Set By: MV

Teachers:

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KNOX GRAMMAR SCHOOL
MATHEMATICS DEPARTMENT

2003
TRIAL HSC EXAMINATION

Mathematics Extension 2

Total marks (**120**)

- Attempt Questions 1–8
- All questions are of equal value
- Use a **SEPARATE** Writing Booklet for each question

General Instructions

- Reading time – 5 minutes
- Working time – 3 hours
- Write using blue or black pen
- Board-approved calculators may be used
- A table of standard integrals is provided
- All necessary working should be shown in every question

NAME:____________________________ **TEACHER:**____________________________

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Total marks (120)
Attempt questions 1 – 8
All questions are of equal value

Answer each question in a SEPARATE writing booklet. Extra writing booklets are available.

Question 1 (15 marks) Use a SEPARATE writing booklet **Marks**

(a) Using the method of integration by parts, find $\int xe^{2x} \, dx$. **2**

(b) (i) Find real constants A and B such that $\frac{7x-4}{2x^2-3x-2} = \frac{A}{2x+1} + \frac{B}{x-2}$. **3**

(ii) Hence, find $\int \frac{7x-4}{2x^2-3x-2} \, dx$. **2**

(c) Using an appropriate diagram or otherwise, evaluate $\int_0^{\frac{3}{2}} \sqrt{9-x^2} \, dx$. **4**

(d) Use the substitution $t = \tan\frac{\theta}{2}$ to show that $\int_0^{\frac{\pi}{2}} \frac{d\theta}{2+\cos\theta} = \frac{\pi}{3\sqrt{3}}$. **4**

Question 2 (15 marks) Use a SEPARATE writing booklet **Marks**

(a) Given $z = 1 - i$, find:

(i) $\text{Im}\left(\frac{1}{z}\right)$ **2**

(ii) $|z|$ **1**

(iii) $\arg(z)$ **1**

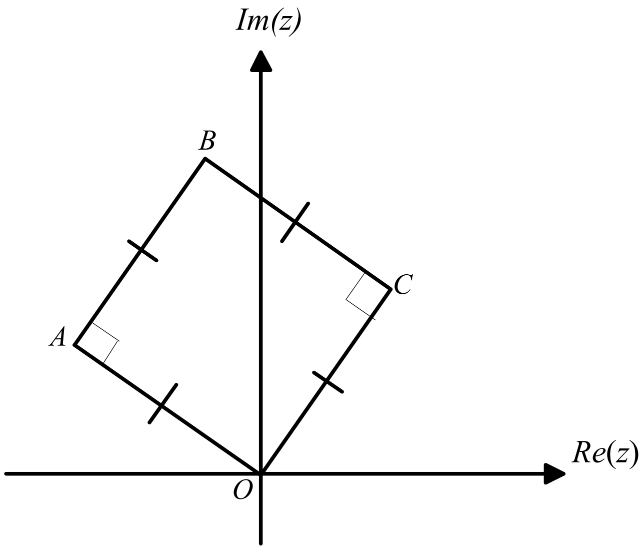
(iv) z^8 in the form $x + yi$ **2**

(v) the two values for ω such that $\omega^2 = 3\bar{z} + i$. **3**

(b) Illustrate on an Argand diagram, the region that satisfies both **3**

$$0 \le \arg(z+4) \le \frac{2\pi}{3} \quad and \quad |z+4| \le 4.$$

(c)



The diagram above represents a square *OABC*. The point *C* represents the complex number 2 + 3*i*.

(i) Find the coordinates of the point *A*. **1**

(ii) Hence, or otherwise determine the coordinates of the point *B*. **2**

Question 3 (15 marks) Use a SEPARATE writing booklet **Marks**

(a) Consider the function $y = f(x)$ where $f(x) = 9 - x^2$.

On separate number planes, sketch the following graphs showing any intercepts on the coordinate axes and the equations of any asymptotes:

(i) $y = f(x)$ **1**

(ii) $y = |\, f(x) \,|$ **1**

(iii) $|\, y \,| = f(x)$ **2**

(iv) $y = [f(x)]^2$ **2**

(v) $y = \sqrt{f(x)}$ **2**

(vi) $y = \log_e f(x)$. **2**

(b) Consider the function $y = f(x)$ where $f(x) = \sin\left(\cos^{-1} x\right)$.

