Leaving Certificate

MATHEMATICS I HONOURS

EXAMINATION PAPERS

Contents:

SPECIMEN MATHEMATICS : HONOURS PAPER ISSUED BY THE EXAMINERS.

1. A point moves in a straight line so that its distance at time 't' from a given point O of the line is x, where $x = t^2 \sin t + 6t \cos t - 12 \sin t$. Find its velocity at time 't' and prove that the acceleration is then -

$$-t^2 \sin t - 2t \cos t + 2 \sin t.$$

Determine the time ($t > 0$) at which the acceleration has a turning value, distinguishing between maxima and minima.

2. (i) Integrate the functions: (a) $\dfrac{\cos^3 x}{\sin^2 x}$ (b) $\dfrac{1}{x + \sqrt{x}}$

(ii) Evaluate $\displaystyle\int_2^{12} \frac{2x + 5}{(2x - 3)(2x + 1)}\,dx$ to three significant figures. ($\log_{10} e = 0.4343$).

3. A lighthouse AB of height c feet stands on the edge of a vertical cliff OA of height b feet above sea level. From a small boat at a variable distance x feet from O, the angle subtended by AB is Θ. Prove that $\tan\Theta = \dfrac{cx}{x^2 + b(b + c)}$ and that if ϕ is the maximum value of Θ, then: $\tan\phi = \dfrac{c}{2\sqrt{b(b + c)}}$

4. Write down the series for $\log_e(1 + x)$ and state for what range of values of x the expansion is valid. Prove that within the range of validity of the expansions

$$(1 - x)\log_e(1 - x) + (1 + x)\log_e(1 + x) = 2\sum_{n=1}^{\infty} \frac{x^{2n}}{2n(2n - 1)}$$

5. (i) Prove that if $f(x) = \left(\dfrac{1 + x}{1 - x}\right)^n$ then $\dfrac{d}{dx}f(x) = \dfrac{2n}{1 - x^2} f(x)$

Thence or otherwise prove that if $f(x) = a_0 + a_1 x + a_2 x^2 + \ldots\ldots$

then $(r + 1)\, a_{r+1} = 2n\, a_r + (r - 1)\, a_{r-1}$.

(ii) The numbers 1, 2, ..., n are arranged in random order. What are the probabilities that:- (a) the number 2 immediately follows the number 1 ; and (b) the number 2 follows the number 1 and there are r, $\leq n - 2$, numbers between them. Sum the values of the probabilities obtained in (b) for the values $r = 0, 1, 2, \ldots, n - 2$, and compare your result with the probability that 2 should appear later in the arrangement than 1.

6. (i) By adding together suitable multiples of the columns show that $a + wb + w^2c$ is a factor of the determinant $\begin{vmatrix} a & b & c \\ c & a & b \\ b & c & a \end{vmatrix}$ w being any of the roots of the equation $w^3 = 1$.

Express the determinant as the product of factors linear in a, b and c.

(b) In the sequence of numbers $a_1, a_2, \ldots,$ $a_n > 0$, for all sufficiently large n, and

$\dfrac{a_n}{1 + a_n} \rightarrow 0$, as $n \rightarrow \infty$. Prove that $a_n \rightarrow 0$, as $n \rightarrow \infty$

7. (i) If f (x) is a function of x, give an approximation to f (a + h) correct to the first order in h. Show how this result may be used to improve on an approximation x = a to a root of the equation f (x) = 0.

(ii) Without using tables calculate $\sqrt[3]{8.05}$ to four significant figures.

(iii) Show, graphically or otherwise, that if n is a large positive integer, there is a root of the equation $x \sin x = 1$ nearly equal to $2n\pi$ Show that a better approximation is $2n\pi + \dfrac{1}{2n\pi}$.

8. Find the turning points on the graph of the function $y = \dfrac{2x}{x^2 + 1}$, and the equations of the inflexional

tangents. Sketch the graph and calculate the area enclosed by the curve, the x-axis and the ordinates to the curve at $x = 1$ and $x = 2$.

9. (i) State what is meant by the modulus and argument (amplitude) of a complex number. If z is a complex number for which $|z| = 1$ and $\arg z = \theta$, find the values of $\left|\frac{2}{1 - z^2}\right|$ and $\arg \frac{2}{1 - z^2}$

(ii) If $|z_1 - z_2| = |z_1 + z_2|$, prove that $\arg z_1$ and $\arg z_2$ differ by $\frac{\pi}{2}$ or $\frac{3\pi}{2}$.

(N.B. Arg w denotes the principal value of the argument of w).

10. Give the formula for integration by parts and evaluate the definite integral $\int_0^{\frac{1}{2}\pi} x^2 \cos x \, dx$.

The portion of the curve $y = \sin x$ from $x = 0$ to $x = \frac{1}{2}\pi$ revolves around the y-axis. Prove that the volume described by the area between the curve, the y-axis and the line $y = 1$ is $\frac{1}{4}\pi(\pi^2 - 8)$.

TRIAL LEAVING. **HOMEBUSH BOYS' HIGH SCHOOL** SEPT, 1956

MATHEMATICS 1 HONOURS

1. An empty dam, of depth 30 feet, has a horizontal circular cross-section whose diameter diminishes uniformly with depth. At the top, the cross-section is 1600 square feet and at the bottom it is 100 square feet. If water is pumped in at the rate of 1 cubic foot per second, find (a) how long it takes to fill the dam, (b) at what rate the level is rising just as it reaches the top of the dam.

2. If f (x) is a function of x, give an approximation to f (a + h) correct to the first order in h. Show how this result may be used to improve on an approximation $x = a$ to a root of the equation $f(x) = 0$.
(a) Without using tables, show that $\cos 31^\circ \doteqdot 0.857$ (b) Determine to two places of decimals that root of the equation $\frac{x^{5/3}}{x + 2} = 3.2104$ whose value is nearly 8.

3. (a) Express in partial fractions: $\frac{1}{(1 - y)(1 - y^2)}$, show that $\frac{1}{(1 - x^3)(1 - x^6)} = 1 + x^3 + 2x^6 + 2x^9 + 3x^{12} + \ldots\ldots$

(b) If a, b and c all have different values and $\begin{vmatrix} a & a^2 & a^3 - 1 \\ b & b^2 & b^3 - 1 \\ c & c^2 & c^3 - 1 \end{vmatrix} = 0$

prove that $abc = 1$.

4. If $|x| < 1$, write down in ascending powers of x a series for $\log(1 + x)$ and show that

$$\log \frac{1 + x}{1 - x} = 2\sum_1^{\infty} \frac{x^{2r - 1}}{2r - 1}$$

Deduce that if $n > 0$,
$$\log \frac{n + 1}{n} = 2\sum_1^{\infty} \frac{1}{(2r - 1)(2n + 1)^{2r - 1}}$$

Hence show that
$$\log 2 = 2\left\{\frac{1}{3} + \frac{1}{3 \cdot 3^3} + \frac{1}{5 \cdot 3^5} + \frac{1}{7 \cdot 3^7}\right\} + R \quad \text{where } R < \frac{1}{4 \cdot 3^9}$$ and

establish that $0.69312 < \log 2 < 0.69316$.

5. (a) If α, β and γ are the roots of the equation $x^3 - x - 1 = 0$, find the equation whose roots are $\frac{1 + \alpha}{1 - \alpha}, \frac{1 + \beta}{1 - \beta}$ and $\frac{1 + \gamma}{1 - \gamma}$. Hence write down the value of $\sum \frac{1 + \alpha}{1 - \alpha}$.

(b) Prove that if $y = (\sin^{-1} x)^2$

i. $(1-x^2)\left(\frac{dy}{dx}\right)^2 = 4y$. ii. $(1-x^2)\frac{d^2y}{dx^2} - x\frac{dy}{dx} = 2$.

6. (a) Show that $\int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx$. Deduce that $\int_0^{\pi/4} \frac{1-\sin 2x}{1+\sin 2x}\,dx = \int_0^{\pi/4} \tan^2 x\,dx$, and evaluate the integral.

(b) Evaluate $\int_0^{\pi/2} \frac{\cos x - \sin x}{1 + \sin x \cos x}\,dx$.

7. (a) Two fixed points A and B, and a variable point P represent the complex numbers α, β, and z. Find the locus of P in each of the following cases illustrating with appropriate sketches:

i. $|z-\alpha| = |\beta|$ ii. $|z-\alpha| = |z-\beta|$ iii. $\text{am}(z-\alpha) = \text{am}\,\beta$.

(b) $p\left(\frac{1+i}{1-i}\right)$ and $q\left(\frac{\sqrt{2}}{1-i}\right)$ are two complex numbers. Express each in the form $x + iy$, and find their moduli and amplitudes. Show each on the one Argand Diagram. Hence, without tables, show that the argument of $\frac{1+\sqrt{2}+i}{1-i}$ is $\frac{3\pi}{8}$.

8. (a) Prove that if $0 < x < \pi$, $x\sin x + \cos x > 1 + \frac{1}{2}x^2\cos x$.

(b) Prove that the slope of the curve whose equation is $y = 1 + x + \frac{1}{2}x^2 + \frac{1}{6}x^3$ is always positive. Show that the curve has a point of inflexion where $x = -1$ and find the equation of the inflectional tangent. Prove also that the tangent at the point (0, 1) meets the curve again at the point (-3, -2). Use the information obtained in order to sketch the curve.

9. (a) INTEGRATE the following functions:

i. $e^{3x}\cos 3x$ ii. $\frac{1-x}{\sqrt{x+2}}$ iii. $\frac{1}{5-3\cos x}$ iv. $\frac{1}{x^2\sqrt{1-x^2}}$

(b) Sketch that part of the curve $y = 2x^2$ which lies within the area bounded by the x- and y-axes, $x = 2$ and $y = 8$. Calculate the volume formed by the rotation of:

i. the area bounded by the curve, the line $x = 2$ and the x-axis about the x-axis, and
ii. the area bounded by the curve, the line $y = 8$ and the y-axis, about the y-axis.

10. (a) If $x = 1 + at$ where t is small, show that:

$$e^x - e^{-x} = (e - \tfrac{1}{e}) + at(e + \tfrac{1}{e}) + \tfrac{1}{2}a^2t^2(e - \tfrac{1}{e}) \quad \text{(Correct to terms in } t^2\text{)}$$

(b) i. Discuss the convergence of the series $\sum \frac{1}{n}$ and $\sum (-1)^{n+1}\frac{1}{n}$

ii. Sum the series: $\sum_1^\infty \frac{1}{n(n+1)}$ and $\sum_1^\infty \frac{1}{n(n+1)(n+2)}$.

FORT STREET BOYS' HIGH SCHOOL 1956.

MATHEMATICS I. HONOURS

1. (i) Show that $-(a+b+c)$ is a root of the equation $\begin{vmatrix} x+a & b & c \\ b & x+c & a \\ c & a & x+b \end{vmatrix} = 0$ and solve the equation completely.

(ii) If α, β, γ are the roots of the equation $x^3 + px^2 + qx + r = 0$, express $\begin{vmatrix} 1 & \alpha\beta & \gamma\alpha \\ \alpha\beta & 1 & \beta\gamma \\ \gamma\alpha & \beta\gamma & 1 \end{vmatrix}$ in terms of p, q, r.

2. Integrate with respect to x a. (i) $\int x^2 \log x \, dx$ (ii) $\int \frac{1}{x^3 - 1} dx.$

b. Prove that $\int_\alpha^\beta \sqrt{(x - \alpha)(\beta - x)} \, dx = \frac{1}{8}\pi (\beta - \alpha)^2$ HINT substitute $x = \alpha \cos^2\theta + \beta \sin^2\theta$

3. (a) Prove that the coefficient of x^n in the expansion of $\left(\frac{1 + x}{1 - x}\right)^2$ is 4n, except for n = O and n = 1, and find it in those cases

(b) Write down the first five terms in the expansion of $pe^{qx} + qe^{px}$ in ascending powers of x. If p, q are the roots of the quadratic equation $\lambda^2 - a\lambda + b = 0$, show that the first four terms of the expansion are $a + 2bx + \frac{1}{2} abx^2 + \frac{1}{6} b(a^2 - 2b) x^3$ and express the coefficient of x^4 in terms of a and b.

4. (i) Given $\frac{dx}{dt} = k(a - x)$ and x = O when t = O, prove that $x = a(1 - e^{-kt})$ and $k = \frac{1}{t} \log \frac{a}{a - x}$

(ii) If $\frac{dx}{dt} = k(a - x)(b - x)$ and x = O when t = O find similar results for x and k.

5. A wall 7'. high is 5' from a building. A ladder rests with one end on the ground and the other on the building, and also touches the wall. Prove the length of the shortest ladder possible is $(7^{2/3} + 5^{2/3})^{3/2}$ and evaluate this correct to one decimal place. (Let the ladder make an angle θ with the ground).

6. (i) If w is an imaginary cube root of unity, prove that $(a + wb + w^2c)^3 + (a + w^2b + wc)^3 = (2a - b - c)(2b - c - a)(2c - a - b)$

(ii) If z = x +iy , show that this complex number may be represented in the form $r(\cos\theta + i\sin\theta)$.
Prove that if z_1, z_2 are 2 complex numbers then $\left|\frac{z_1}{z_2}\right| = \frac{|z_1|}{|z_2|}$ and $\text{amp}\left(\frac{z_1}{z_2}\right) = \text{amp } z_1 - \text{amp } z_2$.

A point z varies so as to satisfy the equations (i) $\left|\frac{z - 1}{z + 1}\right| = \frac{1}{2}$ (ii) $\text{amp}\left(\frac{z - 1}{z + 1}\right) = \frac{1}{2}\pi$
Find the locus of z in each case, on the argand diagram

7. Prove that $\int \sqrt{16a^2 - x^2} \, dx = \frac{1}{2} x \sqrt{16a^2 - x^2} + 8a^2 \sin^{-1} \frac{x}{4a}$, by the substitution $x = 4a \sin\theta$ in the integral.

Prove that the area common to the circle $x^2 + y^2 = 4a^2$ and the ellipse $\frac{x^2}{16a^2} + \frac{5y^2}{16a^2} = 1$ is

$$a^2 \left(\frac{32}{\sqrt{5}} \sin^{-1} \frac{1}{4} + \frac{8}{3}\pi\right).$$

8. Write down the series for $\log_e (1 + x)$. Find the range of convergence of x, stating carefully the tests applied.
Expand $\log_e (1 - x - x^2)$ as far as x^4, by writing it as $\log_e (1 - \overline{x + x^2})$, it being given that x is positive. Between what numbers may x be in order that the expansion may be valid?

9. Given $y = e^{-at} \sin(bt + c)$, prove that $\frac{dy}{dt} = -e^{-at}\sqrt{a^2 + b^2} \sin(bt + c - \theta)$ where $\tan\theta = b/a$.

Express $\frac{d^2y}{dt^2}$ in the same form, and write down a result for $\frac{d^ny}{dt^n}$, the result obtained by n differentiations.

