Northern Beaches Secondary College

Manly Selective Campus

2012
HIGHER SCHOOL CERTIFICATE
TRIAL EXAMINATION

Mathematics Extension 2

General Instructions

- Reading time – 5 minutes
- Working time – 3 hours
- Write using blue or black pen
- Board-approved calculators and templates may be used
- A table of standard integrals is provided at the back of this paper
- All necessary working should be shown in every question

Total marks – 100

- Attempt Questions 1-1 6
- Multiple Choice – questions 1-10 – one mark per question.
- Questions 11-16 – 15 marks per question.

Marks

Questions 1 to 10 - Multiple Choice – to be answered on given answer sheet. ***10 Marks***

Q1. Let $z=1+i$ and $w=1-2i$. What is the value of zw?

(A) $-1-i$

(B) $-1+i$

(C) $3-i$

(D) $3+i$

Q2. Which of the following complex numbers equals $(\sqrt{3}+i)^4$?

(A) $-2+\frac{2}{\sqrt{3}}i$

(B) $-8+\frac{8}{\sqrt{3}}i$

(C) $-2+2\sqrt{3}i$

(D) $-8+8\sqrt{3}i$

Q3. Consider the ellipse with the equation $\frac{x^2}{16}+\frac{y^2}{9}=1$.

What is the eccentricity of the ellipse?

(A) $\frac{\sqrt{7}}{4}$

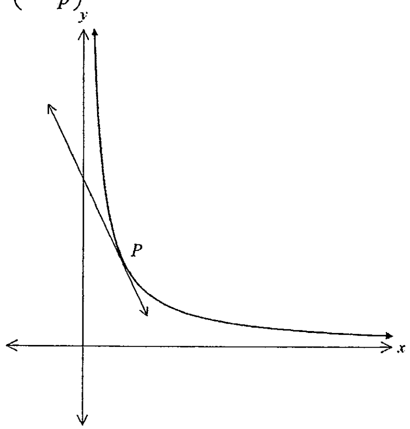
(B) $\frac{5}{4}$

(C) $\frac{\sqrt{7}}{16}$

(D) $\frac{4}{5}$

Marks

Q4. The point $P\left(cp, \frac{c}{p}\right)$ lies on the hyperbola $xy = c^2$.



Which of the following is the equation of the tangent to the hyperbola at P?

(A) $x^2 - p^2y = 2cp$

(B) $x^2 - p^2y = 2c^2$

(C) $x + p^2y = 2cp$

(D) $x + p^2y = 2c^2$

Q5. Which of the following is an expression for? $\int \frac{1}{\sqrt{x^2 - 6x + 10}}\,dx$

(A) $\ln\left(x - 3 - \sqrt{x^2 - 6x + 10}\right) + c$

(B) $\ln\left(x + 3 - \sqrt{x^2 - 6x + 10}\right) + c$

(C) $\ln\left(x - 3 + \sqrt{x^2 - 6x + 10}\right) + c$

(D) $\ln\left(x + 3 + \sqrt{x^2 - 6x + 10}\right) + c$

Marks

Q6. What is the value of $\int_0^1 \frac{\cos^{-1}x}{\sqrt{1 - x^2}}\,dx$?

(A) $\frac{\pi^2}{4}$ (B) $-\frac{\pi^2}{4}$

(C) $\frac{\pi^2}{8}$ (D) $-\frac{\pi^2}{8}$

Q7. A volume is formed by rotating the region enclosed by $y = \sin x$, the x-axis, $x = \frac{\pi}{2}$ and $x = \pi$, around the y-axis. Which is the correct integral for calculating this volume using the method of cylindrical shells?

(A) $V = \pi \int_{\frac{\pi}{2}}^{\pi} \sin^2 x - \frac{\pi^2}{4}\,dx$

(B) $V = 2\pi \int_0^1 y \sin y\,dy$

(C) $V = 2\pi \int_{\frac{\pi}{2}}^{\pi} x \sin x\,dx$

(D) $V = \pi \int_0^1 (\sin^{-1}y)^2 - \frac{\pi^2}{4}\,dy$

Q8. A particle of mass m is moving in a horizontal circle, with constant tangential speed v, around the curved surface of a cone with semi apex angle θ. The particle is attached to the lower end of a taut string, whilst the upper end remains fixed at the apex of the cone. Taking up as the positive direction, which equation represents the sum of the vertical components of the forces acting on the particle.

(A) $T\sin\theta - N\cos\theta + mg = 0$ (B) $T\sin\theta + N\cos\theta - mg = 0$

(C) $T\cos\theta + N\cos\theta - mg = 0$ (D) $T\cos\theta + N\sin\theta - mg = 0$

Marks

Q9. The equation $48x^3 - 64x^2 + 25x - 3 = 0$ has roots α, β and γ.
If $\alpha = \beta\gamma$, one possible value of α is ?