(i) Show that $f(x)$ is an even function. **2**

(ii) Find the domain and range of $y = f(x)$. **2**

(iii) Hence, sketch the graph of $y = f(x)$. **1**

Question 4 (15 marks) Use a SEPARATE writing booklet **Marks**

(a) For the ellipse $\frac{x^2}{9}+\frac{y^2}{4}=1$, find:

(i) its eccentricity **1**

(ii) the coordinates of the foci **1**

(iii) the equations of the directrices. **1**

(b) Show that the condition for which the line $y=mx+c$ is a tangent to the ellipse $\frac{x^2}{9}+\frac{y^2}{4}=1$ is given by $c^2=9m^2+4$. **3**

(c) (i) Show that the equation of the tangent to the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ at the point $P(a\sec\theta, b\tan\theta)$ is given by $\frac{x\sec\theta}{a}-\frac{y\tan\theta}{b}=1$. **3**

(ii) The equation of the normal at the point $P(a\sec\theta, b\tan\theta)$ is given by:

$$\frac{ax}{\sec\theta}+\frac{by}{\tan\theta}=a^2+b^2.$$

A line through P parallel to the y axis meets the asymptote $y=\frac{bx}{a}$ at Q. The tangent at P meets the same asymptote at the point R. The normal at P meets the x axis at the point G.

(α) Find the coordinates of Q and G. **2**

(β) Show that $\angle RQG = 90°$. **2**

(γ) Explain why the points R, Q, P, and G are concyclic? **2**

Question 5 (15 marks) Use a SEPARATE writing booklet **Marks**

(a) The polynomial $P(x) = x^4 - 6x^3 + 12x^2 - 10x + 3$ has a rational zero of multiplicity three. **3**

Find all the roots of $x^4 - 6x^3 + 12x^2 - 10x + 3 = 0$.

(b) Consider the polynomial $P(x) = x^4 - 8x^3 + 29x^2 - 52x + 40$.

(i) $P(x) = 0$ has complex roots of the form $a + bi$ and $a - 2bi$ (where a and b are real numbers). **1**

State why $a - bi$ and $a + 2bi$ are also roots of $P(x) = 0$?

(ii) Find the values of a and b. **3**

(iii) Hence, or otherwise, express $P(x)$ as a product of two quadratic factors with real coefficients. **2**

(c) The equation $x^3 + 2x - 1 = 0$ has roots α, β, and γ. Find:

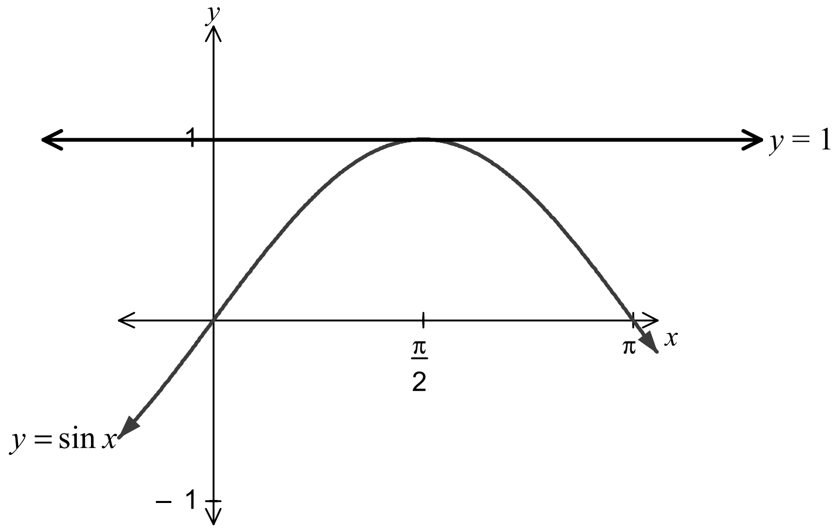
(i) the value of $\alpha^2 + \beta^2 + \gamma^2$ **2**

(ii) the cubic equation whose roots are $-\alpha, -\beta, -\gamma$ **2**

(iii) the value of $\alpha^3 + \beta^3 + \gamma^3$. **2**

Question 6 (15 marks) Use a SEPARATE writing booklet **Marks**

(a)



The area defined by $y \geq \sin x$, $0 \leq x \leq \dfrac{\pi}{2}$ and $0 \leq y \leq 1$ is rotated about the straight line $y = 1$.

