evaluate I_3. **2**

(b) Use the Principle of Mathematical Induction to show that for all positive integers $n \geq 1$, that: **4**

$$a^2 + (a+d)^2 + (a+2d)^2 + \ldots + (a+(n-1)d)^2 = \frac{n}{6}\left(6a^2 + 6ad(n-1) + d^2(n-1)(2n-1)\right)$$



(c) $ABCD$ is a cyclic quadrilateral. DA is produced and CB produced meet at P. T is a point on the tangent at D to the circle through A, B, C and D. PT cuts CA and CD at E and F respectively. $TF = TD$.

Copy this diagram into your writing booklet.

(i) Show that $AEFD$ is a cyclic quadrilateral. **3**

(ii) Show that $PBEA$ is a cyclic quadrilateral. **3**

END OF EXAMINATION

STANDARD INTEGRALS

$$\int x^n \, dx = \frac{1}{n+1} x^{n+1}, \quad n \neq -1; x \neq 0, \text{ if } n < 0$$

$$\int \frac{1}{x} \, dx = \ln x \, , \quad x > 0$$

$$\int e^{ax} \, dx = \frac{1}{a} e^{ax}, \quad a \neq 0$$

$$\int \cos ax \, dx = \frac{1}{a} \sin ax, \quad a \neq 0$$

$$\int \sin ax \, dx = -\frac{1}{a} \cos ax, \quad a \neq 0$$

$$\int \sec^2 ax \, dx = \frac{1}{a} \tan ax, \quad a \neq 0$$

$$\int \sec ax \tan ax \, dx = \frac{1}{a} \sec ax, \quad a \neq 0$$

$$\int \frac{1}{a^2 + x^2} \, dx = \frac{1}{a} \tan^{-1} \frac{x}{a}, \quad a \neq 0$$

$$\int \frac{1}{\sqrt{a^2 - x^2}} \, dx = \sin^{-1} \frac{x}{a}, \quad a > 0, \ -a < x < a$$

$$\int \frac{1}{\sqrt{x^2 - a^2}} \, dx = \ln\left(x + \sqrt{x^2 - a^2}\right), \quad x > a > 0$$

$$\int \frac{1}{\sqrt{x^2 + a^2}} \, dx = \ln\left(x + \sqrt{x^2 + a^2}\right)$$

Note $\ln x = \log_e x, \quad x > 0$