(A) $-\frac{1}{2}$

(B) $\frac{1}{4}$

(C) $\frac{1}{2}$

(D) $\frac{3}{8}$

Q10. What is the solution to the inequation $\frac{x(5-x)}{x-4} \geq -3$?

(A) $2 \leq x < 4$ or $x \geq 6$
(B) $1 \leq x < 4$ or $x \geq 5$
(C) $4 < x \leq 6$ or $x \leq 2$
(D) $4 > x \leq 5$ or $x \leq 1$

Marks

Question 11 ***(Answer in a separate booklet)*** **15**

(a) $\displaystyle\int_0^{\frac{\pi}{3}} \sec^3 x \tan x \, dx$ (2)

(b) $\displaystyle\int \sqrt{\frac{5-x}{5+x}} \, dx$ (3)

(c) (i) Find real numbers A, B and C such that $\frac{10}{(3+x)(1+x^2)} \equiv \frac{A}{3+x} + \frac{Bx+C}{1+x^2}$. (2)

(ii) Hence use the substitution $t = \tan\theta$ to find $\displaystyle\int \frac{10}{3+\tan\theta} d\theta$ (3)

(d) If α, β and γ are roots of the polynomial equation $x^3 - 2x^2 + x + 3 = 0$

(i) Evaluate $\alpha^2 + \beta^2 + \gamma^2$. (1)

(ii) Form an equation whose roots are α^2, β^2 and γ^2. (2)

(e) Given that $(x + 2i)$ is a factor of $P(x) = x^3 - 3x^2 + 4x - 12$,
factorise $P(x)$ over the complex field. (2)

Marks

Question 12 *(Answer in a separate booklet)* **15**

(a) Let $z = 3 + 4i$ and $w = 1 - 2i$. Find in the form $x + iy$,

(i) Re(z)-Im(w) *(1)*

(ii) $\frac{z}{iw}$ *(2)*

(iii) $\sqrt{z}$ *(2)*

(b) On the Argand diagram, let O be the origin and A be the point representing the complex number, $\alpha = \frac{1}{\sqrt{2}} + i\left(\frac{1}{\sqrt{2}}\right)$.

(i) If point B represents the complex number β, where $\beta = \alpha \times \text{cis}\left(\frac{\pi}{3}\right)$, express β in modulus-argument form. *(1)*

(ii) Hence find the area of ΔOAB. *(1)*

(c) Sketch, on the same Argand diagram, the locus specified by,

(i) $|z-9| = |z+1|$ *(2)*

(ii) $\arg(z - 2i) = \frac{\pi}{4}$ *(2)*

(iii) Hence write down all the values of z which simultaneously satisfy

$|z-9| = |z+1|$ and $\arg(z - 2i) = \frac{\pi}{4}$ *(1)*

(d) The vertices of a hyperbola are located at the points (–3,0) and (3,0).

The equations of the hyperbola's asymptotes are $y = \frac{5x}{3}$ and $y = \frac{-5x}{3}$.

(i) Calculate the eccentricity of the hyperbola. *(1)*

(ii) State the equation of either directrix. *(1)*

(iii) State the co-ordinates of either foci. *(1)*

Marks

Question 13 *(Answer in a separate booklet)* **15**

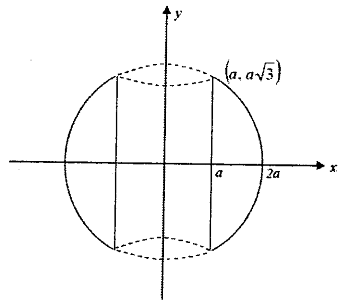
(a) A sequence $u_1, u_2, u_3, \ldots$ is defined by $u_1 = 2$, $u_2 = 12$ and $u_n = 6u_{n-1} - 8u_{n-2}$ *for* $n \geq 3$.

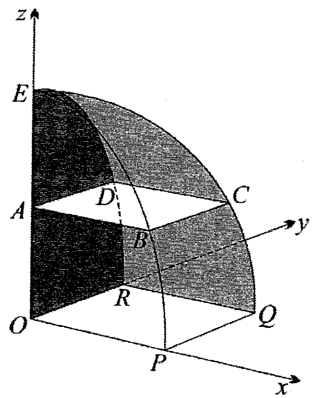
Use Mathematical induction to show that $u_n = 4^n - 2^n$ *for* $n \geq 1$ *(3)*

(b) (i) For a $a \geq 0$ *and* $b \geq 0$, show that $a + b \geq 2\sqrt{ab}$. *(1)*

(ii) Hence or otherwise show that $\frac{a+b+c+d}{4} \geq \sqrt[4]{abcd}$ *(2)*

(c) A cylindrical hole of radius a cm is bored through the centre of a sphere of radius $2a$ cm. Show that the volume of the remaining solid is $4\sqrt{3}\,a^3\pi\ cm^3$ *(4)*





(d) The solid in the diagram above has a horizontal square base $OPQR$ with diagonal $OQ = r$. The thin horizontal slice height z above the base is also square with $OC = r$.