(i) Copy the diagram above into your writing booklet and shade in the region defined by the simultaneous inequalities $y \geq \sin x$, $0 \leq x \leq \dfrac{\pi}{2}$ and $0 \leq y \leq 1$. **1**

(ii) Find the total volume of the solid formed, by taking slices perpendicular to the axis of rotation. **4**

(b) The horizontal base of a solid is an ellipse defined by the equation $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$. **4**

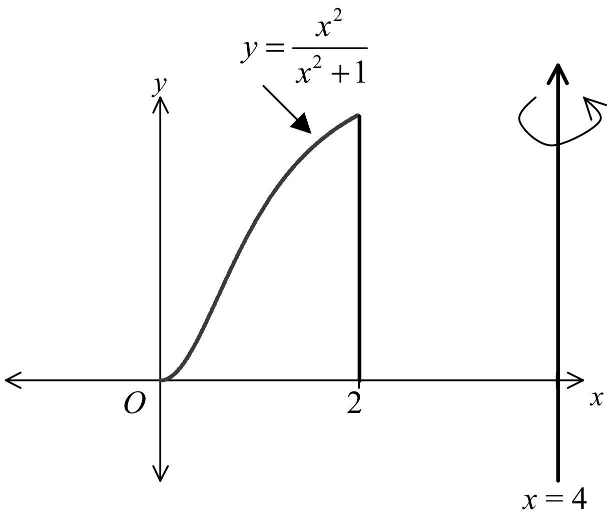
Vertical cross-sections taken perpendicular to the y axis are squares with one side in the horizontal base of the solid.

Find the volume of the solid formed in terms of a and b.

Question 6 continues on the next page …

Question 6 continued …

(c) The region bounded by the curve $y = \dfrac{x^2}{x^2+1}$, the x axis and $0 \le x \le 2$, is rotated about the line $x = 4$ to form a solid.



(i) Using the method of cylindrical shells, explain why the volume δV of a typical shell distant x units from the origin and with thickness δx is given by **3**

$$\delta V = 2\pi(4-x)\left(1-\frac{1}{1+x^2}\right)\delta x.$$

(ii) Hence, find the total volume of the solid formed. **3**

Question 7 (15 marks) Use a SEPARATE writing booklet **Marks**

(a) (i) Evaluate $\int_0^{\frac{\pi}{4}} \tan x \quad dx$. **2**

(ii) If $I_n = \int_0^{\frac{\pi}{4}} \tan^n x \quad dx$, $n = 0, 1, 2, 3, \ldots$, show that for $n = 2, 3, 4, \ldots$ **3**

$$I_n = \frac{1}{n-1} - I_{n-2}$$

(iii) Hence, evaluate I_5. **2**

(b) The equation of the tangent to the rectangular hyperbola $xy = 4$ at the point $T\left(2t, \frac{2}{t}\right)$ is given by $x + t^2 y - 4t = 0$. This tangent cuts the x axis at the point Q.

(i) State the coordinates of Q in terms of t. **1**

(ii) Find the equation of line through Q which is perpendicular to the tangent at T. **2**

(iii) The line through Q and perpendicular to the tangent at T meets the hyperbola at the points R and S. **1**

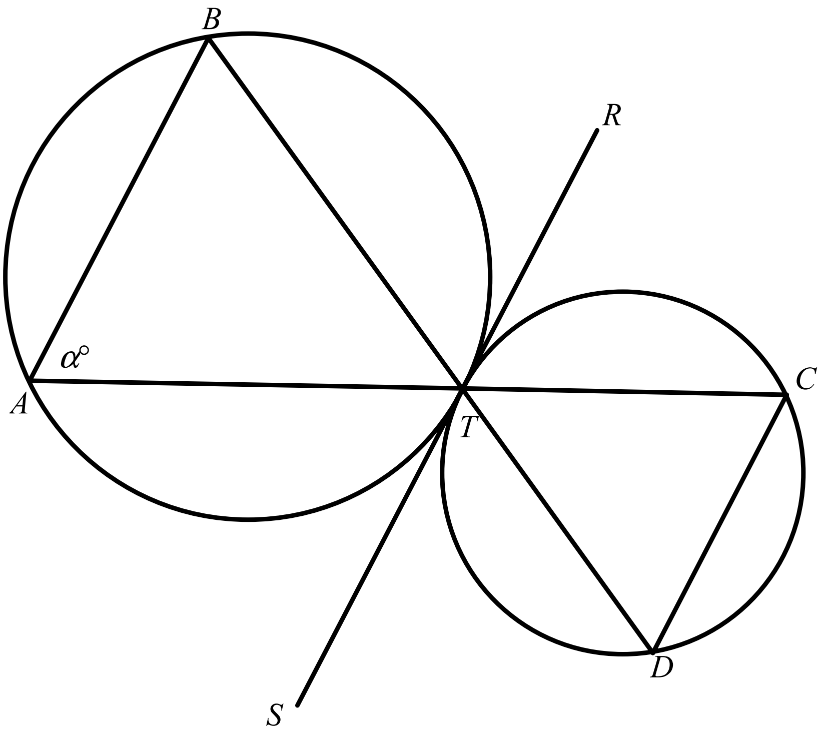
Show that the x coordinates of the points R and S are the roots of the quadratic equation $t^2x^2 - 4t^3x - 4 = 0$.