Denoting $\frac{\theta - c}{b}$ by ϕ, show that the turning points for y occur at intervals of t which are in arithmetic progression, and the corresponding maxima and minima are in geometric progression.

Give a rough sketch of the curve, without consideration of the actual dimensions of a, b, c.

10. In work on bending of beams under uniform loads E is Young's Modulus, I is the moment of inertia constant for the given case, and w the load per unit length of the beam.

A beam of length L is clamped at both ends. Axes x'ox, y'oy are taken along the beam, and along the perpendicular bisector of the beam. The beam is subjected to a load of w per unit length, and it is found that the beam bends. It is known that the equation representing the curvature of the beam is is given by

$$EI\frac{d^4y}{dx^4} = w.$$ Explain why the initial conditions in this case are:

(i) $y = 0$ when $x = \pm\frac{1}{2}L$. (ii) $\frac{dy}{dx} = 0$ when $x = \pm\frac{1}{2}L$.

Use these initial conditions to prove that the equation of the curve assumed by the beam is $EIy = \frac{1}{384}w(4x^2 - L^2)^2$, and prove that the maximum deflection of the beam is $\frac{wL^4}{384EI}$.

EXTERNAL LEAVING CERTIFICATE

1956

MATHEMATICS 1 HONOURS

1. (i) State the comparison test for convergence and use it to prove that the series $1 + 1 + \frac{1}{2!} + \frac{1}{3!} + \dots$ is convergent to a sum less than 3

(ii) By considering the binomial expansion of $\left(1 + \frac{1}{n}\right)^n$ for positive integer values of n, show that $\left(1+\frac{1}{n}\right)^n < 3$ and so $\left(1 + \frac{1}{n}\right)^n < n$ for $n = 3, 4, 5, \dots$ Deduce that $n^{1/n}$ continually decreases as n runs through the values 3, 4, 5, ...

2. (i) (a) Show that $\begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ a^2 & b^2 & c^2 \end{vmatrix} = (b - c)(c - a)(a - b)$

(b) Solve the simultaneous equations

$$x + y + z = 1 \ ; \quad ax + by + cz = d \ ; \quad a^2x + b^2y + c^2z = d^2$$

(ii) If $z = x + iy$, express $w = \frac{1 + z}{1 - z}$ in the form X + iY where x, y, X and Y are real numbers. Find the path of the point which represents w in the Argand diagram when the point which represents z moves along the y-axis, $x = 0$.

3. If for $x > 0$ $y = \frac{x}{1 + x^2}\log x$, find $\frac{dy}{dx}$. Examine the behaviour of y and $\frac{dy}{dx}$ as $x \to 0$ and as $x \to \infty$. By considering rough graphs of the two functions $\frac{x^2 + 1}{x^2 - 1}$, $\frac{\log x}{x}$ show that $\frac{dy}{dx}$ is zero for just two positive values of x. Hence draw a sketch of the graph $y = \frac{x}{1 + x^2}\log x$.

4. (i) The figure ABXY is a trapezium and the angles ABX, BAY are right angles. AX and BY meet at C and $\angle ACB = \alpha$. By considering the variation of AY + BX as C moves in a circular arc through A, B, prove that $AY + BX > 2AB\cot\frac{\alpha}{2}$ unless AY = BX.

(ii) A set of twelve cards contains three cards each of four different colours, white, black, red, and yellow. Apart from colour the cards are indistinguishable. Three cards are drawn at random from the set. Find the chances that the cards are (i) all white, (ii) all of one colour, (iii) of three different colours.

5. If the functional relation $y = f(x)$ can be expressed in the form $x = \phi(y)$, what is the relation between the derived functions $f'(x)$ and $\phi'(y)$?

If $y = \frac{1}{2}(e^x - e^{-x})$, find $\left(\frac{dy}{dx}\right)^2 - y^2$. By expressing x in terms of y, deduce that

$$\int \frac{dy}{\sqrt{1+y^2}} = \log\left(y + \sqrt{1+y^2}\right).$$

6. Find the indefinite integrals: i. $\int \frac{1+x}{1+x^2}dx$ ii. $\int \frac{dx}{x(x^2+1)}$ iii. $\int \sqrt{\frac{1-x}{x}}\,dx$ iv. $\int x \sec^2 x\,dx$

7. Evaluate the definite integrals: - i. $\int_0^1 \frac{dx}{x^2 - x + 1}$ ii. $\int_{-\frac{1}{4}\pi}^{\frac{1}{4}\pi} x \sin x\,dx$

iii. $\int_0^1 \log x\,dx$ iv. $\int_{-1}^{+1} 2^x\,dx$

8. Explain how an approximation to the value of a definite integral may be obtained by the formation of a suitable finite sum. By considering the integral $\int_0^1 \sqrt{x(1-x)}\,dx$

prove that

$$\lim_{n \to \infty} \sum_{r=1}^{n-1} \frac{\sqrt{r(n-r)}}{n^2} = \frac{\pi}{8}$$

9. A quadrant of a circle of radius a is rotated about its chord. Show that the volume of the solid of revolution which is formed is $\left(\frac{5}{3} - \frac{\pi}{2}\right)\frac{\pi a^3}{\sqrt{2}}$

10. Explain the method of integration by parts. If $u_n = \frac{1}{n!}\int_0^a x^n e^{-x}\,dx$, prove that

$\frac{a^n}{n!} = e^a(u_{n-1} - u_n)$, and that $e^a - \left(1 + a + \frac{a^2}{2!} + \ldots\ldots + \frac{a^n}{n!}\right) = \frac{e^a}{n!}\int_0^a x^n e^{-x}dx$

For $a > 0$, show that $\int_0^a x^n e^{-x}dx < \frac{a^{n+1}}{n+1}$ and deduce that the infinite series

$1 + a + \frac{a^2}{2!} + \ldots\ldots\ldots$ converges to the sum e^a.

HOMEBUSH BOYS' HIGH SCHOOL

TRIAL LEAVING CERTIFICATE. AUGUST, 1957.

MATHEMATICS I HONOURS

1. (a) Discuss the series $\sum_{n=1}^{\infty} u_n$ where $u_n = \frac{(x-1)^n}{n}$.

(b) Investigate the convergence or divergence of the series whose n^{th} term is $(n^3+1)^{1/3} - (n^3-1)^{1/3}$.

(c) Show that $1 + \frac{1.2}{3!} + \frac{1.2.4.5}{6!} + \frac{1.2.4.5.7.8}{9!} + \ldots = e^{1/3}$

2. (a) Prove that, if $y = e^{\tan x}$, then : i. $\frac{dy}{dx} = y(1 + t^2)$; ii. $\frac{d^2y}{dx^2} = (1+t)^2 \frac{dy}{dx}$ where t = tan x.

(b) Integrate the following functions: i. $x \sec^2 x$. ii. $\frac{x^2}{1 - x^6}$. iii. $\frac{1}{\sqrt{x(2-x)}}$.

3. (a) A point $z = x + iy$ in the Argand Diagram is such that $|z| = 2$, $x = 1$ and $y > 0$. Determine the point and find its distance from the point $\frac{1}{2}z^2$.

(b) If $\begin{vmatrix} x+a & b & c \\ c & x+b & a \\ a & b & x+c \end{vmatrix} = 0$, find x in terms of a, b, and c.

4. (a) Explain how the calculus may be used to determine the volume of a solid of revolution.
The radius of a sphere is 5". Two parallel planes are drawn at distances 2" and 3" respectively from the centre and 1" apart. Use calculus methods to determine the volume of the slice of the sphere between the two planes.
(b) Find an approximation to that root of the equation $x^3 - 4x + 1 = 0$ which lies between 0 and 1.

5. If $y = \frac{x(1-x)}{1+x^2}$, find the maximum and minimum values of the function and any points of inflexion on the curve.

Use the information obtained in order to sketch the graph, showing clearly the points determined and the position of the curve relative to the line $y = x$.

6. Evaluate the following integrals: (a) $\int_0^1 x^2 e^x \, dx$; (b) $\int_0^2 \frac{4\,dx}{x^2+4}$ (c) $\int_{-\frac{1}{2}\pi}^{\frac{1}{2}\pi} x \sin x \, dx$.

(d) $\int_1^2 \frac{dx}{x^2(x+1)}$. (e) $\int_{-a}^{a} \sqrt{a^2 - x^2} \, . \, dx$

7. (a) Show that $\int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx$. Hence prove that

$\int_0^{\frac{1}{2}\pi} (a\cos^2 x + b\sin^2 x)\,dx = \int_0^{\frac{1}{2}\pi} (a\sin^2 x + b\cos^2 x)\,dx$, and deduce that each integral equals

$\frac{1}{4}\pi(a+b)$.

(b) If x and y are positive so that $x + y = 1$ and $a = 1 + x + x^2 + \ldots$;
$b = 1 + y + y^2 + \ldots$;
$c = 1 + xy + x^2y^2 + \ldots$

prove that $ab = a + b$ and $abc = a + b + c$.

8. (a) Show that

$$\lim_{n \to \infty} \sum_{r=0}^{n-1} \frac{\frac{1}{n}}{1 + \left(\frac{r}{n}\right)^2} = \int_0^1 \frac{dx}{1+x^2}.$$

Deduce that:

$$n \sum_{r=0}^{n-1} \frac{1}{n^2 + r^2} \to \frac{1}{4}\pi \quad \text{when } n \to \infty.$$

(b) Prove that if x is positive the value of sin x lies between $x - \frac{x^3}{3!}$ and $x - \frac{x^3}{3!} + \frac{x^5}{5!}$.

9. Prove that if $a \neq 0$, then $\log_e \frac{a - x}{a(1 - ax)} = \frac{a^2-1}{a} x + \frac{a^4-1}{2a^2} x^2 + \frac{a^6-1}{3a^3} x^3 + \dots$

For what range of values of x does this series converge?

Show that $\log \frac{\tan 3x}{3 \tan x} = \frac{8}{3} \tan^2 x \left\{ 1 + \frac{5}{3} \tan^2 x + \dots \right\}$. and find the next term in the expansion.

Prove that when $-\frac{\pi}{6} < x < \frac{\pi}{6}$ the series converges.

10. Prove that if $n > 1$, $\int_0^\infty x^n e^{-x^2} dx = \frac{1}{2}(n - 1) \int_0^\infty x^{n-2} e^{-x^2} dx$. Hence show that if n is a positive integer, $\int_0^\infty x^{2n+1} e^{-x^2} dx = \frac{1}{2} n!$

FORT STREET BOYS' HIGH SCHOOL 1957.

1. (i) Solve the equation $4x^3 - 24x^2 + 23x + 18 = 0$, given that the roots are in A.P.

(ii) Show that the equation $x^3 + 3x - 7 = 0$ has a root between 1 and 2, and find an approximation for it correct to two decimal points.

2. Prove that a series $\sum u_n$ is convergent if (a) its terms are alternating in sign, (b) $|u_n|$ continually decreases as n increases, (c) u_n approaches 0 as n approaches infinity. Investigate the convergence of the series:-

(i) $\frac{3}{\sqrt{(1^3)}} + \frac{5}{\sqrt{(2^3)}} + \frac{7}{\sqrt{(3^3)}} + \dots + \frac{2n+1}{\sqrt{(n^3)}} + \dots$

(ii) $\frac{2.x}{1^2} + \frac{3.x^2}{2^2} + \frac{4.x^3}{3^2} + \dots\dots + \frac{n+1}{n^2}. x^n + \dots$

3. (a) Prove that $\begin{vmatrix} a-b-c & 2a & 2a \\ 2b & b-c-a & 2b \\ 2c & 2c & c-a-b \end{vmatrix}$ is a perfect cube.

(b) Write $\begin{vmatrix} ap + bq & ar + bs \\ cp + dq & cr + ds \end{vmatrix}$ as the sum of four determinants and hence prove that it equals

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} \times \begin{vmatrix} p & q \\ r & s \end{vmatrix}$$

(c) Express in the modulus - amplitude form $(1 + i)^n$

(d) If P represents the complex number z, what facts about the position of P are expressed by
(i) $|z| = 5$ (ii) $|z - i| = 2$ (iii) $|z + 2| = 3$ (iv) $|2z - 1| = 3$
(v) $|z - 2 - 3i| = 4$ (vi) $\text{am}(z) = 0$.

4. (a) From 20 tickets marked with the first 20 numerals, one is drawn at random: find the chance that it is a multiple of 3 or of 7.

4. (b) 2m white counter and 2n red counters are arranged in a straight line on each side of a central mark. Find how many of the arrangements are symmetrical with respect to this mark.

(c) Sum the series $\frac{1}{1 \cdot 2 \cdot 3} + \frac{1}{2 \cdot 3 \cdot 4} + \frac{1}{3 \cdot 4 \cdot 5} + \ldots\ldots\ldots$

5. (i) Write down the expansion of log $(1 + x)$ in ascending powers of x and find the range of convergence. Prove the theorem you use in investigating the convergence for the case $x = 1$.

(ii) Obtain partial fractions for $\frac{1}{n(n+1)(n+2)(n+3)}$ and find the sum to infinity of the series for which this expression is the n th. term.

6. Differentiate the following functions: (a) $ae^{-kx} \sin kx$ (b) $\sin^{-1} \log x$ (c) $\log \frac{x}{a+\sqrt{(a^2 - x^2}}$

(d) $\sin^{-1}\left(\frac{e^x - e^{-x}}{e^x + e^{-x}}\right)$

7. Find the indefinite integrals (a) $\int \cos x \sqrt{\sin x}\, dx$. (b) $\int \frac{dx}{5-3\cos x}$ (c) $\int \log x\, dx$.
(d) $\int x e^{ax} dx$.

8. (a) Find the volume of the solid generated by the rotation about the x-axis of the part of the curve $y = \sin x$ between $x = 0$ and $x = \pi$

(b) The circle $x^2 + y^2 = 4$ cuts the positive side of the x-axis at A and cuts the parabola $3y = x^2$ in the first quadrant at P. Find the coordinates of P. Then find, in surd form, the volume generated when the area AOP revolves through π radians about the x-axis

9. (i) Show that $\int_0^{\pi} \frac{x \sin x\, dx}{1 + \cos^2 x} = \int_0^{\pi} \frac{(\pi - x) \sin x}{1 + \cos^2 x} dx$, and hence evaluate the integral on the left hand side.

(ii) Suppose $a + b = 1$, $a > 0$, $b > 0$. If $f(x) = bx \log x - (a + bx) \log (a + bx)$, show $f'(x) = b \log \frac{x}{a + bx}$ and deduce the minimum value of $f(x)$ is zero.

10. (i) By making the substitution $x = \frac{1 + u^2}{3 - u^2}$, find (as a function of u) the indefinite integral $\int \frac{dx}{x\sqrt{3x^2 + 2x - 1}}$, and hence verify that the value of the definite integral taken between the limits $\frac{1}{2}$ and 1 is $\pi/6$.

(ii) If $I_n = \int \sin^n x\, dx$, prove that $I_n = -\frac{\sin^{n-1} x \cos x}{n} + \frac{n-1}{n} I_{n-2}$.