The line OAE is vertical. The curve QCE is a arc of a circle with centre O and radius r.

(i) Show that the area of ABCD is $\frac{1}{2}(r^2 - z^2)$. *(2)*

(ii) Hence find the volume of the solid. *(3)*

Marks

Question 14 ***(Answer in a separate booklet)*** **15**

(a) Suppose that b and d are real numbers and $d \neq 0$.
Consider the polynomial $P(z) = z^4 + bz^2 + d$.

The polynomial has a double root at α.

(i) Prove that $P'(z)$ is an odd function. *(1)*

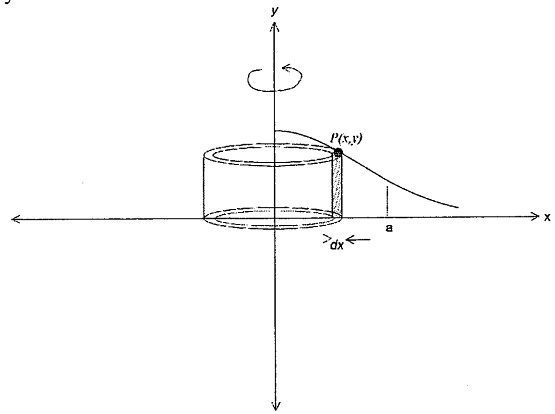
(ii) Prove that $-\alpha$ is also a double root of $P(z)$. *(1)*

(iii) Prove that $d = \dfrac{b^2}{4}$ *(2)*

(iv) For what values of b does $P(z)$ have a double root equal to $\sqrt{3}i$? *(1)*

(v) For what values of b does $P(z)$ have real roots? *(1)*

(b) The graph of $y = e^{-x^2}$ is shown below. The graph has a horizontal asymptote at $y = 0$. The region between the curve for $0 \leq x \leq a$, the y-axis and the x-axis is rotated about the y axis to form a solid. The volume of this solid is to be determined using the method of cylindrical shells.



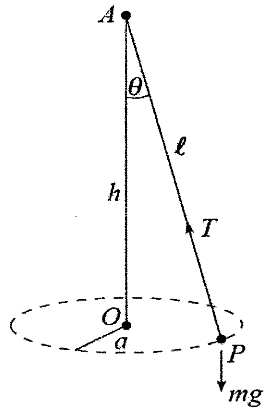
(i) State the volume dV, of a typical cylindrical shell shown in the diagram. *(1)*

(ii) Hence calculate the volume of the solid. *(3)*

(iii) What is the limiting value of the volume of the solid as $a \to \infty$? *(1)*

Marks

Question 14 continued



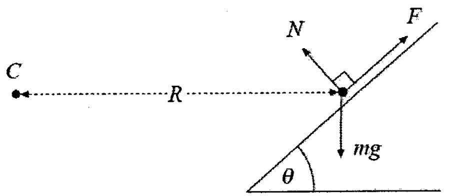
(c) A conical pendulum consists of a bob P of mass m kg and a string of length l metres. The bob rotates in a horizontal circle of radius a and centre O at a constant angular velocity of ω radians per second. The angle OAP is Θ and $OA = h$ metres. The bob is subject to a gravitational force of mg newtons and a tension in the string of T newtons.

(i) Write down the magnitude, in terms of ω, of the resultant force acting on P towards the centre O. *(1)*

(ii) By resolving forces, show that $\omega^2 = \dfrac{g}{h}$. *(3)*

Marks

Question 15 ***(Answer in a separate booklet)*** **15**



(a) A particle of mass m travels at a constant speed v around a circular track of radius R, centre C. The track is banked inwards at an angle Θ, and the particle does not move up or down the track.

The reaction exerted by the track on the particle has a normal component N, and a component F due to friction, directed up or down the bank. The force F lies in the range from $-\mu N$ to μN, where μ is a positive constant and N is the normal component; the sign of F is positive when F is directed up the bank.

The acceleration due to gravity is g.

The acceleration related to the circular motion is of magnitude $\frac{v^2}{R}$, and is directed towards the centre of the track.

