2008

HIGHER SCHOOL CERTIFICATE

TRIAL EXAMINATION

Mathematics Extension 2

General Instructions

- Reading time – 5 minutes
- Working time – 3 hours
- Write using black or blue pen
- Board-approved calculators may be used
- A table of standard integrals is provided at the back of this paper
- All necessary working should be shown in every question
- Start a new booklet for each question

Total marks – 120

- Attempt Questions 1 – 8
- All questions are of equal value

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Attempt Questions 1-8
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Answer each question in a SEPARATE writing booklet. Extra writing booklets are available.

Question 1 (15 marks) Use a SEPARATE writing booklet. **Marks**

(a) $\displaystyle\int \frac{2x}{\sqrt{1-x^4}}\,dx$ **2**

(b) $\displaystyle\int_0^{\frac{\pi}{2}} \frac{dx}{2+\cos x}$ **3**

(c) $\displaystyle\int_1^{e^2} 3x^2 \ln x \quad dx$ **3**

(d) $\displaystyle\int \frac{dx}{\sqrt{x^2 - x \ \ +1}}$ **2**

(e) (i) Show that $\displaystyle\int_0^a f(x)\ dx = \int_0^a f(a-x)\ dx$ **2**

(ii) Use this property to show that $\displaystyle\int_0^1 x^3(1-x)^6\ dx = \frac{1}{840}$ **3**

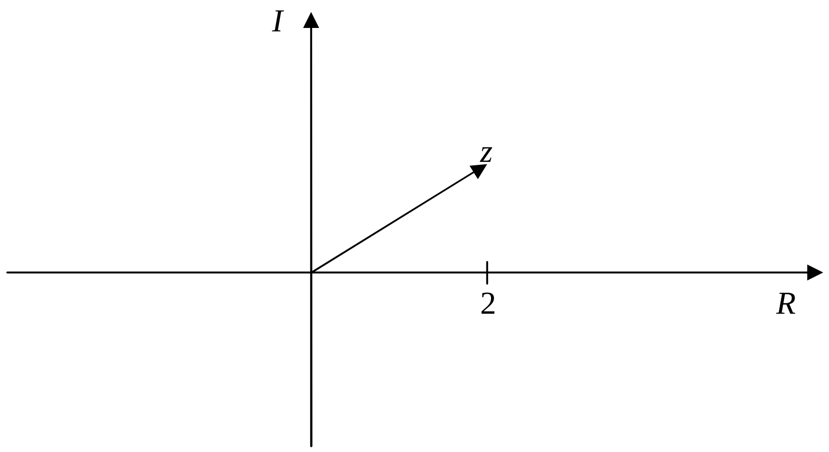
Question 2 (15 marks) Use a SEPARATE writing booklet. **Marks**

(a) A complex number z is given by $z = \sqrt{3} + i$

(i) Evaluate $\bar{z}$. Verify that $z\bar{z}$ is real. **2**

(ii) Find $\frac{1}{z}$ in the form $a + ib$, where a and b are real. **1**

(b) A point z on the Argand Diagram is given below:



Copy this diagram into your examination booklet and use it to plot the following points. Clearly label each point.

(i) $\bar{z}$ **1**

(ii) $2iz$ **1**

(iii) $\frac{1}{z}$ **1**

(c) Express $i - 1$ in modulus argument form, and hence simplify $(i-1)^5$ **2**

Question 2 continues on page 4

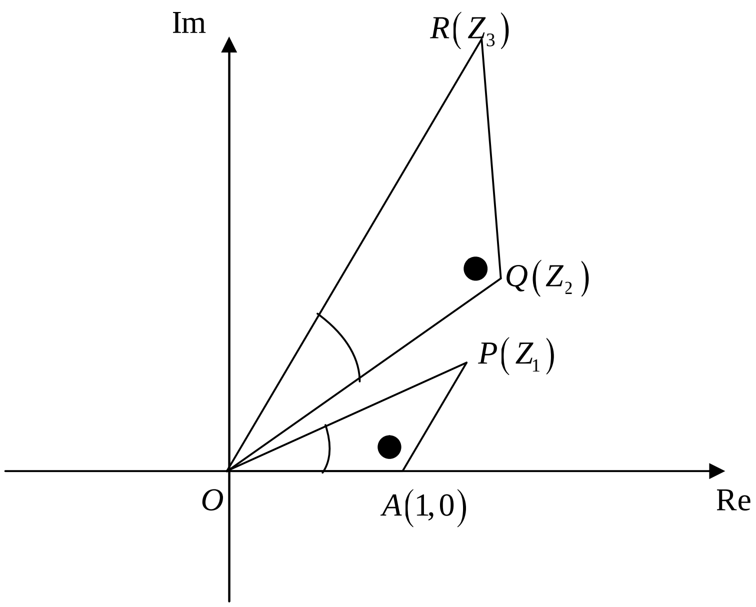
Question 2 (continued)