Hence, or otherwise, find $\int_0^{\pi/2} \sin^8 x\, dx$

EXTERNAL

MATHEMATICS I - HONOURS PAPER 1957.

1. (i) Suppose that z_1, z_2, z_3 are three complex numbers and that Z_1, Z_2, Z_3, are the corresponding points in the Argand diagram. If $z_1 z_2 = z_3^2$ show that OZ_3 bisects the angle $Z_1 0 Z_2$.

(ii) Solve the quadratic equation $z^2 + i2\sqrt{2}z + i2\sqrt{3} = 0$ obtaining each root in the form $x + iy$ where x and y are real.

(iii) Prove that

$$\begin{vmatrix} b_1 + c_1, & c_1 + a_1, & a_1 + b_1 \\ b_2 + c_2, & c_2 + a_2, & a_2 + b_2 \\ b_3 + c_3, & c_3 + a_3, & a_3 + b_3 \end{vmatrix} = 2 \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

2. (i) If $p + q = 1$, prove that -

(a) $\sum_{r=1}^{n} r \; {}^{n}C_r \; p^r q^{n-r} = np;$

(b) $\sum_{r=1}^{n} r^2 \; {}^{n}C_r \; p^r q^{n-r} = npq + n^2p^2.$

(ii) Six dice, each having faces marked 1, ... 6, are thrown together. What is the chance that the dice show six different numbers?

3. If $y = \sqrt{\frac{x^3(4 - x)}{3}}$ find $\frac{dy}{dx}$, $\frac{d^2y}{dx^2}$ and determine the values of x for which these vanish.

Taking 1 inch as unit on both axes sketch carefully the complete curve $3y^2 = x^3(4 - x)$ and mark the turning points and the points of inflexion.

4. (i) A line is drawn through the fixed point (a, b) in the first quadrant, to cut the positive parts of the axes A, B. Find the minimum value of the length AB.

(ii) Sketch the curve $y = e^{-\frac{1}{2}x^2}$. The area between this curve and the positive parts of the co-ordinate axes is rotated about OY. Find the volume of the solid of revolution which is formed.

5. Find the indefinite integrals - (i) $\int \tan^4 x \, dx$; (ii) $\int \frac{e^{2x} \, dx}{\sqrt{1 + e^x}}$; (iii) $\int \frac{x^2 \, dx}{(x^2 + 1)(2x - 1)}$;

(iv) $\int 2^x \, x \, dx$

6(i) Show that, as n tends to infinity through integer values, $(1 + \frac{1}{n})^n$ tends to a limit which lies between between 2 and 3.

(ii) Determine whether the series $\sum_{1}^{\infty} u_n$ converges or diverges in the cases - (a) $u_n = (1 + \frac{1}{n})^{n^2} \cdot \frac{1}{3^n}$

(b) $u_n = \frac{2^n n!}{n^n}$.

7. (i) Prove that $\lim_{n \to \infty} \left(\frac{1}{n+1} + \ldots + \frac{1}{2n} \right) = \int_1^2 \frac{dx}{x}$. Prove also that for all positive integer values of n, $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \ldots + \frac{1}{2n-1} - \frac{1}{2n} = \frac{1}{n+1} + \frac{1}{n+2} + \ldots + \frac{1}{2n}$.

Deduce that the infinite series $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} \ldots$ converges.

7. (ii) Write down the power series expansion for $\log(1 + x)$ and specify the values of x for which the expansion is valid.

Evaluate the limit $\lim_{x \to 1} \frac{1 - x + \log x}{1 - \sqrt{2x - x^2}}$

8. Evaluate the definite integrals - (i) $\int_0^1 \frac{1 + x}{\sqrt{1 - x^2}} dx$; (ii) $\int_0^c x \sin(c - x)\, dx$;

(iii) $\int_0^1 (\log x)^2 dx$; (iv) $\int_1^\infty \frac{dx}{x\sqrt{x^2 + 2x - 1}}$. Put $y = \frac{1}{x}$.

9. (i) Assuming that y is a function of x which satisfies the relation $xy = ce^{y/x}$ where c is a constant show that $x(x - y)\frac{dy}{dx} + y(x + y) = 0$.

(ii) The curve $y = f(x)$ is a parabola whose axis is parallel to OY, and y_1, y_2, y_3, are ordinates corresponding to x_1, x_2, x_3. If $x_2 - x_1 = x_3 - x_2 = h$, show that $\int_{x_1}^{x_3} y\, dx = {}^1/_3 h(y_1 + 4y_2 + y_3)$.

The calculation will be simplified by setting $x_1 = z - h$, $x_2 = z$, $x_3 = z + h$.

10. If $u_n = \int_0^\pi \frac{\cos nx\, dx}{5 - 4\cos x}$, show that, for any value of n, $u_{n+1} + u_{n-1} = \frac{5}{2} u_n$.

Calculate u_0, u_1 directly and hence find u_2 and u_3.

HOMEBUSH BOYS' HIGH SCHOOL

1958.

MATHEMATICS I HONOURS

1. (i) If $a = \frac{1}{2}(-1 + i\sqrt{3})$, find a^{-1} in the form $a + bi$. Hence show that $a^2 - a^{-2} = -i\sqrt{3}$

(ii) Show that $\begin{vmatrix} (b + c) & a & a \\ b & (c + a) & b \\ c & c & (a + b) \end{vmatrix} = 4abc$

then solve the equations

$$\begin{aligned} (b + c)x + ay + az &= 1 \\ bx + (c + a)y + bz &= 1 \\ cx + cy + (a + b)z &= 1 \end{aligned}$$

(iii) Solve the equation $\begin{vmatrix} z & i & 2 \\ 2i & z & 3 \\ z & i & 1 \end{vmatrix} = 2z - i\sqrt{3} - 2$

for z in the form $z_1 = a_1 + ib_1$ and $z_2 = a_2 + ib_2$.

2. (i) Find the sum to "n" terms of the series $1 + 2(1 - a) + 3(1 - a)^2 + 4(1 - a)^3 + \ldots\ldots$.

(ii) Show that $\frac{5}{1.2.3} + \frac{6}{2.3.4} + \frac{7}{3.4.5.} + \ldots\ldots\ldots\ldots + \frac{n + 4}{n(n + 1)(n + 2)} + \ldots$ to infinity is equal to $1\frac{1}{2}$

(iii) An urn contains 30 black marbles and 20 white marbles. What is the probability of drawing (a) a white marble then a black marble in succession (b) three black marbles in succession.

3. What are the limits of $y = \frac{e^x}{1+x}$ as $x \to +\infty$, $x \to -\infty$, and $x \to 0$ respectively.

Find $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ hence find any stationary values of the curve $y = \frac{e^x}{1+x}$. Draw this curve. On the same graph page draw $y = -e^{-x}$. By using the above two graphs, on a separate graph page, sketch the graph of $y = \frac{e^x}{1+x} - e^{-x}$. Mark any maximum or minimum points clearly.

4. (i) Write the series for $\log(1 + x)$ where $0 < x < 1$. Hence by substituting $y = \frac{x}{1+x}$ show that $\log(1+x) = y + \frac{y^2}{2} + \frac{y^3}{3} + \frac{y^4}{4} + \ldots\ldots\ldots$

(ii) If $m = \log a$ express a^x in terms of "m" and "x" where $a > 0$. Hence prove $a^x = 1 + \frac{x \log a}{1!}$ $\frac{x^2(\log a)^2}{2!} + \frac{x^3(\log a)^3}{3!} + \ldots\ldots$

(iii) Test for convergence the series (a) $\sum \frac{e^n}{3^n + 1}$; (b) $\frac{1}{2}x + \frac{1.3}{2.4}x^2 + \frac{1.3.5}{2.4.6}x^3 + \frac{1.3.5.7}{2.4.6.8}x^4 + \ldots$

5. Integrate the following (i) $\int \frac{x\,dx}{(x^2+1)(x-2)}$ (ii) $\int x^2 \log x dx$ (iii) $\int \sqrt{a^2 - x^2}\, dx$.

(iv) $\int \frac{e^{2x}}{e^x - 1} dx$ (v) $\int x e^{-x} dx$

6. A square of side "2a" with a semi-circle on one side is rotated about the opposite side. Show that the volume of revolution generated is $V = \frac{2}{3}\pi\ (26 + 3\pi)\, a^3$.

7. (i) $\int_{-1}^{0} \frac{dx}{x^2 + 2x + 2}$ (ii) $\int_0^{\pi/2} \frac{dx}{5 + 4\cos x}$ (iii) $\int_0^1 \log x\, dx$

(iv) $\int_0^1 2^x dx$ (v) $\int_0^{\pi/2} e^x \sin x\, dx$.

8. (i) Prove that $\lim_{n \to \infty} \sum_{r=1}^{n-1} \frac{1}{\sqrt{n^2 - r^2}} = \pi/2$

(ii) Prove $\int_0^t f(x)\, dx = \int_0^t f(t - x)\, dx$. Use this principle to evaluate $\int_0^t x^2 (t - x)^{3/2} dx$.

9. (i) Show that $\frac{1}{n+1} + \frac{1}{n+2} + \frac{1}{n+3} + \ldots\ldots + \frac{1}{2n}$ converges to $\log 2$ as $n \to \infty$

(ii) Evaluate $\lim_{x \to 1} \frac{x \log x + x - 1}{(x-1)\log x}$.

10. Prove $\int x^{n-1} (\log x)^m dx = \frac{y^m e^{ny}}{n} - \frac{m}{n}\int y^{m-1} e^{ny} dy$ where $x = e^y$. Hence integrate $\int x^2 (\log x)^3 dx$.

FORT ST. BOYS' HIGH SCHOOL

1958.

MATHEMATICS I. HONOURS PAPER

1. (a) Prove that if x, y, z are not all zero, then the result obtained by eliminating x, y, z from the equations

$$x = cy + bz$$
$$y = az + cx$$
$$z = bx + ay$$

is $a^2 + b^2 + c^2 + 2abc = 1$, and hence that $\frac{x^2}{1-a^2} = \frac{y^2}{1-b^2} = \frac{z^2}{1-c^2}$

(b) Divide $x^3 - 2 - 2i$ by $x + 1 - i$ and hence prove that the 3 cube roots of 2 + 2i are

$i - 1$ and $\frac{1}{2}\left\{1 - i \pm (\sqrt{3} + i\sqrt{3})\right\}$

(c) Prove that $\begin{vmatrix} 1 & 1 & 1 \\ x & y & z \\ x^3 & y^3 & z^3 \end{vmatrix} = (x + y + z)(x - y)(y - z)(z - x)$

2. (i) Two persons A and B throw a die with the understanding that the one who first throws an ace is to receive a prize of £1. Prove that A's expectation of winning to B's is as $1 : \frac{5}{6}$, if A throws first.

(ii) If 'a' is small, show that the equation cos x = ax has 2 roots in the range $0 \leq x \leq 2\pi$, and that a closer approximation to the root near $\pi/2$ is $\pi/2(1 - a)$. Find to the same order of 'a', an approximation to the second root

(iii) Two similar triangles have corresponding vertices in the order given, $(\alpha_1, \alpha_2, \alpha_3)$ and $(\beta_1, \beta_2, \beta_3)$ (and in the same sense) in the Argand Diagram.

Prove that $\begin{vmatrix} 1 & 1 & 1 \\ \alpha_1 & \alpha_2 & \alpha_3 \\ \beta_1 & \beta_2 & \beta_3 \end{vmatrix} = 0$

3. If $y = \frac{\log x}{x}$ find $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$. Find the limits as $x \to \infty$ of y, $\frac{dy}{dx}$. Draw a graph of the function $y = \frac{\log x}{x}$ and mark on it the maximum and the point of inflection. If the part of this curve for $x > 1$ be rotated about the x-axis, show that the volume of the solid of revolution which is generated is $\pi\int_1^\infty \left(\frac{\log x}{x}\right)^2 dx$. Evaluate this result by writing $x = e^u$ or otherwise.

4. Find the indefinite integrals (i) $\int \frac{x^2}{\sqrt{1+x}}\,dx$; (ii) $\int \tan^5\theta\, d\theta$;

(iii) $\int \frac{(3x+1)\,dx}{x^2 + 2x + 5}$ (iv) $\int x^2 \cdot 3^x\, dx$

5.(i) Write down the series for $\log_e(1 + x)$ and $\log_e(1 - x)$ and state carefully for what values of x, each series is valid.

Prove that $\frac{1}{3}\left(1 - \frac{1}{2^3}\right) - \frac{1}{4}\left(1 + \frac{1}{2^4}\right) + \frac{1}{5}\left(1 - \frac{1}{2^5}\right) - \ldots = \frac{1}{8}$

(ii) Find the range of values of x for which the series $\sum_{n=1}^{\infty} (n + 1)(2n + 1)\frac{x^n}{n}$ is convergent. Find also the sum to infinity of this series when x lies within this range.

6. Find the definite integrals (i) $\int_0^\pi \frac{d\theta}{2 + \cos\theta}$; (ii) $\int_1^3 \left\{(x - 1)(3 - x)\right\}^{-\frac{1}{2}} dx.$

(iii) $\int_1^{\infty} \frac{dx}{x^2(1+x)}$ (iv) $\int_0^1 x \sin^{-1}x \, dx$

7. (i) Considering the definition of $\log x = \int_1^x \frac{du}{u}$ $(x > 0)$ prove that $1 + \frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{n-1} > \log n > \frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{n}$. Hence show that if $\gamma_n = 1 + \frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{n-1} - \log n$ then $1 > \gamma_{n+1} > \gamma_n > 0$. Explain how this inequality implies the convergence of the series γ_n as $n \to \infty$.

(ii) The acceleration through water of a ship is k times the square of the speed of the ship. Write down an equation connecting the speed (v) with the distance (x) run after the engines are stopped. If the speed was u when the engines were stopped, and is v when x = a, find k. What is the sign of k?

8. (i) A sphere is cut by 2 parallel planes distance h feet apart in circles of radii a and b. Prove that the volume of the sphere included between the planes is $\frac{\pi h}{6}\left\{h^2 + 3(a^2 + b^2)\right\}$.

(ii) If $x \equiv c \sin(4t - \alpha)$ and $4\frac{d^2x}{dt^2} + 9\frac{dx}{dt} + 100x = 36 \sin 4t$ $(0 < \alpha < \frac{\pi}{2})$ find the values of the constants c and α.

9. By integration by parts, if $I_n = \int_0^1 x^n e^{x^2} dx$ prove that $2I_n = e - (n-1) I_{n-2}$. Hence evaluate

$$\int_0^1 x^5 e^{x^2} dx.$$

10. By considering the graph of $x = \tan\theta$, and defining the inverse function $y = \tan^{-1}x$ for values between $-\pi/2$ and $\pi/2$, draw a rough graph of the inverse function. Find its derivative. Show that the functions

(i) $\tan^{-1}\left\{\frac{1}{2}(e^x - e^{-x})\right\}$ (ii) $2\tan^{-1}(e^x) - \frac{1}{2}\pi$ (iii) $2\tan^{-1}\left\{\frac{e^{x/2} - e^{-x/2}}{e^{x/2} + e^{-x/2}}\right\}$

all have the same derivative and deduce that they are equal for all values of x.