(i) By resolving forces horizontally and vertically, show that

$$\frac{v^2}{Rg} = \frac{N\sin\theta - F\cos\theta}{N\cos\theta + F\sin\theta}$$ (3)

(ii) Show that, if the optimal speed for the track occurs when there is no friction, then the optimal speed v_0, is $v_0 = \sqrt{Rg\tan\theta}$ (2)

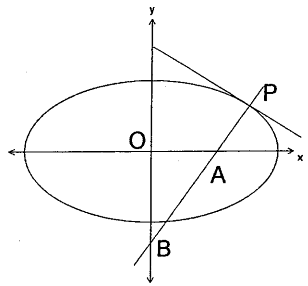
(iii) Given that the minimum speed v_{min}, at which the particle can travel without slipping down the track, occurs when $F = \mu N$, show that v_{min} is given by $v_{min} = \sqrt{\frac{Rg(\tan\theta - \mu)}{1 + \mu\tan\theta}}$ (2)

(iv) Using $g = 9{\cdot}8ms^{-2}$, calculate the difference between the optimal speed and the minimum speed for a track where $R = 500m$, $\theta = 15^\circ$ and $\mu = {\cdot}1$. Give your answer to nearest whole number. (1)

Marks

Question 15 continued

(b) $P(a\cos\theta, b\sin\theta)$, where $0 < \theta < \frac{\pi}{2}$, is a point on the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ where $a > b > 0$.



The normal at P cuts the x axis at A and the y axis at B.

(i) Show that the normal at P has the equation $ax\sin\theta - by\cos\theta = (a^2 - b^2)\sin\theta\cos\theta$ (2)

(ii) Show that triangle OAB has area $\frac{(a^2 - b^2)^2\sin\theta\cos\theta}{2ab}$ (2)

(iii) Find the maximum area of the triangle OAB and the coordinates of P when this maximum occurs. (3)

Marks

Question 16 *(Answer in a separate booklet)* **15**

(a) Consider the integral $I_n = \int_0^1 \frac{x^n}{\sqrt{1+x}}\,dx$ for $n \ge 0$.

(i) Show that $I_0 = 2\sqrt{2} - 2$. *(1)*

(ii) Given that $I_n + I_{n-1} = \int_0^1 x^{n-1}\sqrt{1+x}\,dx$,

show that $I_n = \frac{2\sqrt{2} - 2nI_{n-1}}{2n+1}$ *(2)*

(iii) Hence evaluate I_2 in exact form. *(2)*

(b) If $f(x) = x - \ln\left(1 + x + \frac{x^2}{2}\right)$

(i) Determine the value of $f(0)$ *(1)*

(ii) Show that $f(x)$ is an increasing function of x for $x < 0$. *(1)*

(iii) Hence show that $e^x < 1 + x + \frac{x^2}{2}$ *for* $x < 0$. *(2)*

(c) The number c is real and non-zero. It is also known that $(1 + ic)^5$ is real.

(i) Use binomial theorem to expand $(1 + ic)^5$. *(1)*

(ii) Show that $c^4 - 10c^2 + 5 = 0$ *(2)*

(iii) Hence show that $c = \sqrt{5 - 2\sqrt{5}}$, $-\sqrt{5 - 2\sqrt{5}}$, $\sqrt{5 + 2\sqrt{5}}$, $-\sqrt{5 + 2\sqrt{5}}$ *(1)*

Let $(1 + ic) = r\text{cis}\Theta$.

(iv) Use deMoivre's Theorem to show that the smallest positive value of Θ is $\frac{\pi}{5}$ *(1)*

(v) Hence evaluate $\tan\left(\frac{\pi}{5}\right)$ *(1)*

END OF EXAMINATION

Marks

STANDARD INTEGRALS

$\int x^n\,dx = \frac{1}{n+1}x^{n+1}, \quad n \ne -1; \quad x \ne 0, \text{ if } n < 0$

$\int \frac{1}{x}\,dx = \ln x, \quad x > 0$

$\int e^{ax}\,dx = \frac{1}{a}e^{ax}, \quad a \ne 0$

$\int \cos ax\,dx = \frac{1}{a}\sin ax, \quad a \ne 0$

$\int \sin ax\,dx = -\frac{1}{a}\cos ax, \quad a \ne 0$

$\int \sec^2 ax\,dx = \frac{1}{a}\tan ax, \quad a \ne 0$

$\int \sec ax \tan ax\,dx = \frac{1}{a}\sec ax, \quad a \ne 0$

$\int \frac{1}{a^2+x^2}\,dx = \frac{1}{a}\tan^{-1}\frac{x}{a}, \quad a \ne 0$

$\int \frac{1}{\sqrt{a^2-x^2}}\,dx = \sin^{-1}\frac{x}{a}, \quad a > 0, \quad -a < x < a$

