Total Marks – 120
Attempt Questions 1-8
All questions are of equal value

Answer each question in a SEPARATE writing booklet. Extra writing booklets are available.

Question 1 (15 marks) Use a SEPARATE writing booklet. **Marks**

(a) Show that $\displaystyle\int_1^e \frac{(\log_e x)^3\,dx}{x} = \frac{1}{4}$ **2**

(b) (i) Write $1 - \dfrac{1}{1+x^2}$ as a single fraction. **1**

(ii) Use integration by parts to find $\int 2x\tan^{-1} x\,dx$ **2**

(c) Use the table of standard integrals to help evaluate $\displaystyle\int \frac{dx}{\sqrt{x^2 - 6x + 25}}$ **2**

(d) (i) Find A and B such that $\dfrac{2x^2+7x-10}{(x-4)(x+1)^2} = \dfrac{A}{x-4} + \dfrac{B}{(x+1)^2}$ **2**

(ii) Hence find $\displaystyle\int \frac{2x^2+7x-10}{(x-4)(x+1)^2}\,dx$ **2**

(e) Use the substitution $t = \tan\frac{x}{2}$ to evaluate $\displaystyle\int_{\frac{\pi}{3}}^{\frac{\pi}{2}} \operatorname{cosec} x\,dx$ **4**

Question 2 (15 marks) Use a SEPARATE writing booklet. **Marks**

(a) Let $z = 3 - 2i$ and $w = 4 + i$.
Find in the form $x + iy$,

(i) $z\bar{z}$ **1**

(ii) $\frac{1}{w}$ **1**

(b) On an Argand Diagram the point A is represented by $z = 1 + i$ and the point B is represented by $\frac{1}{z}$.

(i) Express z in mod-arg form. **1**

(ii) Show clearly the points A and B on the Argand Diagram. **2**

(iii) Find the area of the triangle OAB where O is the origin.
Justify your answer carefully. **2**

(c) (i) Find the locus in the Argand Diagram satisfied by $z\bar{z} - 2\text{Re}(z) = 0$ **2**

(ii) Draw a neat sketch of this locus. **1**

(iii) On the same diagram, draw the locus $\arg z = \frac{\pi}{4}$. **1**

(iv) Find the complex number satisfied by $\arg z = \frac{\pi}{4}$ and $z\bar{z} - 2\text{Re}(z) = 0$. **2**

(d) If ω is a complex cube root of unity,

(i) Write down the value of $1 + \omega + \omega^2$. **1**

(ii) Simplify $\omega^4 + \omega^5 + \omega^6$. **1**

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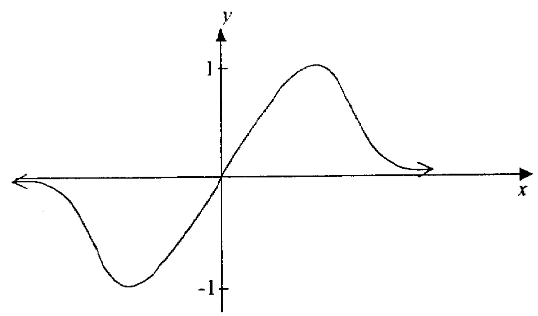
Question 3 (15 marks) Use a SEPARATE writing booklet. **Marks**

(a) The diagram shows $y = f(x)$ which is an odd function.
There is a turning point at $(1,1)$.



Draw a separate sketch of each of the following graphs.
Use about one third of a page for each graph. Show all significant features.

(i) $y = f(-x)$ **2**

(ii) $y = \frac{1}{f(x)}$ **2**

(iii) $y = f(|x|)$ **2**

(iv) Draw $y = f(x)$ and $y = \sqrt{f(x)}$ on the same number plane. **2**

(v) $y = e^{f(x)}$ **2**

(vi) $y = (f(x))^2$ **2**

(vii) $y = f(x) \times \sin^{-1} x$ (show the coordinates of the endpoints) **3**

Question 4 (15 marks) Use a SEPARATE writing booklet. **Marks**

(a) If α is a root of $ax^4 + bx^3 + cx^2 + dx + e = 0$ where a, b, c, d, e are real and α is complex, prove that $\overline{\alpha}$ is also a root. **3**