EXTERNAL MATHEMATICS I HONOURS 1958 LEAVING.

1. P is the point on the Argand diagram for which OP = a. Angle XOP = α. A circle is drawn on OP as diameter, and on it Q and R are the points such that angle POQ = angle QOR = θ. The points P, Q, R, represent the complex numbers z_1, z_2, z_3. Express each of the numbers z_i in the form $z = r(\cos\alpha + i\sin\alpha)$.
Prove that $z_1 z_3 \cos^2\theta = z_2^2 \cos 2\theta$.

2. Prove that the determinant $\Delta = \begin{vmatrix} a & b & c \\ c & a & b \\ b & c & a \end{vmatrix}$ vanishes when $a + wb + w^2c = 0$. Where $w^3 = 1$.
Thence express Δ as a product of three linear functions of a, b, c.

Prove also that $\begin{vmatrix} b & c & 1 \\ a & b & w \\ c & a & w^2 \end{vmatrix}$ is a product of two of these factors.

Given $\Delta \neq 0$, solve the equations

$$ax + by + cz = 1$$
$$cx + ay + bz = w$$
$$bx + cy + az = w^2$$

3. (i) If a, b are real and a = c + d, b = c - d, prove that cd is less than zero is a necessary and sufficient condition for $|a| < |b|$

(ii) a, b, c, d are real numbers. Prove that .. (a) $a^4 + b^4 \geq 2a^2b^2$ (b) $a^4+b^4+c^4+d^4 \geq 4abcd$
(c) If $a^4 + b^4 + c^4 + d^4 \leq 4$, then $a^{-4} + b^{-4} + c^{-4} + d^{-4} \geq 4$,
(d) If $a^4 + b^4 + c^4 + d^4 > 4$ what statement can be made about $a^{-4} + b^{-4} + c^{-4} + d^{-4}$?

4. (i) What is the number, $\Phi(n, r)$, of different sets of r pairs that can be formed from n different objects?

(ii) Either using this answer, or independently, prove that, for $r \geq 1$,

$\Phi(n, r) = \Phi(n-1, r) + (n-1)\,\Phi(n-2, r-1)$ taking $\Phi(n, 0) = 1$).

(iii) using this formula, or otherwise, complete the following table of values of $\Phi(n, r)$:-

	r	0	1	2	3
n = 2		1	1	0	0
3		1	3		
4		1			
5		1			
6		1			
7		1			

5. Write down the series for $\log_e(1 + x)$ and state its range of convergence. Find a, b, c, d such that $(1 + ax + bx^2)\log_e(1 + x) = x + cx^2 + dx^5 + \ldots$. Show that the coefficient of x^{n+1} in the series on the right hand side is $\frac{(-1)^n (n-2)(n-3)}{6n(n^2-1)}$. Using these results show that $\log_e(\frac{3}{2}) = \frac{15}{37}$ with an error less than 2×10^{-4}.

6. $u_1, u_2, \ldots\ldots, u_n, \ldots$ is a sequence of positive numbers (i) State precisely what is meant by the statements: (a) $u_n \to 0$ as $n \to \infty$
(b) $\sum u_n$ converges, with sum 1.

(ii) Prove that if $u_n \to 0$ as $n \to \infty$ then $\frac{1}{n}(u_1 + u_2 + \ldots u_n) \to 0$ as $n \to \infty$

(iii) Show that if $u_n \to 0$ as $n \to \infty$ and $u_n > u_{n+1}$ for all n, then the series $u_1 - \frac{1}{2}(u_1+u_2) + \frac{1}{3}(u_1 + u_2 + u_3) - \ldots\ldots$ is convergent

7. Prove that the curves $6y = x^3 + 3x^2 - 9x - 27$, and $3y = x^3 - 3x^2 + 9x - 27$, have one common point, that they have a common tangent at the point, and that they cross one another there.

Sketch the parts of the curves within the rectangle bounded by x = 3.5, x = -3.5, y = -10, and y = 1. Find the equation of the common tangent and the coordinates of all the points in which it meets each curve.

8. Evaluate the definite integrals (i) $\int_{-1}^{1} (1-x^2)\cos\frac{\pi x}{2}\,dx$ (ii) $\int_0^{\frac{1}{2}\pi} \frac{\cos x - \sin x}{1 + \sin x \cos x}\,dx$

(iii) $\int_0^{\infty} x e^{-\frac{1}{2}x^2}\,dx$

9. (i) Find the area enclosed by the x-axis, the ordinates x=2 x=3 and the arc of the curve

$y = \frac{3}{(x-1)(4-x)}$

(ii) Find the volume of the solid of revolution obtained by rotating this area bout the x - axis.

10. (i)

If $U_m = \int_0^x \frac{1 - (-1)^m t^{2m}}{1 + t^2}\,dt$ show that $U_m - U_{m-1} = (-1)^{m-1}\frac{x^{2m-1}}{2m-1}$; Deduce a power series for $\tan^{-1}x$ (valid for $-1 < x < 1$)

(ii) Find the indefinite integral, for $a > 0$, $\int \frac{\sqrt{a + x}}{\sqrt{a - x}}\,dx$; use this result to find $\int \frac{\sqrt{b + u}}{\sqrt{c - u}}\,du$ stating any assumptions you have made about b and c.

HOMEBUSH BOYS' HIGH

TRIAL LEAVING CERTIFICATE AUGUST 1959

MATHEMATICS I HONOURS

1. (a) Solve for x and y : $\begin{vmatrix} 1 & i & 0 \\ x & y & 2 \\ y & -x & 1 \end{vmatrix} = 4 + 3i$

(b) Factorise: $\begin{vmatrix} x - a & 1 & 1 \\ (x - a)^2 & x & x^2 \\ x^2 - a^2 & x + a & (x + a)^2 \end{vmatrix}$

(c) Two complex numbers $z = (x + iy)$ and $w = (u + iv)$ are connected by the relation $w = \frac{2 + z}{2 - z}$ Prove that when $x = 0$, the locus of w is a circle and find its radius.

2. Differentiate : i. $y = x^2 \cos 3x$.

ii. $y = \sin^{-1}\left[\frac{1}{\sqrt{(1 + x^2)}}\right]$

(b) Prove that when $y = e^{ax} \sin bx$, $\frac{d^2y}{dx^2} - 2a\frac{dy}{dx} + (a^2 + b^2)y = 0$. Show that if

$e^{ax} \sin bx = \sum_{n=1}^{\infty} \frac{c_n}{n!} x^n$, then $c_{n+2} - 2ac_{n+1} + (a^2 + b^2)\,c_n = 0$, and find the values of

c_1, c_2, and c_3.

3. Show that when $-1 < x < 1$, $\frac{1}{2}\log\frac{1 + x}{1 - x} = x + \frac{1}{3}x^3 + \frac{1}{5}x^5 + \ldots$ and deduce that when $u > 1$ $\log\sqrt{u} = \frac{u - 1}{u + 1} + \frac{1}{3}\left(\frac{u - 1}{u + 1}\right)^3 + \frac{1}{5}\left(\frac{u - 1}{u + 1}\right)^5 + \ldots$ Hence prove that

$\log_3^{\frac{1}{}} \sqrt{11} = 0.100335$ approximately

4. (a) Prove that $\phi(t) = \frac{t}{e^t - 1} + \frac{t}{2}$ is an even function of t, and show that

$\phi(2t) = \phi(t) + \frac{t^2}{4\phi(t)}$ (b) Show that $(e^x - 1)^2 = \sum_{n=2}^{\infty} \frac{(2^n - 2)\,x^n}{n!}$

5. (a) Determine whether the series $\frac{\sqrt{3}}{1^2} + \frac{\sqrt{5}}{2^2} + \frac{\sqrt{7}}{3^2} + \ldots$ converges or diverges.

5. (b) Investigate the convergence of the series $x + \frac{x^2}{3} + \frac{x^3}{5} + \frac{x^4}{7} + \dots$ and state

(i) the range of convergence and (ii) the range of absolute convergence.

(c) Find partial fractions for the 'n'th term of the series $\frac{4}{1.2} - \frac{7}{2.3} + \frac{10}{3.4} - \frac{13}{4.5} + \dots$ and find the sum of an infinite number of terms.

6. (a) Evaluate the following integrals:

(i) $\int_0^{\infty} x^2 e^{-x} dx.$ (ii) $\int_0^{1/\sqrt{3}} \frac{dx}{\sqrt{2 - 3x^2}}$ (iii) $\int_{-\frac{1}{2}\pi}^{\frac{1}{2}\pi} \sin^4 x \cos x \, dx$ (iv) $\int_0^2 \frac{x \, dx}{(x+1)(x^2+4)}$

(b) Evaluate the limit: $\lim_{x \to 1} \left(\frac{x}{x-1} - \frac{1}{\log x} \right)$

7. Explain how an approximation to a definite integral may be obtained by the formation of a suitable finite sum.

By considering $\int_0^1 \frac{dx}{1+x^2}$ show that $\lim_{n \to \infty} \sum_{r=0}^{n-1} \frac{1/n}{1 + (r/n)^2} = \frac{1}{4}\pi$

8. (a) Find the indefinite integrals: i. $\int \sin x \sqrt{1 + \cos x} \, dx$ ii. $\int 2^x x \, dx$

(b) If $u_n = \int_0^{\infty} \frac{dx}{(1+x^2)^n}$ where $n > \frac{3}{2}$, show that $(2n-2) u_n = (2n-3) u_{n-1}$

Hence, or otherwise calculate $\int_0^{\infty} \frac{dx}{(1+x^2)^{9/2}}$

9. (a) Prove that $\int_0^a f(x) \, dx = \int_0^a f(a-x) \, dx.$ Hence prove that $\int_0^{\pi} x \sin x \, dx = \frac{\pi}{2} \int_0^{\pi} \sin x \, dx,$

and evaluate the integral.

(b) Evaluate $\int_0^{\frac{1}{2}\pi} x^2 \cos x \, dx.$

The portion of the curve $y = \sin x$ from $x = 0$ to $x = \frac{1}{2}\pi$ revolves around the axis of y. Prove that the volume described by the area between the curve, the y - axis and the line $y = 1$ is $\frac{1}{4}\pi(\pi^2 - 8)$

10. Prove that the curve $y = \frac{x}{1+x^2}$ ..... (i) has a maximum at $(1, \frac{1}{2})$ and a minimum at $(-1, -\frac{1}{2})$.

The curve $y = ax^3 + bx^2 + cx + d$ .... (ii) has its maximum and minimum values at the same points as those of curve (i). Find a, b, c, and d.

Show that (i) has a point of inflection at the origin, and find the difference between the gradient of (i) at this point of inflection and the gradient of (ii) at its point of inflection.

Give a sketch of curve (ii).

FORT STREET BOYS' HIGH SCHOOL

1959

MATHEMATICS I. HONOURS

1. (a) If $1 + i$ is a root of $x^3 - x^2 + 2 = 0$, find the other two roots in the form $A + iB$.

(b) A point z moves so as to satisfy the equations

(i) $\left| \frac{z-1}{z+1} \right| = \frac{1}{2}$ (ii) $\text{amp}\left(\frac{z-i}{z+i} \right) = \frac{\pi}{2}$. Find the locus of z in each case on the Argand diagram.

2. (a) If the equations $ax + by = 1$ and $cx^2 + dy^2 = 1$ have only one solution prove that

   (i) $\frac{a^2}{c} + \frac{b^2}{d} = 1$ (ii) $x = \frac{a}{c}$, $y = \frac{b}{d}$.

   (b) If α, β, γ are the roots of the equation $x^3 + px^2 + qx + r = 0$

   express $\begin{vmatrix} 1 & \alpha\beta & \gamma\alpha \\ \alpha\beta & 1 & \beta\gamma \\ \gamma\alpha & \beta\gamma & 1 \end{vmatrix}$ in terms of p, q, r.

3. (a) Differentiate x^{x^x}.

   (b) If $y = m \cos(\log x) + n \sin(\log x)$ prove that $x^2 \frac{d^2y}{dx^2} + x \frac{dy}{dx} + y = 0$.

   (c) If $f(x) = \int_x^{10} 2^{x^3}\, dx$, evaluate $f'(2)$.

4. (a) Factorise $\begin{vmatrix} 1 & 1 & 1 \\ bc & ca & ab \\ b^2c^2 & c^2a^2 & a^2b^2 \end{vmatrix}$

   (b) Find whether the series $1 - \frac{1}{2} + \frac{1.3}{2.4} - \frac{1.3.5}{2.4.6} + ..$ is convergent or divergent.

5. (a) If $y = \frac{x}{1!} - \frac{x^2}{2!} + \frac{x^3}{3!} - \dots$ show that $x = y + \frac{y^2}{2} + \frac{y^3}{3} + \dots$ y being numerically less than unity.

   (b) (i) Using ω, an imaginary cube root of unity express $a^3 + b^3 + c^3 - 3abc$ as the product of three factors,

   (ii) If $a = 1 + \frac{x^3}{3!} + \frac{x^6}{6!} + \dots$, $b = x + \frac{x^4}{4!} + \frac{x^7}{7!} + \dots$ and $c = \frac{x^2}{2!} + \frac{x^5}{5!} + \frac{x^8}{8!} + \dots$,

   evaluate $a^3 + b^3 + c^3 - 3abc$

6. For the curve $y = e^{-x^2}(x + x^3)$ find $\frac{dy}{dx}$, $\frac{d^2y}{dx^2}$.

Find the maximum value of y for $x > 0$ and find the co-ordinates of the point of inflexion on the curve. Sketch the curve for $x > 0$ and find the area between this part of the curve and the x-axis.

7. Evaluate (a) $\int_{-3}^{3} \sin x\, e^{-x^2}\, dx$ (b) $\int_0^{\infty} \frac{dx}{a^2e^x + b^2e^{-x}}$ (c) $\int_0^1 \sin(\log x)\, dx$.

8. (a) (i) Explain briefly why a function f (x) is continuous at x = a if $\lim_{\gamma \to 0} f(a - \gamma) = f(a) = \lim_{\gamma \to 0} f(a + \gamma)$.

   (ii) The function f (x) is defined as follows: $f(x) = -x^2$ when $x \le 0$
   $f(x) = 5x - 4$ when $0 < x \le 1$
   $f(x) = 4x^2 - 3x$ when $1 < x < 2$.

Show that f (x) is not continuous at x = 0 but is continuous at x = 1.

   (b) If $f(p) = \int_0^1 x^{m-1}(1 - x^n)^p\, dx$ prove $(m+np)f(p) = npf(p-1)$.

9. (a) Any two particles of masses m_1, m_2 at a distance r apart experience equal forces of attraction of magnitude $\frac{\gamma m_1 m_2}{r^2}$ where γ is a constant.

9. (a) contd.

A particle of mass M is situated at a distance b from one end of a homogeneous rod of mass m and length L in the same line as the rod and outside it.