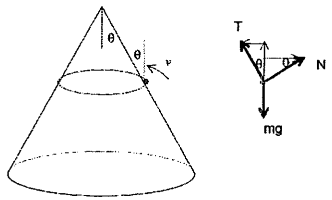
$\int \frac{1}{\sqrt{x^2-a^2}}\,dx = \ln\left(x + \sqrt{x^2-a^2}\right), \quad x > a > 0$

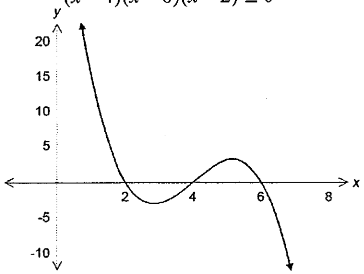
$\int \frac{1}{\sqrt{x^2+a^2}}\,dx = \ln\left(x + \sqrt{x^2+a^2}\right)$

NOTE : $\ln x = \log_e x, \quad x > 0$

Multiple Choice

Q1 – Q10	C D A C C C C D B C		
Q1	C	$zw=(1+i)(1-2i)$ $=1-i-2i^2$ $=3-i$	
Q2	D	$\sqrt{3}+i=2(\frac{\sqrt{3}}{2}+\frac{1}{2}i)$ $=2(\cos\frac{\pi}{6}+i\sin\frac{\pi}{6})$ $(\sqrt{3}+i)^4=2^4(\cos 4\times\frac{\pi}{6}+i\sin 4\times\frac{\pi}{6})$ $=16(\cos\frac{2\pi}{3}+i\sin\frac{2\pi}{3})$ $=-8+8\sqrt{3}i$	
Q3	A	$b^2=a^2(1-e^2)$ $9=16(1-e^2)$ $\frac{9}{16}=1-e^2$ $e^2=\frac{7}{16}$ $e=\frac{\sqrt{7}}{4}$	
Q4	C	To find the gradient of the tangent. $xy=c^2$ $x\frac{dy}{dx}+y=0$ $\frac{dy}{dx}=-\frac{y}{x}$ At P $\left(cp,\frac{c}{p}\right)$, $\frac{dy}{dx}=-\frac{c}{p}\div cp=-\frac{c}{p}\times\frac{1}{cp}=-\frac{1}{p^2}$ Equation of the tangent at P $\left(cp,\frac{c}{p}\right)$	

		$y-\frac{c}{p}=-\frac{1}{p^2}(x-cp)$ $-p^2y+cp=x-cp$ $x+p^2y=2cp$	
Q5	C	$\int\frac{dx}{\sqrt{x^2-6x+10}}=\int\frac{dx}{\sqrt{x^2-6x+9+1}}=\frac{dx}{\sqrt{(x-3)^2+1}}$ $=\ln\left(x-3+\sqrt{(x-3)^2+1}\right)+c$ $=\ln\left(x-3+\sqrt{x^2-6x+10}\right)+c$	
Q6	C	$\int_0^1\frac{\cos^{-1}x}{\sqrt{1-x^2}}dx$ $=-\int_0^1\frac{\cos^{-1}x}{-\sqrt{1-x^2}}dx$ $=-\int_0^1(\cos^{-1}x)d(\cos^{-1}x)$ $=-\left[\frac{(\cos^{-1}x)^2}{2}\right]_0^1$ $=-\frac{1}{2}\left(0-\left(\frac{\pi}{2}\right)^2\right)$ $=\frac{\pi^2}{8}$	
Q7	C	Cylindrical shells radius is x and height $\sin x$ $V=2\pi\int_{\frac{\pi}{2}}^{\pi}x\sin x\,dx$	
Q8	D	 Consider the forces acting vertically. $T\cos\theta+N\sin\theta-mg=0$	

Q9	B	Product of the roots $\alpha\beta\gamma = -\frac{d}{a}$ $\alpha\alpha = \frac{1}{16}$ $\alpha^2 = \frac{1}{16}$ $\alpha = \pm\frac{1}{4}$	
Q10	C	$\frac{x(5-x)}{x-4} \geq -3$ NB $x \neq 4$ $x(5-x)(x-4) \geq -3(x-4)^2$ $x(5-x)(x-4) + 3(x-4)^2 \geq 0$ $(x-4)[x(5-x) + 3(x-4)] \geq 0$ $(x-4)[5x - x^2 + 3x - 12] \geq 0$ $(x-4)[-x^2 + 8x - 12] \geq 0$ $-(x-4)(x-6)(x-2) \geq 0$  $x \leq 2$ and $4 < x \leq 6$	