(b) (i) Given that $1 - \sqrt{3}i$ is a root of $P(x) = 0$ where $P(x) = x^4 - 2x^3 + 5x^2 - 2x + 4$, write down two of the linear factors of $P(x)$. **2**

(ii) Hence factorise $P(x)$ completely into real factors. **2**

(c) (i) Show that the solutions of $z^6 + z^3 + 1 = 0$ are contained in the solutions of $z^9 - 1 = 0$. **2**

(ii) Sketch the nine solutions of $z^9 - 1 = 0$ on an Argand Diagram.
(about one third of a page in size) **2**

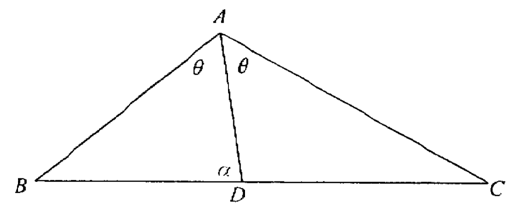
(iii) Mark clearly on your diagram, the six roots $z_1, z_2, z_3, z_4, z_5, z_6$ of $z^6 + z^3 + 1 = 0$. **1**

(iv) Show that the sum of the six roots of $z^6 + z^3 + 1 = 0$ can be given by **3**

$$2\left(\cos\frac{2\pi}{9} + \cos\frac{4\pi}{9} - \cos\frac{\pi}{9}\right)$$

Question 5 (15 marks) Use a SEPARATE writing booklet. **Marks**

(a)



In triangle BAC, DA bisects $\angle BAC$.
$\angle BAD = \angle DAC = \theta$ and $\angle BDA = \alpha$.

(i) Use trigonometry to prove that $\dfrac{BD}{DC} = \dfrac{AB}{AC}$. **3**

(ii) If $\dfrac{AB}{AC} = r$, show that $\dfrac{\text{Area } \Delta ABD}{\text{Area } \Delta ADC} = r$ **2**

(b) $\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = 1$ is the equation of a hyperbola, and $P(x_1, y_1)$ is a point on the hyperbola.

(i) Write down the coordinates of the foci S, S'. **1**

(ii) Show that the equation of the tangent at $P(x_1, y_1)$ is given by $\dfrac{xx_1}{a^2} - \dfrac{yy_1}{b^2} = 1$ **2**

(iii) Find Q, the point at which the tangent cuts the x-axis. **1**

(iv) Find the distances,

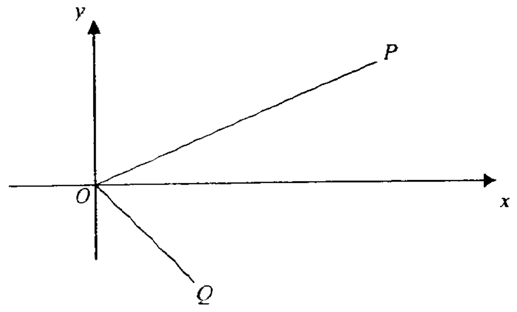
(α) PS **1**
(β) PS' **1**

(v) Show that $\dfrac{PS'}{PS} = \dfrac{QS'}{QS}$ **2**

(vi) Using your proof in (a), what geometrical fact can you deduce in the triangle PSS'? **2**

Question 6 (15 marks) Use a SEPARATE writing booklet. **Marks**

(a)



In the Argand Diagram above, point P corresponds to complex number z.
The triangle OPQ is a right-angled triangle and $OP=3OQ$.

(i) What is the complex number that corresponds to point Q? **1**

(ii) $QOPR$ is a rectangle. Write down the complex number that corresponds to R. **2**

(b) (i) Prove the identity $\cos(a-b)x-\cos(a+b)x=2\sin ax\sin bx$ **2**

(ii) Hence find $\int \sin 3x \sin 2x\,dx$ **2**

(c) If $u_1=8$, $u_2=20$ and $u_n=4u_{n-1}-4u_{n-2}$ for $n\geq 3$.