Find the force of attraction on the particle due to the rod.

(b) If $U_m = \int_0^\infty x^m e^{-ax}\,dx$ where m and a are both positive prove $U_m = \frac{m}{a} U_{m-1}$.

Hence if m is a positive integer evaluate $\int_0^\infty x^m e^{-ax}\,dx$.

10. (a) Prove $\frac{x}{1+x} < \log(1+x) < x$ and hence show that $\frac{\pi}{8} - \frac{1}{4}\log 2 < \int_0^1 \frac{\log(1+x)dx}{1+x^2} < \frac{1}{2}\log 2$

(b) By using the two transformations $x = \tan\beta$ and $x = \tan\left(\frac{\pi}{4} - \gamma\right)$ and equating the results obtained prove $\int_0^1 \frac{\log(1+x)}{1+x^2}\,dx = \frac{\pi}{8}\log 2.$

EXTERNAL LEAVING CERTIFICATE

1959

MATHEMATICS I HONOURS

1. Find formulae for (i) the sum and (ii) the sum of the squares of the numbers 1 to n.

If $s_1(n,a) = \sum_{r=0}^{n-1} (a+r)$, $s_2(n,a) = \sum_{r=0}^{n-1} (a+r)^2$ prove that

$$ns_2(n,a) - \{s_1(n,a)\}^2 = {}^1/_3 s_1(n-1,1)\, s_1(n,1).$$

2. Prove that a single solution for x, y, z of the three equations

$$ax + 2y - 6z = 1$$
$$-2x + ay + 3z = m$$
$$6x - 3y + az = n$$

can be found, whatever the values of l, m, n, provided a does not take any one of three particular values (possibly complex).

Prove also that if a takes one of these values, then the equations can be solved if

$$(a^2+9)\,l - 2(a-9)m + 6(a+1)n = 0.$$

Given that $l = 0$, $m = 3$, $n = (a-9)/(a+1)$, and that in addition to the relations above x, y, z satisfy $x - 2y - 6z = 0$, solve the equations for each of these values of a.

3. Find (as surds) the values of x at which occur the stationary points and inflexions of the curve $y = \frac{5(x-1)}{x^2+1}$. Sketch the curve for the range $-5 \leq x \leq 5$, marking in approximately the positions of the stationary points and inflexions.

4. Find the indefinite integrals - (i) $\int \frac{du}{\sqrt{u}-1}$; (ii) $\int r^m \log r\,dr$, $m \neq -1$;

(iii) $\int \frac{dx}{\sqrt{(5-4x-4x^2)}}$; (iv) $\int \frac{1-x^2}{1+x^2+x^4}\,dx$

5. (i) Find the definite integral $\int_0^1 e^{\lambda t} \sin\pi t\;dt$.

(ii) Find the indefinite integral $\int \sec^3\theta\;d\theta$, and verify that the value of the definite integral

5. (ii) contd. $\int_0^{\frac{1}{4}\pi} \sec^3\theta \, d\theta$ is $\frac{1}{2}\left\{ \sqrt{2} + \log(\sqrt{2} + 1)\right\}$.

6. Given that x and a are positive and that $x^{r(1+\delta)} = a^r$ where δ is small, show that

$x = a \left\{ 1 - \lambda\delta + (\lambda + \frac{1}{2}\lambda^2)\delta^2 - \dots \right\}$ where $\lambda = \log_e a$. Find the term in δ^3 in the expansion above

Find the solution of the equation $x^{10.1} = e^{10}$ as accurately as you can, given the approximate value 2.71828 of e.

7. Define the derived function $\frac{dy}{dx}$ of a function y 0f x.

If x and y are given as functions of a parameter t, prove from your definition that, if

$\dot{x} \neq 0$, $\frac{dy}{dx} = \frac{\dot{y}}{\dot{x}}$, where $\dot{y} = dy/dt$, $\dot{x} = dx/dt$. Find the equations of the tangent and the normal to the curve $x = x(t)$, $y = y(t)$ at the point of parameter t_o.

The tangent and normal at the point t_o to the curve $x = ct^2$, $y = ct^{-1}$ meet the x axis in P and Q. Find an expression in t_o for the length of PQ, and prove that this length is shortest when the normal to the curve at t_o passes through the origin.

8. For each of the following series state whether it converges or diverges; prove your statements carefully.

(i) $1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \dots$ (ii) $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \dots$

(iii) $(1 + \frac{1}{2}) - (\frac{1}{3} + \frac{1}{4}) + (\frac{1}{5} + \frac{1}{6}) - \dots$

(two terms in each bracket; alternate + and - signs between the brackets).

(iv) $1 - (\frac{1}{2} + \frac{1}{3}) + (\frac{1}{4} + \frac{1}{5} + \frac{1}{6}) - (\frac{1}{7} + \frac{1}{8} + \frac{1}{9} + \frac{1}{10}) + \dots$

(successivly 1, 2, 3, 4, . . . terms in the brackets; alternate signs between the brackets).

9. A function $y = f(x)$, for which $f(a) < 0$ and $f(b) > 0$, satisfies the following conditions in the range $a \leq x \leq b$; whenever $x_2 > x_1$ then $f(x_2) > f(x_1)$ and $f'(x_2) > f'(x_1)$. Sketch a graph which represents such a function.

Prove that under these conditions: if $f(p) > 0$ and $q = p - f(p)/f'(p)$, then $0 < f(q) < f(p)$.

Show that the function $x^3 - 18x + 11$ satisfies the conditions in the range $3 \leq x \leq 4$, and find a better approximation than 3.9 to the largest root of the equation $x^3 - 18x + 11 = 0$.

10. (i) Transform the indefinite integral $\int \frac{d\theta}{(1 + k\cos\theta)^r}$ where $0 < k < 1$, into an integral with respect to u by means of the substitution $\tan\frac{1}{2}\theta = \sqrt{\frac{1+k}{1-k}} \tan\frac{1}{2}u$.

(ii) Find the area of the region bounded by the curve $y = \frac{1}{5 + 4\cos x}$, the ordinates $x = 0$ and $x = \pi$, and the x-axis.

(iii) Find the volume of the solid of revolution formed by rotating this region about the x-axis.

HOMEBUSH BOYS' HIGH SCHOOL

TRIAL LEAVING EXAMINATION 1960

MATHEMATICS I HONOURS

1. (i) $z = i(1 - i\sqrt{3})(\sqrt{3} + i)$. Express z in the form a + ib and $r(\cos\theta + i\sin\theta)$. Hence find $z^2 + z^{-2}$ in a + ib form.

Write down the two square roots of z in the $r(\cos\theta + i\sin\theta)$ form and illustrate them on an Argand Diag.

EXAMINATION PAPERS

1. (ii) Solve $\begin{vmatrix} x & i & 2 \\ 1-i & 2i & 3 \\ ix & 1-1 & i+1 \end{vmatrix} = 8 + 2i$ for x in the form x = a + ib.

(iii) Show that the equations

$$(1+w)^2x + wy + wz = 0$$
$$x + (w+w^2)^2y + z = 0$$
$$w^2x + w^2y + (w^2+1)^2z = 0$$

have non-zero roots for x, y and z. (w is a cube root of unity).

2. (i) Show that $\frac{1}{1.2} - \frac{1}{2.3} + \frac{1}{3.4} - \frac{1}{4.5} + \dots\dots\dots\dots = \log_e\left(\frac{4}{e}\right)$

(ii) If $a = x - \frac{1}{2}x^2 + \frac{1}{3}x^3 - \frac{1}{4}x^4 + \dots$ show that $x = a + \frac{a^2}{2!} + \frac{a^3}{3!} + \frac{a^4}{4!} + \dots$, if $-1 < x \leq 1$

(iii) If m and n are the roots of $x^2 + px + q = 0$ show that $\log_e(1-px + qx^2) = (m+n)x - \frac{1}{2}(m^2+n^2)x^2 + \frac{1}{3}(m^3 + n^3)x^3 - \frac{1}{4}(m^4+n^4)x^4 + \dots$, if $-1/m < x \leq 1/m$ and $-1/n < x \leq 1/n$

3. Write the nth term of each of the following series, then test each for convergence.

(a) $\frac{1}{2} + \frac{1}{6} + \frac{1}{12} + \frac{1}{20} + \frac{1}{30} + \frac{1}{42} + \dots$ (b) $\frac{1}{3} + \frac{2x}{2.3^2} + \frac{3x^2}{2^2.3^3} + \frac{4x^3}{2^3.3^4} + \dots$

(c) $(1 + \frac{1}{2}) - (\frac{1}{3} + \frac{1}{4}) + (\frac{1}{5} + \frac{1}{6}) - (\frac{1}{7} + \frac{1}{8}) + (\frac{1}{9} + \frac{1}{10}) + \dots$

Two terms in each bracket with alternate + and - signs between brackets.

(d) $(x-1) + \frac{(x-1)^2}{\sqrt{2}} + \frac{(x-1)^3}{\sqrt{3}} + \frac{(x-1)^4}{\sqrt{4}} + \dots\dots$

4. (i) Test for maximum and minimum values the function $y^2 = x^2\left(\frac{4+x}{4-x}\right)$. Hence draw the curve of the function

(ii) The above curve is rotated about the x-axis from x = -4 to x = 0. Show that the volume of revolution formed is $128\pi\left(\log 2 - \frac{2}{3}\right)$

5. Find each of the following indefinite integrals:

(i) $\int x^n \log x \, dx$; (ii) $\int \frac{dx}{x^4 - x^3}$; (iii) $\int \sin 2x \cos 3x \, dx$; (iv) $\int \frac{dx}{\sqrt{2x - x^2}}$; (v) $\int \frac{1-\cos 2x}{1+\cos 2x} dx$

6. (i) Show that $\int_0^\pi f(x)dx = \int_0^\pi f(\pi - x)dx$. Use this to evaluate $\int_0^\pi \frac{e^x - e^{(\pi - x)}}{x^2 + (\pi - x)^2} dx$

(ii) Evaluate $\int_{-1}^{+1} \sqrt{\frac{1-x}{1+x}} dx$ (iii) Evaluate $\int_0^\infty e^{-x} \cos(x + \frac{\pi}{4}) dx$

(iv) Evaluate $\int_0^{\frac{\pi}{2}} (\sec\theta - \tan\theta) d\theta$ Put $t = \tan\frac{\theta}{2}$

7. (i) A variable line passes through the fixed point (a, b) and meets the x- and y-axes in P and Q respectively. If O is the origin, find an expression for OP + OQ. Show that the minimum value of OP + OQ is $a + 2\sqrt{ab} + b$.

(ii) Taking x = 0.5 radians as a first approximation, solve x + sin x = 1 correct to 2 decimal places.

8. (i) Show by graphical consideration that if f(x) steadily diminishes as x increases from zero to infinity, the series f(1) + f(2) + f(3) + f(4) + ... + f(x) + ... is convergent, and its sum is between I and I + f(1), provided

$$I = \int_1^\infty f(x)\,dx \text{ is finite.}$$

8. (i) Contd. Show that if a is large $\frac{1}{a^2+1^2} + \frac{1}{a^2+2^2} + \frac{1}{a^2+3^2} + \dots + \frac{1}{a^2+x^2} + \dots$ is approximately $\frac{\pi}{2a}$.

(ii) What is the probability of drawing two red marbles in succession out of a bag containing 3 red and 2 white marbles. What are the odds against this happening?

9. (i) If $p = \frac{e^x + e^{-x}}{2}$ and $q = \frac{e^x - e^{-x}}{2}$ (a) Prove $x = \frac{1}{2}\log_e \frac{p+q}{p-q}$ (b) If $p = \sec\theta$ show that $\tan\theta = q$. (c) If $p = \sec\theta$ show that $x = \log_e \tan\left(\frac{\pi}{4} + \frac{\theta}{2}\right)$

(ii) The ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ has a variable chord PQ drawn parallel to the x-axis. O is the centre of the ellipse. What is the maximum area of the triangle OPQ.

10. (i) If $u_n = \int_0^{\frac{\pi}{2}} \theta \cos^n\theta \, d\theta$ show that $u_n = \frac{-1}{n^2} + \left(\frac{n-1}{n}\right)u_{n-2}$. Hence prove that $u_3 = .2694\dots$

(ii) A particle moves according to the law $\frac{ds}{dt} = v_0 e^{-kt}\cos nt$. Prove that the space described from time t = 0 until it first comes to rest is $\left[\frac{ne^{-\frac{k\pi}{2n}} + k}{n^2 + k^2}\right] v_0$.

FORT STREET BOYS' HIGH SCHOOL

MATHEMATICS I HONOURS

1960

1. The sides AB, BC, CD, DA of a deformable but plane quadrilateral are of fixed lengths a, b, c, d respectively. Show that its area is greatest when the shape is such that A, B, C, D are concyclic.

2. If n straight lines are drawn in a plane so that no two of them are parallel and no three of them pass through the same point, the number of regions into which the plane is divided is denoted by f(n). What are the values of f(1), f(2), f(3)?
Prove that $f(n+1) - f(n) = n + 1$ and deduce that $f(n) = \frac{1}{2}(n^2 + n + 2)$.

3. (i) Show that $\lim_{x\to 0} x\log x = 0$ (ii) Evaluate $\lim_{x\to\infty} \frac{x}{\log x}$

(iii) Evaluate $\lim_{n\to\infty} na^n$ for 'a' positive and < 1 (iv) Evaluate $\lim_{n\to\infty} \frac{b^n}{n^a}$ for $b > 1$ and $a > 0$

4. Find the stationary values and points of inflexion for the curve $y = x^2 e^{-x^2}$. (Use tables to give answers correct to 2 decimal places.) Sketch the curve.

5. (a) Show that the series $\frac{1}{1^p} + \frac{1}{2^p} + \frac{1}{3^p} + \dots + \frac{1}{n^p} +$ is convergent if $p > 1$.

(b) Prove $\sum \frac{n^4 + 5n^2 - 6}{n^6 + 11}$ is convergent

(c) Find the range of values of x for which the following series is convergent:

$1 + ax + \frac{a+1}{2}x^2 + \dots + \frac{a+n-1}{n}x^n + .$ Prove, also, that the series is divergent when x = 1 and that it is non-convergent when x = -1.

6. (a) Prove $\int_0^{\pi} \sin mx \sin nx \, dx = 0 \quad (m \neq n)$

(b) Find: (i) $\int \sec \theta \, d\theta$

(ii) $\int \frac{dx}{1 + x^2 + x^4}$

(iii) $\int \left(\frac{1}{x^2} - 1\right)^{\frac{1}{2}} dx$

7. In the Argand Diagram ABC is a triangle with vertices at the points $e^{i\alpha}$, $e^{i\beta}$, e^{iy} where α, β, y are real numbers.

Assuming $\cos\alpha = \frac{e^{i\alpha} + e^{-i\alpha}}{2}$ and $\sin\alpha = \frac{e^{i\alpha} - e^{-i\alpha}}{2i}$, prove that $e^{\frac{1}{2}i(\beta + y)}$ is a unit vector perpendicular to BC.