(d) Sketch the locus and state its equation:

(i) $|z-2| = |z-2i|$ **2**

(ii) $z\bar{z} - 3(z+\bar{z}) \le 0$ **2**

(e)



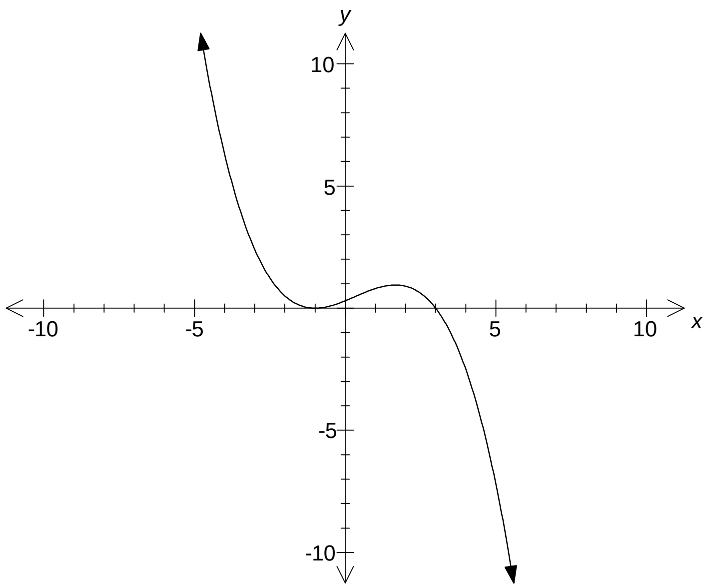
In the figure above, the points P, Q and A represent the complex numbers Z_1, Z_2 and $(1,0)$ respectively. Given $\angle OAP = \angle OQR$ and $\angle AOP = \angle QOR$.

Explain why $R(Z_3)$ represents the complex number Z_1Z_2. **3**
You must support your answer with clear and complete mathematical reasons.

End of Question 2

Question 3 (15 marks) Use a SEPARATE writing booklet. **Marks**

(a)



The graph of $f(x) = \dfrac{1}{10}(x+1)^2(3-x)$ is drawn above.

On separate diagrams, draw a neat sketch showing the main features of each of the following:

(i) $y = f(x-1)$ **1**

(ii) $y = f(|x|)$ **1**

(iii) $y = \{f(x)\}^2$ **2**

(iv) $y = xf(x)$ **2**

(v) $y^2 = f(x)$ **2**

(vi) $y = e^{f(x)}$ **2**

(b) Given that $I_n = \displaystyle\int_0^{\frac{\pi}{4}} \sec^n x \; dx$ show that :

(i) $I_n = \dfrac{1}{n-1}\left((\sqrt{2})^{n-2} + (n-2)I_{n-2}\right)$ **4**

(ii) Hence or otherwise evaluate I_4 **1**

End of Question 3

Question 4 (15 marks) Use a SEPARATE writing booklet. **Marks**

(a) (i) If $P(x) = x^4 + x^3 - 3x^2 - 5x - 2$, show that $P(x) = 0$ has a multiple root, find this root and its multiplicity. **3**

(ii) Hence factorise $P(x) = x^4 + x^3 - 3x^2 - 5x - 2$ into its linear factors. **1**

(b) The equation $x^3 + 2x - 1 = 0$ has roots α, β, γ. Find the monic equations with roots

(i) $\alpha^2, \beta^2, \gamma^2$. **2**

(ii) $\alpha\beta,\ \beta\gamma, \alpha\gamma$ **3**

(iii) Evaluate $\alpha^3 + \beta^3 + \gamma^3$ **2**

(c) A point $P\left(ct, \dfrac{c}{t}\right)$ lies on the rectangular hyperbola $xy = c^2$.

(i) Show that the equation of the tangent at the point $P\left(ct, \dfrac{c}{t}\right)$ on the rectangular hyperbola is given by $x + t^2 y = 2ct$. **2**

(ii) Prove that the area bounded by the tangent and the asymptotes of the rectangular hyperbola is a constant. **2**

End of Question 4

Question 5 (15 marks) Use a SEPARATE writing booklet. **Marks**

(a) ABC is an equilateral triangle, inscribed in a circle. X is a point on the minor arc BC.