Question 11

	$\int_0^{\frac{\pi}{3}} \sec^3 x \tan x . \, dx$ $= \int_0^{\frac{\pi}{3}} \sec x \tan x \sec^2 x \, dx$ $= \frac{1}{3}[\sec^3 x]$ $= \frac{1}{3}[8-1] = \frac{7}{3}$	2 marks – correct solution 1 mark -- correct integration
	$\int \sqrt{\frac{5-x}{5+x}} \, dx$ $= \int \frac{(\sqrt{5-x})(\sqrt{5-x})}{(\sqrt{5+x})(\sqrt{5-x})} dx$ $= \int \frac{5-x}{\sqrt{25-x^2}} dx$ $= 5\int \frac{1}{\sqrt{25-x^2}} dx + \frac{1}{2}\int -\frac{2x}{\sqrt{25-x^2}} dx$ $= 5\sin^{-1}\left(\frac{x}{5}\right) + \sqrt{25-x^2} + C$	3 marks – correct solution 2 marks – 1 mark – rationalizing numerator
	$\frac{10}{(3+x)(1+x^2)} \equiv \frac{A}{3+x} + \frac{Bx+C}{1+x^2}$ $A(1+x^2) + (Bx+C)(3+x) = 10$ $(A+B)x^2 + (3B+C)x + (A+3C) = 10 \quad \ldots ①$ $\therefore \quad A + B = 0 \qquad B = -A$ $3B + C = -3A + C = 0 \qquad C = 3A$ $A + 3C = 10 \qquad A + 9A = 10 \; \therefore$ $A = 1$ $B = -1$ $C = 3$	2 marks – correct solution 1 mark – correct equation (1)

	$\int \frac{10}{3+\tan\theta}\,d\theta$ $t = \tan\theta \therefore \theta = \tan^{-1}t$ $d\theta = \frac{1}{1+t^2}$ $\therefore \int \frac{10}{(3+t)(1+t^2)}dt$ $= \int \frac{1}{3+t} + \frac{3-t}{1+t^2}dt$ $= \int \frac{1}{3+t} + \frac{3}{1+t^2} - \frac{t}{1+t^2}dt$ $= \ln(3+t) + 3\tan^{-1}t - \frac{1}{2}\ln(1+t^2) + C$ $= \ln(3+\tan\theta) + 3\tan^{-1}(\tan\theta) - \frac{1}{2}\ln(1+\tan^2\theta) + C$ $= \ln\frac{3+\tan\theta}{\sec\theta} + 3\theta + C$	3 marks – correct solution *(nb. to second last line of solution.)* 2 marks – correct integral 1 mark – correct substitution using results from c(i). nb. *A large number of students left final answer in terms of t rather than original variable Θ*
d	$x^3 - 2x^2 + x + 3 = 0$ $\alpha^2 + \beta^2 + \gamma^2 = (\alpha+\beta+\gamma)^2 - 2(\alpha\beta + \alpha\gamma + \beta\gamma)$ $\alpha+\beta+\gamma = -\frac{b}{a} = 2$ $\alpha\beta + \alpha\gamma + \beta\gamma = \frac{c}{a} = 1$ $\therefore \alpha^2 + \beta^2 + \gamma^2 = 4 - 2 = 2$	1 mark – correct solution
	$x^3 - 2x^2 + x + 3 = 0$ $(\sqrt{x})^3 - 2(\sqrt{x})^2 + (\sqrt{x}) + 3 = 0$ $\sqrt{x}\,(x+1) = 2x - 3$ $x(x+1)^2 = (2x-3)^2$ $x^3 + 2x^2 + x = 4x^2 - 12x + 9$ $x^3 - 2x^2 + 13x - 9 = 0$	2 marks – correct solution 1 mark – correct substitution of $\sqrt{x}$ *for x*

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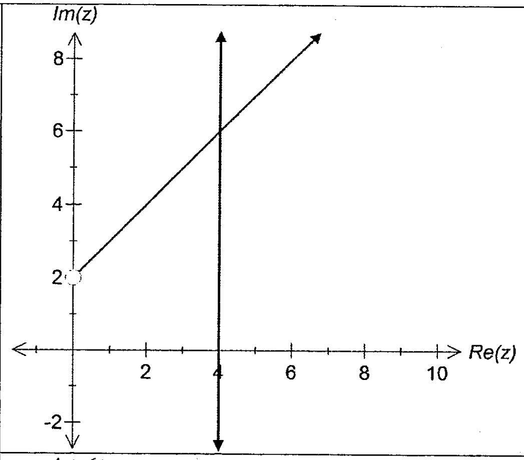
2012 Mathematics Extension 2 Trial - solutions

	$P(x) = x^3 - 3x^2 + 4x - 12$ Cubic equation therefore three possible factors. Coefficients are real therefore complex roots are in conjugate pairs. i.e. *(x + 2i)* and *(x-2i)*. $P(x) = x^3 - 3x^2 + 4x - 12$ $= (x+2i)(x-2i)(ax+b)$ $= (x^2+4)(x-3)$ $= (x+2i)(x-2i)(x-3)$	2 marks – correct solution 1 mark – two complex factor correct. Ie *(x +2i)(x -2i)*.