8. If n is a positive integer, show that $\begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ a^{n+2} & b^{n+2} & c^{n+2} \end{vmatrix}$

has the value (b - c)(c - a)(a - b) S where $S = \sum a^r b^s c^t$ summed over all values r, s, t satisfying $r + s + t = n$.

9. (i) If $y = \left\{x + \sqrt{1 + x^2}\right\}^k$, prove that $\frac{dy}{dx}\sqrt{1 + x^2} = ky$ and that $\frac{d^2y}{dx^2}(1 + x^2) + x\frac{dy}{dx} = k^2 y$.

(ii) If $u_n = \int_0^{\pi/2} \theta \cos^n\theta \, d\theta$, prove $u_n = -\frac{1}{n^2} + \frac{n-1}{n} u_{n-2}$

10. (i) If log (1 + u) where (1 + u) > 0 is defined as $\int_1^{1+u} \frac{dx}{x}$, prove $\frac{u}{1+u} < \log(1 + u) < u$.

Taking n to be positive and > 1, show that $(1 + \frac{1}{n})^n < e < (1 - \frac{1}{n})^{-n}$

(ii) Write down the $(r + 1)^{th}$ term in the expansion of $(1 + \frac{1}{n})^n = 1 + \frac{1}{1!} + \frac{1 - \frac{1}{n}}{2!} + \dots$ if n > 1 and show that $(1 + \frac{1}{n})^n < (1 + \frac{1}{n+1})^{n+1}$

EXTERNAL MATHEMATICS I HONOURS 1960

1. (i) A sequence $u_1, u_2, u_3 \dots$ is defined by the relations $u_1 = 1$, $u_2 = 5$, $u_n = 5u_{n-1} - 6u_{n-2}$, for n = 2, 3, ...

Prove carefully, using the method of induction or otherwise, that $u_n = 3^n - 2^n$

(ii) Prove that, for n a positive integer, $7^{2n} - 48n - 1$ is divisible by 2304.

2. Show that on the Argand diagram a general circle can be represented by $|z - a| = r$;

What is the locus represented by $\arg(z - k) = \alpha$?

State clearly which of the constants, a, k, r, α are real and which complex, and what restrictions there are on their ranges of values.

2. (contd) The two circles

$$|z - a| = r$$
$$|z - b| = s$$

are given. What geometrical relations have the circles (i) when $|a - b| = r + s$?
(ii) when $|a - b| = \sqrt{(r^2 + s^2)}$?

Find an inequality which is necessary and sufficient to ensure that the circle $|z - a| = r$ lies entirely inside the circle $|z - b| = s$.

3. Sketch the curve $x = a\cos^3 t$, $y = a\sin^2 t$ between $t = 0$ and $t = \frac{1}{2}\pi$

Find (i) the area enclosed by the curve and the two co-ordinate axes,
(ii) the volume of the solid generated by rotating this area about the x-axis.

4. Find the indefinite integrals (i) $\int \cos^{-1}x\,dx$, (ii) $\int \frac{dx}{x(1 + \log x)}$, (iii) $\int \frac{dx}{x(4 + 9x^2)}$.

5. Find the values of the definite integrals

(i) $\int_0^{\frac{1}{4}\pi} \tan^4 x\,dx$, (ii) $\int_1^e x\log x\,dx$, (iii) $\int_0^1 \frac{x^4}{\sqrt{(1 - x^2)}}\,dx$, (iv) $\int_{-\frac{3}{2}}^{\frac{3}{2}} \sqrt{\left(\frac{3 - 2x}{3 + 2x}\right)}dx$

6. Write out the first three terms and the general term of the series for $\log_e(1 - x)$, stating the range of convergence. Deduce that for two positive numbers p, q

$$\log p - \log q = 2\left\{\left(\frac{p - q}{p + q}\right) + \frac{1}{3}\left(\frac{p - q}{p + q}\right)^3 + \ldots\right\}.$$

If $p > q$, and $\log p - \log q = 2\frac{p - q}{p + q} + \delta$, prove that $\delta < \frac{(p - q)^3}{6pq(p + q)}$.

By taking $p = \frac{71}{70}$, $q = \frac{72}{71}$, or otherwise, find $\log_e 71$ to six decimal places, given that

$\log_e 2 = 0.693\ 147\ 2$, $\log_e 3 = 1.098\ 612\ 3$, $\log_e 5 = 1.609\ 437\ 9$, $\log_e 7 = 1.945\ 910\ 1$.

7. Find the stationary point and inflexion of the curve $y = a\left(\frac{x + a}{x}\right)^2$, and sketch the curve (assuming $a > 0$) over a sufficient range to make its shape clear.

Find the equations of the tangent and normal at the point of abscissa x_0.

Prove that two normals pass through the origin, namely those at the points whose abscissae are the roots of $x^2 - \sqrt[3]{2}.a(x + a) = 0$.

8. ABCD is a square of side a. Points H, K, L, M are taken on the sides AB, BC, CD, DA respectively such that AH = BK = CL = DM = x. $\angle BAK = \theta$.

AK meets HD and BL respectively in P and Q, and CM meets LB and DH respectively in R and S. Express the area, T, of the square PQRS
(i) as a function of x, and (ii) as a function of θ.

Find $\frac{1}{a}\int_0^a T\,dx$, and $\frac{4}{\pi}\int_0^{\frac{1}{4}\pi} T\,d\theta$

9. Given that $S_m = \frac{1}{(1 - a)(1 - a^2)} + \frac{a}{(1 - a^2)(1 - a^3)} + \ldots$

$+ \frac{a^r}{(1 - a^{r+1})(1 - a^{r+2})} + \ldots + \frac{a^m}{(1 - a^{m+1})(1 - a^{m+2})}$, prove that

9. contd.

$$S_m = \frac{1}{a(1-a)}\left\{\frac{1}{1-a} - \frac{1}{1-a^{m+2}}\right\}.$$ Prove that, if $0 < a < 1$,

$$\lim_{m \to \infty} S_m = \frac{1}{(1-a)^2}.$$ Find the value of $\lim_{m \to \infty} S_m$ when $a > 1$.

Discuss the convergence of the series $\sum_0^\infty \frac{a^m x^m}{(1-a^{m+1})(1-a^{m+2})}$, a being assumed positive and not equal to 1.

10. If, for $1 \leq x \leq n$, $f(x) \geq 0$ and $f''(x) < 0$, show, by comparing the area under the curve $y = f(x)$ between $x = 1$ and $x = n$ with the area of a region consisting of a suitably chosen sequence of trapezia, that

$$\int_1^n f(x)\,dx > \sum_{r=2}^{n-1} f(r) + \tfrac{1}{2}f(1) + \tfrac{1}{2}f(n).$$

Taking $f(x) = \log_e x$, deduce that for n a positive integer, $n! < n^{n+\frac{1}{2}}.e^{-n+1}$.

LEAVING CERTIFICATE 1961

1. Find the indefinite integrals—

(i) $\int \frac{x\,dx}{(1-x)(1+x^2)}$; (ii) $\int x\cos^2 x\,dx$; (iii) $\int \frac{dx}{(1-x)\sqrt{1+x}}$.

2. Calculate the definite integrals—

(i) $\int_0^{\pi/2} \frac{dx}{\cos^2 x + 2\sin^2 x}$; (ii) $\int_1^2 \frac{(x+1)dx}{\sqrt{-2+3x-x^2}}$.

3. Write down the power series expansion for $\log(1+x)$. For what values of x does this series converge?

Express the sums of the series

$$a = \frac{1}{10} + \frac{1}{2.10^2} + \frac{1}{3.10^3} + \ldots,$$

$$b = \frac{4}{100} + \frac{4^2}{2.100^2} + \frac{4^3}{3.100^3} + \ldots,$$

$$c = \frac{1}{80} - \frac{1}{2.80^2} + \frac{1}{3.80^3} - \ldots,$$

as logarithms, to base e, of rational numbers.

Deduce that $\log_e 10 = 23a - 6b + 10c$.

Calculate the numerical value of $\log_e 10$ to five decimal places.

4. A sequence $u_0, u_1, u_2, \ldots$ is derived from its first term u_0 by means of the relation

$$u_{n+1} - u_n = (1-k)(A - u_n),\ n = 0, 1, 2, \ldots$$

where $A \neq u_0$. Prove that $\frac{u_n - A}{u_0 - A} = k^n$.

(i) If the sequence contains terms less than A and also terms greater than A show that $k < 0$;

(ii) If u_n approaches a limit as $n \to \infty$ show that $-1 < k < 1$.

5. Find the coordinates of the stationary points and the inflexions of the curve $y = \frac{18x^2}{(x+1)^3}$.

Draw a sketch of this curve, on a scale and over a range, which shows its essential features. On your sketch mark the maximum and minimum points, the points of inflexion and the asymptotes.

6. (i) If $y = 0$ when $x = 0$ and if

$$(1 - x^2)\frac{dy}{dx} = 2,$$

find the value of x for which $y = 1$.

Taking $e = 2{\cdot}718$ calculate this value of x to three significant figures.

(ii) Find a definite integral which represents approximately the sum

$$S_n = \sum_{r=1}^{n} \frac{r\sqrt{(n^2 - r^2)}}{n^3}$$

when n is a large positive integer. Deduce that $\lim_{n \to \infty} S_n = \frac{1}{3}$.

7. Sketch the curve

$$y = e^{-x}\sin x$$

for $0 \leqq x \leqq 4\pi$.

Show that y has infinitely many maxima which lie on the curve

$$y = \frac{1}{\sqrt{2}}e^{-x}.$$

Show also that the successive areas enclosed between the curve and the x-axis form a geometric progression, and that the sum of the absolute values of all these areas on the positive side of the y-axis is

$$\frac{1}{2}\frac{e^{\pi} + 1}{e^{\pi} - 1}$$

8. (i) Find the limit

$$\lim_{n \to \infty} n\left\{1 - \left(1 + \frac{1}{n}\right)^{-p}\right\}.$$

(ii) If u_n, v_n are positive terms and if $\frac{u_n}{v_n}$ approaches a finite limit as $n \to \infty$, show that the series

$$u_1 + u_2 + u_3 + \ldots$$

converges, whenever the series

$$v_1 + v_2 + v_3 + \ldots$$

converges.

(iii) By taking

$$v_n = \frac{1}{n^p} - \frac{1}{(n+1)^p}$$

and making use of the results in (i) and (ii), show that the series

$$\frac{1}{1^{1+p}} + \frac{1}{2^{1+p}} + \frac{1}{3^{1+p}} + \ldots$$

converges when p is positive.

9. (i) The numbers z_1, z_2, z_3 are represented in the Argand diagram by the points Z_1, Z_2, Z_3. Explain carefully the geometric meaning of

$$\left|\frac{z_1 - z_3}{z_2 - z_3}\right| \text{ and } \arg\left(\frac{z_1 - z_3}{z_2 - z_3}\right)$$

in terms of distances and angles in the diagram.

(ii) The points which correspond to z_1, z_2, z_3 and w_1, w_2, w_3 in the Argand diagram form two similar and similarly situated triangles. Show that

$$\begin{vmatrix} z_1 & w_1 & 1 \\ z_2 & w_2 & 1 \\ z_3 & w_3 & 1 \end{vmatrix} = 0.$$

(iii) Show that the points which correspond to z_1, z_2, z_3 form an equilateral triangle if and only if

$$z_1^2 + z_2^2 + z_3^2 = z_2 z_3 + z_3 z_1 + z_1 z_2.$$

10. Through the vertices A, B, C of a given triangle are drawn lines B′C′, C′A′, A′B′ respectively to form an equilateral triangle A′B′C′ which circumscribes the triangle ABC. If $\angle ACB' = \theta$, $\angle ABC' = \varphi$ show that

$$B'C' = \frac{2}{\sqrt{3}}(b \sin\theta + c \sin\varphi).$$

Prove that if the triangle A′B′C′ has maximum area then

$$b\cos\theta - c\cos\varphi = 0.$$

Deduce that this maximum area is

$$2S + \frac{1}{2\sqrt{3}}(a^2 + b^2 + c^2)$$

where a, b, c are the sides and S is the area of the triangle ABC.

LEAVING CERTIFICATE EXAMINATION, 1962

1. (i) Show that $a + b + c$ is one factor of

$$\begin{vmatrix} a - b - c & 2a & 2a \\ 2b & b - c - a & 2b \\ 2c & 2c & c - a - b \end{vmatrix}$$

and find the value of the determinant in its simplest form.

(ii) Show that the equations

$$\begin{aligned} x + y + 2z &= 2 \\ 2x - y + 3z &= 2 \\ 5x - y + az &= 6, \end{aligned}$$

have a unique solution if a is not equal to 8. Find *all* the solutions if $a = 8$.

2. (i) Show that if m and n are positive integers and $m \neq n$,

$$\int_0^{\pi} \cos mx \cos nx \, dx = 0.$$

What is the value of this integral when $m = n$?

(ii) Differentiate $\sin^{n-1}\theta \cos\theta$ and express the result in terms of $\sin\theta$ only. Deduce that $\int_0^{\pi/2} \sin^n\theta d\theta = \frac{n-1}{n}\int_0^{\pi/2} \sin^{n-2}\theta d\theta$. Evaluate $\int_0^{\pi/2} \sin^6\theta d\theta$.

3. Find the indefinite integrals

$$\text{(i)} \int x^2 e^{-x} dx, \quad \text{(ii)} \int \frac{7 + x}{1 + x + x^2 + x^3}\, dx, \quad \text{(iii)} \int \frac{e^{-2x}}{e^{-x} + 1}\, dx.$$

4. (i) Sketch the graph of the function e^{kx}, where k is a positive constant.

Find the equation of the tangent to the curve $y = e^{kx}$ which passes through the origin. Deduce that the equation $e^{kx} = x$ has 0, 1 or 2 real roots according as k is greater than, equal to or less than $1/e$.

(ii) Show that if $|x| < 1$, $e^{-x} - \dfrac{1 - x}{(1 - x^2)^{\frac{1}{2}}(1 - x^3)^{\frac{1}{3}}} = ax^5 + bx^6 + \dots,$

where a and b are constants. Find the value of a.

5. (i) Show that the greatest value taken by the function $1 + 2x - e^x$ is 0·386, to three decimal places.

(ii) Prove that the volume, V, the area of curved surface, S, and the radius of the base, r, of a right circular cone are connected by the equation

$$9V^2 = r^2(S^2 - \pi^2 r^4).$$

Show that the maximum volume for a given area of curved surface S, is $\dfrac{2^{\frac{1}{2}}S^{\frac{3}{2}}}{\pi^{\frac{1}{2}}3^{\frac{7}{4}}}$.

6. Sketch, referred to the same axes, the curves

$$y = \frac{a^3}{x^2 + a^2}, \qquad y = \frac{ax^2}{x^2 + a^2}$$

where a is a positive constant. The finite area between these curves is rotated

(i) about the x-axis,
(ii) about the y-axis.

Find, in each case, the volume generated.