(i) Prove that $\Delta BDX \parallel\!| \, \Delta ACX$ **3**

(iii) Prove that $XB + XC = XA$ **3**

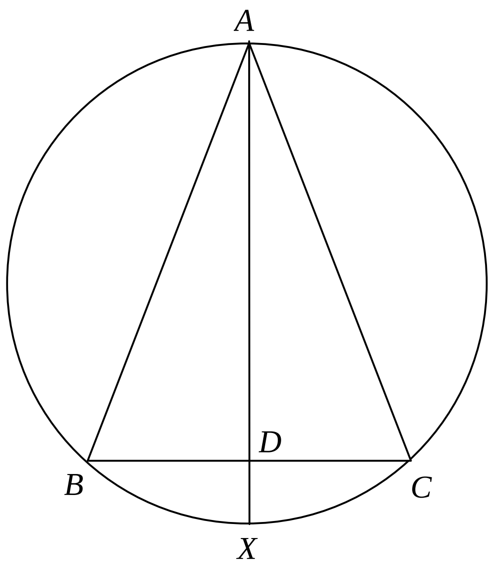


DIAGRAM NOT TO SCALE

(b) State whether each of the following are true or false giving brief reasons for your answers:

(i) $\displaystyle\int_0^{\pi} \sin 9x \; dx = 0$ **1**

(ii) $\displaystyle\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} x \sin x \, dx = 0$ **2**

(c) Find the equation of the tangent to the curve $\cos 2x + \sin y = 1$ at the point $x = \dfrac{\pi}{6}$. **3**

(d) Use the substitution $x = a\sin\theta$ to show that **3**

$$\int \sqrt{(a^2 - x^2)} \; dx = \frac{1}{2} a^2 \sin^{-1} \frac{x}{a} + \frac{1}{2} x \sqrt{(a^2 - x^2} + C$$

End of Question 5

Question 6 (15 marks) Use a SEPARATE writing booklet. **Marks**

(a) (i) Given $a\alpha^2 + b\alpha + c = 0$ where $a,b,c \in \mathbb{R}$ and $\alpha \in \mathbb{C}$, prove that $a\left(\overline{\alpha}\right)^2 + b\overline{\alpha} + c = 0$ **2**

(ii) A polynomial $P(x)$ with real coefficients , has two of its zeros $3i$ and $1+2i$. Find in expanded form, a possible polynomial $P(x)$. **3**

(b) Use De Moivres Theorem and binomial expansion to find an expression for $cos\,4\theta$ in terms of $cos\,\theta$. **3**

(c) (i) Given $z = cos\,\theta + i\,sin\,\theta$, prove $z^n + \dfrac{1}{z^n} = 2\,cos\,n\theta$ **2**

(i) Hence by considering the expansion $\left(z + \dfrac{1}{z}\right)^4$ show that **3**

$$\cos^4\theta = \frac{1}{8}\cos 4\theta + \frac{1}{2}\cos 2\theta + \frac{3}{8}$$

(iii) Hence evaluate $\displaystyle\int_0^{\frac{\pi}{2}} cos^4\theta\, d\theta$ **2**

End of Question 6

Question 7 (15 marks) Use a SEPARATE writing booklet. **Marks**

(a) The roots of the polynomial $p(x) = x^3 + ax^2 + bx + c = 0$ are three consecutive terms of an arithmetic series. Prove that the relationship between the coefficients is given by $2a^3 - 9ab + 27c = 0$ **4**
Hint: make an appropriate choice for the roots in arithmetic progression.

(b) A point $P(a\cos\theta, b\sin\theta)$ lies on the ellipse $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$ where $a > 0$ and $b > 0$.
The equation of the normal at the point $P(a\cos\theta, b\sin\theta)$ is given by
$xa\sin\theta - yb\cos\theta = (a^2 - b^2)\sin\theta\cos\theta$

(i) Show that the ellipse intersects the rectangular hyperbola $xy = c^2$ in four points if $ab > 2c^2$ **3**

(ii) Show that for $0 < \theta < \dfrac{\pi}{2}$, the normal at P on the ellipse intersects the hyperbola in two distinct points, say A and B. **3**

(iii) If M is the mid-point of AB, show that the coordinates of M are given by **2**

$$\left(\frac{(a^2 - b^2)\cos\theta}{2a}, -\frac{(a^2 - b^2)\sin\theta}{2b} \right)$$

(iv) Hence find the locus of M as θ varies. **3**

End of Question 7

Question 8 (15 marks) Use a SEPARATE writing booklet. **Marks**

(a) For the function $y = \cos^{-1}(e^x)$,

(i) Find the domain and the range. **2**

(ii) Draw a neat sketch the graph of $y = \cos^{-1}(e^x)$. **2**

(iii) Hence draw a neat sketch of the curve $y = \dfrac{1}{(\cos^{-1}(e^x))}$ **2**

(b) (i) Using induction, show that for each positive integer *n*, there are unique positive integers p_n and q_n such that: $(1+\sqrt{2})^n = p_n + q_n\sqrt{2}$ **4**

(ii) Show also that $p_n{}^2 - 2q_n{}^2 = (-1)^n$. **1**

(c) If $f(xy) = f(x) + f(y)$, for all $x, y \neq 0$, prove that

(i) $f(1) = f(-1) = 0$ **2**

(ii) $f(x)$ is an even function. **2**

End of paper

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