Question 12:

(a)i	$3 - -2 = 5$	*1 mark*
(a)ii	$\dfrac{3+4i}{i(1-2i)} = \dfrac{3+4i}{2+i}$ $= \dfrac{3+4i}{2+i} \times \dfrac{2-i}{2-i}$ $= \dfrac{10+5i}{5}$ $= 2+i$	*2 marks for correct solution* *1 mark for correct multiplication of iw*
(a)iii	Let $\sqrt{3+4i} = (x+iy)$ $3+4i = (x+iy)^2$ $3+4i = x^2 - y^2 + 2xyi$ $\therefore \quad 3 = x^2 - y^2 \qquad 4 = 2xy$ $3x^2 = x^4 - x^2y^2 \qquad 2 = xy$ $0 = x^4 - 3x^2 - 4 \qquad 4 = x^2y^2$ $0 = (x^2-4)(x^2+1)$ $x^2 = 4$ or -1 $x = \pm 2$ (as x is real) substituting into $2 = xy$ gives $y = \pm 1$ $\therefore \quad \sqrt{3+4i} = \pm(2+i)$	*2 marks for correct solution* *1 mark attaining* $3 = x^2 - y^2 \qquad 4 = x^2y^2$
(b)i	$\alpha = \text{cis}\frac{\pi}{4}$ multiply moduli and add arguments $\beta = \text{cis}\frac{\pi}{4} \times \text{cis}\frac{\pi}{3}$ $\beta = \text{cis}\frac{7\pi}{12}$	*1 mark*
(b)ii	$Area = \frac{1}{2}ab\sin C$ $= \frac{1}{2} \times 1 \times 1 \times \sin\frac{\pi}{3}$ $= \frac{\sqrt{3}}{4}\ units^2$	*1 mark*

(c)i *(c)ii*		*ci) 2 marks for correct graph of x=4* *1 mark not showing x-intercept or not indicating arrows on ends of line.* *cii) 2 marks for correct graph of ray y=x+2 for x>0* *1 mark not circling y-intercept or not indicating arrow on end of line.*
(c)iii	$z = 4 + 6i$	*1 mark*
(d)i	$b^2 = a^2(e^2 - 1)$ $25 = 9(e^2 - 1)$ $\frac{25}{9} = e^2 - 1$ $e = \frac{\sqrt{34}}{3}$	*1 mark*
(d)ii	Either $x = \frac{9}{\sqrt{34}}$ or $x = -\frac{9}{\sqrt{34}}$	*1 mark*
(d)iii	Either $(\sqrt{34}, 0)$ or $(-\sqrt{34}, 0)$	*1 mark*

Question 13:

(a)	**DELETED**	
(b)	$(a-b)^2 \geq 0$ $a^2 - 2ab + b^2 \geq 0$ $a^2 + b^2 \geq 2ab$ $a \to \sqrt{a}, \quad b \to \sqrt{b}$ $\therefore \quad a + b \geq 2\sqrt{ab}$	**1 mark – correct demonstration**
	$LHS = \dfrac{a+b+c+d}{4}$ $\geq \dfrac{2\sqrt{ab} + 2\sqrt{cd}}{4}$ $= \dfrac{\sqrt{ab} + \sqrt{cd}}{2}$ $\geq \dfrac{2\sqrt{\sqrt{abcd}}}{2}$ $= \sqrt[4]{abcd}$ $= RHS$ $\therefore \dfrac{a+b+c+d}{4} \geq \sqrt[4]{abcd}$	**2 marks – correct demonstration** **1 mark – one correct use of inequality from part b(i)**

(c)	Method 1 – Annulus (washers) $\Delta V = \pi(R^2 - r^2)h \qquad where\ R = x\,;\, r = a$ $= \pi(x^2 - a^2)\delta y \qquad x^2 + y^2 = 4a^2$ $= \pi(4a^2 - y^2 - a^2)\delta y$ $V = \lim\limits_{\delta y \to 0} \sum\limits_{y=-a\sqrt{3}}^{a\sqrt{3}} \pi(4a^2 - y^2 - a^2)\delta y$ $V = \pi \int_{-a\sqrt{3}}^{a\sqrt{3}} x^2 - a^2 dy$ $= 2\pi \int_{0}^{a\sqrt{3}} x^2 - a^2 dy$ $= 2\pi \int_{0}^{a\sqrt{3}} 4a^2 - y^2 - a^2 dy$ $= 2\pi \left[3a^2 y - \dfrac{y^3}{3} \right]_0^{a\sqrt{3}}$ $= 2\pi \left[3a^3\sqrt{3} - a^3\sqrt{3} - 0 \right]$ $= 4\sqrt{3}\, a^3 \pi\ cm^3$	*4 marks – correct solution fully demonstrated.* *including* *- development of integral:* *- correct substitution for x* *- correct integration* *- correct substitution to reach final result.*

(c) Method 2 – cylindrical shells.