7. (i) If $z = x + iy$, and

$$w = \frac{z - 1}{z},$$

express w in the form $u + iv$ where u and v are real. Hence show that if $|z| = 1$, then $|w - 1| = 1$.

(ii) Give a geometrical description of the locus of the point in the Argand diagram representing the complex number z which satisfies the condition

$$|z + i| = |z + 3 + 4i|.$$

(iii) Find the greatest value of arg z, when

$$|z - i| = \tfrac{1}{2}.$$

(iv) Z_1, Z_2 are the points in the Argand diagram representing the complex numbers z_1, z_2, where

$$z_2 = \frac{1}{z_1 - 4} + 2.$$

What is the locus of Z_2 as Z_1 describes the circle centre 4 and radius 3?

8. For the curve given by the parametric equations

$$x = a(\theta - \sin\theta)$$
$$y = a(1 - \cos\theta),$$

where a is a positive constant, find the equation of the tangent at the point θ and show that the normal at this point has the equation

$$x \sin\frac{\theta}{2} + y \cos\frac{\theta}{2} = a\theta \sin\frac{\theta}{2}.$$

P is the point θ and Q is the point $\theta + \Delta\theta$ on the above curve; R is the point of intersection of the normals at P and Q. Show that as $\Delta\theta$ tends to 0 the limiting position of R is the point

$$\{a(\theta + \sin\theta), a(\cos\theta - 1)\}.$$

9. (i) By considering a definite integral as the limit of a sum, show that

$$\lim_{n \to \infty} \left(\frac{1}{n} + \frac{1}{n+1} \cdots + \frac{1}{2n}\right) = \int_0^1 \frac{dx}{1+x}$$

and give the value of this limit correct to three decimal places.

(ii) By means of a freehand sketch, show that if $f(x)$ is a positive function which is steadily *decreasing* as x increases, then

$$0 < \sum_{r=1}^{n} f(r) - \int_1^{n+1} f(x)dx < f(1).$$

Deduce that

$$0 < 1 + \frac{1}{2} + \ldots + \frac{1}{n} - \log(n+1) < 1.$$

10. (i) Write down the first four terms and a general term of the series for e^x.

Show that

$$\sum_1^\infty \frac{3n-1}{n!} = 3\sum_1^\infty \frac{1}{(n-1)!} - \sum_1^\infty \frac{1}{n!}$$

and deduce the sum of the infinite series on the left.

(ii) State the *comparison test* for the convergence of a series of positive terms. Discuss the convergence of the series having general term—

(*a*) $u_n = \frac{1}{n} \sin \frac{\pi}{\sqrt{n}}$

(*b*) $u_n = \frac{x^n}{n(1 + x^{2n})}$, for positive values of x.

ANSWERS

ANSWERS. SPECIMEN PAPER (page 302)

1. $v = t^2\cos t - 4t\sin t - 6\cos t$; $t = 2n\pi + \frac{\pi}{2}$ give minimum $n = 1, 2, 3, \ldots$

$t = 2n\pi - \frac{\pi}{2}$ give maximum $n = 1, 2, 3 \ldots$

2. (i) $-\operatorname{cosec} x - \sin x$, $2\log(\sqrt{x} + 1)$ (ii) $\log_e \frac{21}{\sqrt{5}} = 2.24$ 5. (ii) (a) $\frac{1}{n}, \frac{n-r-1}{n(n-1)}$

6. (i) 3 factors are $(a + bw + cw^2)(a + b + c)(a + bw^2 + cw)$ 7. (ii) 2.004

8. Min $(-1, -1)$, max $(1, 1)$, inflex $(0, 0)$, $\left(-\sqrt{3}, \frac{-\sqrt{3}}{2}\right)$, $\left(\sqrt{3}, \frac{\sqrt{3}}{2}\right)$ equations of inflexional tangents are $2x - y = 0$, $x + 4y = 3\sqrt{3}$, $x + 4y = -3\sqrt{3}$ Area = log 2.5

9. modulus $= |\operatorname{cosec}\theta|$ and argument $= \frac{\pi}{2} - \theta$ if $0 < \theta < \pi$ or $-\pi/2 - \theta$ if $-\pi < \theta < 0$

10. $(\frac{1}{4}\pi^2 - 2)$

ANSWERS (HOMEBUSH 1956) (page 303)

1. (a) 5 hrs. 50 min. (b) $\frac{1}{1600}$ ft/second 2. 8.03 3. (a) $\frac{1}{4(1-y)} + \frac{1}{2(1-y)^2} + \frac{1}{4(1+y)}$

5. (a) $\sum \frac{1+\alpha}{1-\alpha} = -7$ 6. (a) $1 - \frac{1}{4}\pi$ (b) 0 7. (a) (i) a circle radius B centre A (ii) perpendicular bisector of AB (iii) half line through A parallel to OB and in the same sense

(b) $P \equiv i$ $Q \equiv \frac{1}{\sqrt{2}}(1 + i)$ $\therefore$ mod P = 1 am. P $= \frac{\pi}{2}$, mod Q = 1, am. Q $= \frac{\pi}{4}$

9. (a) (i) $\frac{1}{6}e^{3x}(\cos 3x + \sin 3x)$ (ii) $2\left[3\sqrt{x+2} - \frac{1}{3}\sqrt{(x+2)^3}\right]$ or $\frac{2}{3}\sqrt{x+2}\,(7-x)$.

(iii) $\frac{1}{2}\tan^{-1}(2\tan\frac{x}{2})$ (iv) $-\frac{1}{x}\sqrt{1-x^2}$ (b) (i) $\frac{128\pi}{5}$ (ii) 16π

10. (b) (i) divergent, conditionally convergent. (ii) 1, $\frac{1}{4}$

ANSWERS - 1956 FORT ST. BOYS' HIGH (page 304)

1. (i) $x = -(a+b+c) \pm (a^2 + b^2 + c^2 - ab - ac - bc)^{\frac{1}{2}}$ (ii) $1 - q^2 + 2pr + 2r^2$

2. (a) (i) $\frac{x^3}{9}(3\log x - 1) + C$ (ii) $\frac{1}{6}\log\frac{(x-1)^2}{x^2 + x + 1} + \frac{1}{\sqrt{3}}\tan^{-1}\frac{2x+1}{\sqrt{3}} + C$

3. (a) 1, 4 (b) $(p + q) + 2pqx + \frac{x^2}{2!}pq(p + q) + x^3 pq(p^2 + q^2) + \frac{x^4}{4!}pq(p^3 + q^3) + \ldots$;

$\frac{1}{24}ab(a^2 - 3b)$

4. (ii) $x = \frac{ab(1 - e^{-kt(a-b)})}{a - be^{-kt(a-b)}}$; $k = \frac{1}{t(a-b)}\log\frac{b(a-x)}{a(b-x)}$ 5. 16.89' (Note $\tan\theta = \sqrt[3]{\frac{7}{5}}$)

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1956 FORT ST. BOYS' HIGH Contd.

6. (ii) $3x^2 + 3y^2 - 10x + 3 = 0$; $x^2 + y^2 = 1$ 8. $-x - \frac{3x^2}{2} - \frac{4x^3}{3} - \frac{7x^4}{4} - \ldots$;

$0 \leq x \leq \frac{-1+\sqrt{5}}{2}$ 9. $\frac{d^2y}{dt^2} = e^{-at}(a^2 + b^2) \sin(bt + c - 2\theta)$;

$\frac{d^ny}{dt^n} = (-1)^n e^{-at}(a^2 + b^2)^{n/2} \sin(bt + c - n\theta)$

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2.(i) (b) $x = \frac{(d-b)(c-d)}{(a-b)(c-a)}$, $y = \frac{(a-d)(d-c)}{(a-b)(b-c)}$, $z = \frac{(b-d)(d-a)}{(b-c)(c-a)}$ (ii) locus is circle centre origin, radius 1. 3. As $x \to 0$, $y \to 0$, $\frac{dy}{dx} \to -\infty$; as $x \to \infty$, $y \to 0$, $\frac{dy}{dx} \to 0$

4. (ii) $\frac{1}{220}$, $\frac{1}{55}$, $\frac{27}{55}$ 5. $\left(\frac{dy}{dx}\right)^2 - y^2 = 1$ 6. (i) $\tan^{-1}x + \frac{1}{2}\log(1 + x^2)$

(ii) $\log x - \frac{1}{2}\log(1 + x^2)$ (iii) $\sqrt{x - x^2} + \frac{1}{2}\sin^{-1}(2x - 1)$ (iv) $x \tan x + \log(\cos x)$.

7. (i) $\frac{2\pi}{3\sqrt{3}}$ (ii) $\frac{4 - \pi}{2\sqrt{2}}$ (iii) -1 (iv) $^3/\log 4$

ANSWERS 1957 (HOMEBUSH) (page 307)

1. (a) convergent for $0 \leq x < 2$ (b) series is convergent. 2. (b) (i) $x \tan x + \log \cos x$

(ii) $\frac{1}{6} \log \frac{1 + x^3}{1 - x^3}$ (iii) $\sin^{-1}(x - 1)$ 3. (a) $z = 1 + i\sqrt{3}$, distance = 2 units

(b) $x = 0$ or $-(a + b + c)$ 4. (a) $18\frac{2}{3}\pi$ (b) 0.25 5. mx. (0.41, 0.2)

min (-2.41, -1.3), inflex. (1, 0), and inflex when $x = -2 \pm \sqrt{3}$

6. (i) $e - 2$ (ii) $\frac{1}{2}\pi$ (iii) 2 (iv) $\frac{1}{2} + \log\frac{1}{4}$ (v) $\frac{1}{2}\pi a^2$ 9 $|x| < a$; $\frac{728}{81}\tan^6 x$

ANSWERS 1957 (EXTERNAL) (page 310)

1. (ii) $-1 + i(\sqrt{3} - \sqrt{2})$, $1 - i(\sqrt{3} + \sqrt{2})$ 2. (ii) $\frac{5}{324}$ 3. max. and min. when $x = 3$, inflex. when $x = 1.27$

4. (i) $(a^{2/3} + b^{2/3})^{3/2}$ (ii) 2π

5. (i) $x + \frac{1}{3}\tan^3 x - \tan x$

(ii) $\frac{2\sqrt{1 + e^x}}{3}(e^x - 2)$ (iii) $\frac{1}{10}\log(2x - 1) + \frac{1}{5}\log(x^2 + 1) + \frac{1}{5}\tan^{-1}x$

(iv) $\frac{2^x}{\log 2}\left\{x - \frac{1}{\log 2}\right\}$

ANSWERS 1956 EXTERNAL Cont

6. (ii) (a) convergent (b) convergent. 7. (ii) - 1 8. (i) $\frac{1}{2}\pi + 1$ (ii) c - sin c (iii) 2 (iv) $\frac{1}{4}\pi$ 10. $u_0 = \pi/3$, $u_1 = \pi/6$, $u_2 = \pi/12$, $u_3 = \pi/24$

ANSWERS 1957 (FORT ST. BOYS' HIGH SCHOOL) (page 309)

1. (i) $4\frac{1}{2}$, 2, $-\frac{1}{2}$ (ii) 1.41 2. (i) Divergent (ii) convergent if $|x| < 1$, divergent if $x = 1$, conditionally convergent if $x = -1$. 3. (a) $(a + b + c)^3$ (c) $2^{n/2}(\cos \frac{n\pi}{4} + i\sin \frac{n\pi}{4})$

(d) (i) $x^2 + y^2 = 25$ (ii) $x^2 + (y - 1)^2 = 4$ (iii) $(x + 2)^2 + y^2 = 9$ (iv) $(x - \frac{1}{2})^2 + y^2 = 2\frac{1}{4}$

(v) $(x - 2)^2 + (y - 3)^2 = 16$ (vi) positive x axis. 4. (a) $\frac{2}{5}$ (b) $\frac{(m + n)!}{m!\, n!}$

(c) $S_n = \frac{n(n + 3)}{4(n + 1)(n + 2)}$; $S_\infty = \frac{1}{4}$ 5. (ii) $\frac{1}{18}$ 6. (a) $kae^{-kx}(\cos kx - \sin kx)$ (b) $\frac{1}{x\sqrt{1 - (\log x)^2}}$

(c) $\frac{-a}{x\sqrt{a^2 - x^2}}$ (d) $\frac{2}{e^x + e^{-x}}$ 7. (a) $\frac{2}{3}(\sin x)^{3/2} + C$ (b) $\frac{1}{2}\tan^{-1}(2\tan x/2) + C$

(c) $x(\log x - 1) + C$ (d) $\frac{e^{ax}}{a^2}(ax - 1) + C$ 8. (a) $\frac{\pi^2}{2}$ cu. units (b) P is $(\sqrt{3}, 1)$

vol = $\frac{\pi}{15}(40 - 21\sqrt{3})$ cu. units 9. (i) $\frac{\pi^2}{4}$ 10. (ii) $\frac{35\pi}{256}$

ANSWERS (HOMEBUSH 1958) (page 312)

1. (i) $a^{-1} = \frac{1}{2}(-1 - i\sqrt{3})$ (ii) $x = \frac{b + c - a}{4bc}$, $y = \frac{a + c - b}{4ac}$, $z = \frac{a + b - c}{4ab}$

(iii) $(-1 + \frac{1}{2}\sqrt{6}) + i/\sqrt{2}$, $(-1 - \frac{1}{2}\sqrt{6}) - i/\sqrt{2}$ 2. (i) $\frac{1 - (1 - a)^n(1 + an)}{a^2}$

(iii) (a) $\frac{12}{49}$ (b) $\frac{29}{140}$ 3. ∞, 0, 1 4. (iii) (a) convergent (b) convergent $-1 \leq x < 1$ 5. (i) $\frac{1}{5}\log\frac{(x - 2)^2}{x^2 + 1} + \frac{1}{5}\tan^{-1}x$ (ii) $\frac{1}{9}x^3(3\log x - 1)$

(iii) $\frac{a^2}{2}\left[\frac{x}{a^2}\sqrt{a^2 - x^2} + \sin^{-1}\frac{x}{a}\right]$ (iv) $e^x + \log(e^x - 1)$ (v) $-e^{-x}(x + 1)$.

7. (i) $\frac{\pi}{4}$ (ii) $\frac{2}{3}\tan^{-1}\frac{1}{3}$ (iii) -1 (iv) $1/\log 2$ (v) $\frac{1}{2}(1 + e^{\frac{1}{2}\pi})$ 8. (ii) $\frac{16}{315}t^{9/2}$

9. (ii) ∞ 10. $\frac{x}{3}\left[(\log x)^3 - (\log x)^2 + \frac{2}{3}\log x - \frac{1}{9}\right]$

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2. (ii) $\frac{3\pi}{2}(1 + a)$ 3. Maximum $(e, \frac{1}{e})$; point of inflexion $(e^{3/2}, \frac{3}{2}e^{-3/2})$; as $x \to 0$, $y \to -\infty$ $\frac{dy}{dx} \to +\infty$; as $x \to \infty$, $y \to 0$, $\frac{dy}{dx} \to 0$; vol 2π

4. (i) $\frac{2\sqrt{1 + x}}{15}\left[3x^2 - 4x + 8\right] + C$ (ii) $\frac{1}{4}\tan^4\theta - \frac{1}{2}\tan^2\theta - \log(\cos\theta) + C$

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ANSWERS - 1958 Fort St. Boys' High School. Contd.