(iv) Show that the midpoint M of the interval RS has coordinates $\left(2t, -2t^3\right)$. **2**

(v) Find the equation of the locus of M as T moves on the hyperbola $xy = 4$, noting any restriction(s) that may apply. **2**

Question 8 (15 marks) Use a SEPARATE writing booklet **Marks**

(a) **NOT TO SCALE**



In the diagram above, two circles touch externally at the point T.

The points A and B lie on the circumference of one circle such that $AT = BT$.
The intervals AT and BT produced meet the second circle at C and D respectively.
RS is the common tangent at T as shown. Let $\angle BAT = \alpha$.

Copy or trace this diagram into your writing booklet.

(i) Explain why $\angle BAC = \angle ACD = \alpha$. **2**

(ii) Prove that quadrilateral $ABCD$ is an isosceles trapezium. **3**

Question 8 continues on the next page…

Marks

(b) (i) Prove that $\frac{d}{dx}\left(\frac{1}{2}v^2\right)=\frac{d^2x}{dt^2}$ where v is the velocity and $\frac{d^2x}{dt^2}$ is the acceleration of a particle as a function of time t. **2**

(ii) A particle moves from rest toward the origin O when its displacement from O is d metres (with $d > 0$). At any time t during the motion the particle's acceleration toward O at a displacement x is given by $\frac{k}{x^3}$, where k is a constant greater than zero.

(α) Show that $v^2 = k\left(\frac{1}{x^2} - \frac{1}{d^2}\right)$. **2**

(β) Explain why $v = -\sqrt{k\left(\frac{1}{x^2} - \frac{1}{d^2}\right)}$. **1**

(c) Suppose a function $f(x)$ is defined by $f(x) = \sin(ax)$. **5**

Show by using the Principle of Mathematical Induction, that the n^{th} derivative is given by:

$$f^{(n)}(x) = a^n \sin\left(ax + \frac{n\pi}{2}\right),$$

for all positive integers n.

End of Paper

STANDARD INTEGRALS

$$\int x^n \, dx = \frac{1}{n+1} x^{n+1}, \quad n \neq -1; x \neq 0, \text{ if } n < 0$$

$$\int \frac{1}{x} \, dx = \ln x \ , \quad x > 0$$

$$\int e^{ax} \, dx = \frac{1}{a} e^{ax}, \quad a \neq 0$$

$$\int \cos ax \, dx = \frac{1}{a} \sin ax, \quad a \neq 0$$

$$\int \sin ax \, dx = -\frac{1}{a} \cos ax, \quad a \neq 0$$

$$\int \sec^2 ax \, dx = \frac{1}{a} \tan ax, \quad a \neq 0$$

$$\int \sec ax \tan ax \, dx = \frac{1}{a} \sec ax, \quad a \neq 0$$

$$\int \frac{1}{a^2 + x^2} \, dx = \frac{1}{a} \tan^{-1} \frac{x}{a}, \quad a \neq 0$$

$$\int \frac{1}{\sqrt{a^2 - x^2}} \, dx = \sin^{-1} \frac{x}{a}, \quad a > 0, \ -a < x < a$$

$$\int \frac{1}{\sqrt{x^2 - a^2}} \, dx = \ln\left(x + \sqrt{x^2 - a^2}\right), \quad x > a > 0$$

$$\int \frac{1}{\sqrt{x^2 + a^2}} \, dx = \ln\left(x + \sqrt{x^2 + a^2}\right)$$

Note $\ln x = \log_e x, \quad x > 0$