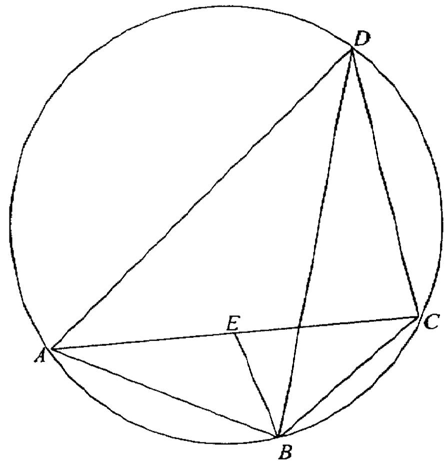
(i) Determine u_3 and u_4. **1**

(ii) Prove by induction that $u_n=(n+3)2^n$ for $n\geq 1$. **4**

(d) If $ax^3+bx^2+d=0$ $(a,b,d\neq 0)$ has a double root, show that $27a^2d+4b^3=0$. **3**

Question 7 (15 marks) Use a SEPARATE writing booklet. **Marks**

(a) In the diagram $ABCD$ is a cyclic quadrilateral. E is the point on AC such that $\angle ABE=\angle DBC$.

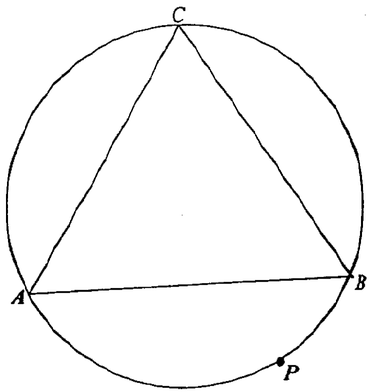


(i) Show that $\Delta ABE \parallel\!| \Delta DBC$ and $\Delta ABD \parallel\!| \Delta EBC$. **3**

(ii) Hence show that $(AB)(DC)+(AD)(BC)=(AC)(DB)$ **3**

(b) In the diagram ABC is an equilateral triangle inscribed in a circle. P is a point on the minor arc AB of the circle. **2**

Use the result from (a) to show that $PC=PA+PB$.



Question 7 (continued)

(c) (i) Show that $\operatorname{cosec}2\theta + \cot 2\theta = \cot\theta$ for all real values of θ. **2**

(ii) Use the result above to :

(α) find in surd form the values of $\cot\frac{\pi}{8}$ and $\cot\frac{\pi}{12}$. **2**

(β) show without using calculators that

$$\operatorname{cosec}\frac{4\pi}{15} + \operatorname{cosec}\frac{8\pi}{15} + \operatorname{cosec}\frac{16\pi}{15} + \operatorname{cosec}\frac{32\pi}{15} = 0$$ **3**

Question 8 (15 marks) Use a SEPARATE writing booklet. **Marks**

(a) (i) Sketch the graph of $y = \sec x$ for $0 \le x \le \frac{\pi}{2}$ indicating any important features. **2**

On the same set of axes, sketch the graph of $y = \sec^{-1} x$, again indicating any important features.

(ii) If $x = \sec y$, find $\frac{dx}{dy}$ and hence if $y = \sec^{-1} x$, find $\frac{dy}{dx}$. **2**

(b) The points $P(a\cos\theta, b\sin\theta), Q(-a\sin\theta, b\cos\theta)$ lie on the ellipse E, given by $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$.

(i) Show that if O is the centre of E, then $OP^2 + OQ^2 = a^2 + b^2$. **2**

(ii) The equations of the tangents at P and Q are : **3**

$$\frac{x\cos\theta}{a} + \frac{y\sin\theta}{b} = 1$$

$$\frac{-x\sin\theta}{a} + \frac{y\cos\theta}{b} = 1$$

Show that the point of intersection T of the two tangents at P and Q is given by $T\big(a(\cos\theta - \sin\theta), b(\sin\theta + \cos\theta)\big)$.

(iv) Show that the locus of T is given by the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 2$ **2**

(v) If α is the angle between the tangents at P and Q, show that $\tan\alpha = 2\frac{\sqrt{1-e^2}}{e^2\sin 2\theta}$ **4**

End of paper