4. (iii) $\frac{3}{2}\log(x^2 + 2x + 5) - \tan^{-1}\left(\frac{x+1}{2}\right) + C$ (iv) $\frac{3^x}{\log 3}\left[x^2 - \frac{2x}{\log 3} + \frac{2}{(\log 3)^2}\right] + C$

5. (ii) convergent if $|x| < 1$, divergent if $x = 1$, non-convergent if $x = -1$; $2x(1-x)^{-2} + 3x(1-x)^{-1} - \log(1-x)$. 6. (i) $\frac{\pi}{\sqrt{3}}$ (ii) π (iii) $1 - \log 2$ (iv) $\frac{\pi}{8}$

7. (ii) $k = \frac{1}{a}\log\frac{V}{v}$ 8. (ii) $C = V/\sqrt{2}$, $\alpha = \pi/4$ 9. $\frac{1}{2}(e - 2)$

ANSWERS (EXTERNAL 1958)(page 315)

1. $z_1 = a(\cos A + i\sin A)$ $z_2 = a\cos x\left[\cos(A + x) + i\sin(A + x)\right]$

$z_3 = a\cos 2x\left[\cos(A + 2x) + i\sin(A + 2x)\right]$

2. $x = \frac{-1}{a + bw + cw^2}$, $y = \frac{-w}{a + bw + cw^2}$, $z = \frac{-w^2}{a + bw + cw^2}$

3. (d) $\leq a^4 + b^4 + c^4 + d^4 < 4$ No conclusion about abcd or $\sum' a^{-4}$.

4. (i) $\phi(n, r) = \frac{n!}{(n-2r)!\,2^r.\,r!}$ (iii)

	0	1	2	3
2	1	1	0	0
3	1	3	0	0
4	1	6	3	0
5	1	10	15	0
6	1	15	45	15
7	1	21	105	105

5. $a = 1$, $b = 1/6$, $c = 1/2$, $d = \frac{1}{180}$ 7. $y = 6(x - 3)$ is common tangent, cuts one curve at $x = 3, -9$. cuts the second at $x = +3, -3$. 8. (i) $\frac{32}{\pi^3}$ (ii) 0 (iii) 1

9. (i) $\log 4$ (ii) $\pi(1 + \frac{4}{3}\log 2)$ 10. (i) $\tan^{-1}x = x - \frac{1}{3}x^3 + \frac{1}{5}x^5$ ------

(ii) $a\sin^{-1}\frac{x}{a} + \sqrt{a^2 - x^2}$, $\frac{1}{2}(b + c)\sin^{-1}\frac{(2u + b - c)}{b + c} + \left[(b + u)(c - u)\right]^{\frac{1}{2}}$ assumption $(b + c) > 0$.

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1. (a) $x = 1$, $y = 2$ (b) $a(a - x)(a + x)(1 - x - a)$ (c) circle has radius 1, centre (0, 0)

2. (a) $2x\cos 3x - 3x^2\sin 3x$, $\frac{-1}{1 + x^2}$ (b) $C_1 = b$, $C_2 = 2ab$, $C_3 = 3a^2b - b^3$

5. (a) convergent (b) range of convergence $-1 \leq x < 1$, range of absolute convergence $-1 < x < 1$ (c) $(-1)^{n-1}\left\{\frac{1}{n} + \frac{2}{n+1}\right\}$, $2 - \log 2$ 6. (a) (i) 2 (ii) $\frac{\pi\sqrt{3}}{9}$ (iii) $\frac{2}{5}$

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6. (iv) $\frac{1}{10}\log\frac{2}{9}+\frac{\pi}{10}$ (b) $\frac{1}{2}$ 8. (a) (i) $-\frac{2}{3}(1+\cos x)^{3/2}$ (ii) $\frac{x.2^x}{\log 2}-\frac{2^x}{(\log 2)^2}$

(b) $\frac{16}{35}$ 9. (a) π (b) $\frac{1}{4}\pi^2-2$

10. $y=-\frac{1}{4}x^3+3/4\,x$ is the curve, difference between gradients is 1.

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1. (a) $1-i,\ -1+0i$ (b) $3x^2+3y^2-10x+3=0$; Semi-circle of $x^2+y^2=1$, in second and third quadrants

2. (b) $1+2pr+2r^2-q^2$ 3. (a) $x^{x^x}\left[x^{x-1}+x^x\log x(1+\log x)\right]$ (b) -256, using result $\frac{d}{dx}\left[\int_a^x f(u)\,du\right]=f(x)$. 4. (a) $-abc(a-b)(b-c)(c-a)$ (b) convergent, by alternating signs test. 5. (b) (i) $(a+b+c)(a+wb+w^2c)$ (ii) 1

6. Max $(1, \frac{2}{e})$; inflexion (1.58, 0.45) approx. using e^x tables. Area = 1 sq. unit.

7. (a) 0 (odd function) (b) $\frac{1}{ab}\left[\pi/2-\tan^{-1}\left(\frac{a}{b}\right)\right]$ (c) $\frac{1}{2}\left[x\left\{\sin(\log x)-\cos(\log x)\right\}\right]_0^1=-\frac{1}{2}$

9. (a) $\frac{\gamma Mm}{b(b+L)}$. Hint $\delta F=\frac{\gamma M\,\delta m}{(b+x)^2}$ and $\frac{\delta m}{\delta x}=\frac{m}{L}$, and thus $F=\int_0^L\frac{\gamma mM}{L(b+x)^2}dx$

(b) $\frac{m!}{a^{m+1}}$

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2. The equations have solutions except when $a=0,\ +7i,\ -7i$.

Solution $x=12t$, $y=-3(a-1)t$, $z=(a+1)t$, where $t=\frac{-1}{a^2-2a+7}$, and $a=0$ or $\pm 7i$

3. Stationary points $x=1\pm\sqrt{2}$, inflexions -1 or $2\pm\sqrt{3}$

4. (i) $2\sqrt{u}+2\log(\sqrt{u}-1)$ (ii) $\frac{r^{m+1}}{m+1}\log r-\frac{r^{m+1}}{(m+1)^2}$ (iii) $\frac{1}{2}\sin^{-1}\left(\frac{2x+1}{\sqrt{6}}\right)$

(iv) $\frac{1}{2}\log\frac{1+x+x^2}{1-x+x^2}$

5. (i) $\frac{\pi(e^{\lambda}+1)}{\pi^2+\lambda^2}$ (ii) $\frac{1}{2}\sec\theta\tan\theta+\frac{1}{2}\log(\sec\theta+\tan\theta)$

6. Coefficient of δ^3 is $(\lambda^2+\lambda+\frac{1}{6}\lambda^3)$ $x=2.69150$.

7. Equation of the tangent is $2ct_o(y-ct_o^{-1})+ct_o^{-2}(x-ct_o^2)=0$

Equation of the normal is $2ct_o(x-ct_o^2)-ct_o^{-2}(y-ct_o^{-1})=0$ length of PQ = $\left|\frac{1}{2}ct_o^{-4}+2ct_o^2\right|$

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8. (i) divergent (ii) conditionally convergent (iii) convergent (iv) convergent.

9. $x = 3.896$ **10.** (i) $I = \int \frac{(1 - k\cos u)^{r-1}}{(1-k^2)^{r-\frac{1}{2}}}\,du$ (ii) $\frac{1}{3}\pi$ (iii) $\frac{5}{27}\pi^2$

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1. (i) $z = 2 + i\,2\sqrt{3} = 4(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3})$ $z^2 + z^{-2} = \frac{1}{32}(-257 + i.255\sqrt{3})$

$\sqrt{z} = 2(\cos\frac{\pi}{6} + i\sin\frac{\pi}{6})$ or $2(\cos\frac{7\pi}{6} + i\sin\frac{7\pi}{6})$ (ii) $x = \frac{8}{41}(1 - 9i)$

3. (a) $U_n = \frac{1}{n^2 + n}$, convergent (b) $U_n = \frac{n\,x^{n-1}}{2^{n-1}3^n}$, $-6 < x < 6$ is range of convergence. (c) $U_n = (-1)^{n-1}\left\{\frac{1}{2n-1} + \frac{1}{2n}\right\} = (-1)^{n-1}\frac{4n-1}{(2n-1)2n}$. Semi-Convergent

(d) $U_n = \frac{(x-1)^n}{\sqrt{n}}$, $0 \leq x < 2$ **4.** (i) Max $(-2{\cdot}47, 1{\cdot}44)$; Min $(-2{\cdot}47, -1{\cdot}44)$

5. (i) $\frac{x^{n+1}}{n+1}\log x - \frac{x^{n+1}}{(n+1)^2}$ (ii) $\log\frac{x-1}{x} + \frac{1}{x} + \frac{1}{2x^2}$ (iii) $\frac{1}{2}(\cos x - \frac{1}{5}\cos 5x)$

(iv) $\sin^{-1}(x-1)$ (v) $\tan x - x$. **6.** (i) 0 (ii) $\pi - \sqrt{2}$ (iii) $1/\sqrt{2}$ (iv) log 2. **7.** (ii) $x = 0.51$

8. (ii) 7 : 3 **9.** (ii) $\frac{1}{2}ab$

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2. $f(1) = 2$, $f(2) = 4$, $f(3) = 7$ **3.** (ii) ∞ (iii) 0 (iv) ∞ **5.** convergent if $|x| > 1$

6. (b) (i) $\log\tan(\pi/4 + \theta/2)$ or $\log(\sec\theta + \tan\theta)$. (ii) $\frac{1}{4}\log\left(\frac{1 + x + x^2}{1 - x + x^2}\right) + \frac{1}{2\sqrt{3}}\left[\tan^{-1}\frac{1-2x}{\sqrt{3}} + \tan^{-1}\frac{1+2x}{\sqrt{3}}\right]$ (iii) $\log\tan\theta/2 + \cos\theta$.

4. Min (0,0); Max $(\pm 1, {\cdot}37)$
Inflexions $(\pm 1{\cdot}51, {\cdot}23)$; $(\pm{\cdot}47, {\cdot}18)$.

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2. Locus is a half line through 'k' with gradient α. α is real, r is real and $r > 0$, k may be complex or real and is unrestricted, 'a' may be complex or real and is unrestricted. The two circles have radii r and s with centres 'a' and 'b' respectively
(i) circles touch (ii) circles cut (orthogonally) (iii) $|a-b| + r < s$.

3. (i) Area = $\frac{2a^2}{5}$ sq. units (ii) Volume = $\frac{8\pi a^3}{35}$

4. (i) $+ x\cos^{-1}x - \sqrt{1 - x^2}$ (ii) $\log(1 + \log x)$ (iii) $\frac{1}{8}\log\left\{\frac{x^2}{4 + 9x^2}\right\}$

5. (i) $\frac{1}{4}\pi - \frac{2}{3}$ (ii) $\frac{1}{4}(e^2 + 1)$ (iii) $\frac{3\pi}{16}$ (iv) $\frac{3\pi}{2}$

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6. $\log 71 = 4.2626782$

7. min. $(-a, 0)$ inflexion $\left(-\frac{3a}{2}, \frac{a}{9}\right)$ Eqns of tangent and normal are

$2a^2(x_0 + a)x + x_0^3 y = ax_0(x_0 + a)(x_0 + 3a)$ and

$x_0^5 x - 2a^2 x_0^2 (x_0 + a)\ y = x_0^6 - 2a^3 (x_0 + a)^3$

8. (i) $\frac{a^2(a^2 - 2ax + x^2)}{a^2 + x^2}$ (ii) $a^2(1 - 2\cos\theta\sin\theta)$; $a^2(1 - \log 2)$,

$a^2\left(1 - \frac{2}{\pi}\right)$

9. $\lim = \frac{1}{a(a-1)^2}$; if $0 < a < 1$, series is cgt if $x < 1/a$, dgt if $x \geq 1/a$

and if $a > 1$, series is cgt if $x < a$, dgt if $x \geq a$

1961 EXTERNAL PAPER

1. (i) $-\frac{1}{2}\log(1-x) + \frac{1}{4}\log(1+x^2) - \frac{1}{2}\tan^{-1}x$ (ii) $\frac{1}{4}x^2 + \frac{1}{4}x\sin 2x + \frac{1}{8}\cos 2x$
(iii) $\frac{1}{\sqrt{2}}\log\frac{\sqrt{2} + \sqrt{1+x}}{\sqrt{2} - \sqrt{1+x}}$ 2. (i) $\frac{\sqrt{2}\,\pi}{4}$ (ii) $\frac{5\pi}{2}$

3. $\log(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \ldots + (-1)^{n+1}\frac{x^n}{n} + \ldots$; $-1 < x \leq 1$, $a = \log {}^{10}/_{9}$, $b = \log {}^{25}/_{24}$,

$c = \log {}^{81}/_{80}$, $\log_e 10 = 2.30258(5)$ 5. Min (0,0); Max $(2, {}^{8}/_{3})$; Inflexions (3.73, 2.36) and (0.27, 0.64)

6. (i) $x = \frac{e-1}{e+1} \doteqdot 0.462$ (ii) $\int_0^1 x\sqrt{1-x^2}\,dx$ 8. (i) p 9. (i) $\frac{\text{length } Z_1Z_3}{\text{length } Z_2Z_3}$; measure of angle $Z_2Z_3Z_1$

1962 EXTERNAL PAPER

1. (i) $(a+b+c)^3$ (ii) If $a = 8$, equations are not independent. Solution $x = \frac{4-5\lambda}{3}$, $y = \frac{2-\lambda}{3}$, $z = \lambda$ for all real values of λ. 2. (i) $\pi/2$ (ii) $\frac{5\pi}{32}$ 3. (i) $-e^{-x}(2 + 2x + x^2)$ (ii) $3\log(1+x) - \frac{3}{2}\log(1+x^2) + 4\tan^{-1}x$ (iii) $-e^{-x} + \log(e^{-x} + 1)$ 4. (i) Eqn of tan is $y = kex$ (ii) $a = 1/5$ 6. (i) $\pi a^3(\pi - 2)$ (ii) $\pi a^3(2\log 2 - 1)$ 7. (i) $u = \frac{x^2+y^2-x}{x^2+y^2}$, $v = \frac{y}{x^2+y^2}$ (ii) right bisector of join of points representing $-i$ and $-3-4i$ (iii) $2\pi/3$ (iv) Circle centre 2 and radius 1/3 8. Eqn of tan is $x\cos\theta/2 - y\sin\theta/2 = a\theta\cos\theta/2 - 2a\sin\theta/2$ 9. (i) .693 10. (i) $2e + 1$ (ii) (a) cges, (use $\sin\frac{\pi}{\sqrt{n}} \to \frac{\pi}{\sqrt{n}}$ as $n \to \infty$) (b) cges if $0 < x < 1$ and if $x > 1$, but dges if $x = 1$.