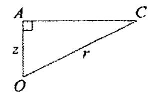
$$x^2+y^2=4a^2 \therefore y=\sqrt{4a^2-x^2}$$

$$\Delta V = 2\pi r.\,h.\,\delta x$$
$$= 2\pi x\, 2y\,\delta x$$
$$= 4\pi x\sqrt{4a^2-x^2}\;\delta x$$
$$V=\lim_{x\to 0}\sum_{x=a}^{2a} 4\pi x\sqrt{4a^2-x^2}\;\delta x$$
$$=4\pi\int_a^{2a} x\sqrt{4a^2-x^2}\,dx \qquad let\ u=4a^2-x^2$$
$$=4\pi\int_{3a^2}^{0} -\sqrt{u}\,.\,du \qquad du=\text{-}2xdx$$
$$=2\pi\left[\left(\frac{2}{3}\right)u^{\frac{3}{2}}\right]_0^{3a^2} \qquad x=a \quad u=3a^1$$
$$=2\pi\times\frac{2}{3}\times\left(\sqrt{3a^2}\right)^2 \qquad x=2a \quad u=0$$
$$=4\sqrt{3}\;a^3\,\pi\, cm^3$$

4 marks – correct solution fully demonstrated.

including

- development of integral.
- correct substitution for x
- correct integration
- correct substitution to reach final result.



In ΔOAC $\quad AC^2=r^2-z^2$ (by Pythagoras)

hence $\quad |ABCD|=\frac{1}{2}AC^2$ (a square is a rhombus)

$$=\tfrac{1}{2}(r^2-z^2).$$

2 marks – correct solution correctly demonstrated.
1mark – correct calculation of AC

$$\Delta V=\frac{1}{2}(r^2-z^2)h$$
$$=\frac{1}{2}(r^2-z^2)\delta y$$
$$V=\lim_{\delta z\to 0}\sum_{z=0}^{r}\frac{1}{2}(r^2-z^2)\delta z$$
$$V=\frac{1}{2}\int_0^r (r^2-z^2)dz$$
$$=\frac{1}{2}\left[r^2z-\frac{z^3}{3}\right]_0^r$$
$$=\frac{1}{2}\left(r^3-\frac{r^3}{3}\right)-0$$
$$=\frac{1}{3}r^3\ units^3$$

3 marks – correct solution fully demonstrated.

including

- derivation of integral for volume
- correct integration
- correct substitution and final answer.

Question 14:

(a)i	$P'(z)=4z^3+2bz$ $P'(-z)=4(-z)^3+2b(-z)$ $=-4z^3-2bz$ $=-P'(z)$ $\therefore P'(z)$ is odd.	*1 mark*
(a)ii	$P(\alpha)=P'(\alpha)=0$(as α is a double root) $P(-\alpha)=(-\alpha)^4+b(\alpha)^2+d$ $=\alpha^4+b\alpha^2+d$ $=0$ $\therefore -\alpha$ is a root $P'(-\alpha)=-P'(\alpha)$(since P'(z) is odd) $P'(-\alpha)=-0$(since $P'(\alpha)=0$ from above) $P'(-\alpha)=0$ $\therefore P(-\alpha)=P'(-\alpha)=0$ and $-\alpha$ is a double root of $P(z)$.	*1 mark*
(a)iii	By product of roots $\frac{e}{a}=\alpha\times\alpha\times-\alpha\times-\alpha$ $d=\alpha^4\ldots$ ① as $P'(\alpha)=0$ $4\alpha^3+2b\alpha=0$ $2\alpha(2\alpha^2+b)=0$ $\alpha\neq 0$ since $P(\alpha)=0$ and $d\neq 0$ $\therefore\quad 2\alpha^2+b=0$ $\alpha^2=-\frac{b}{2}$ squaring both sides $b^2=4\alpha^4\ldots$ ② Alternatively, by sum of roots taken 2 at a time $\frac{c}{a}=\alpha\times-\alpha+\alpha\times-\alpha+\alpha\times-\alpha+\alpha\times-\alpha+\ -\alpha\times-\alpha$ $b=-2\alpha^2$ $b^2=4\alpha^4\ldots$ ② Either way, comparing ① and ② $d=\frac{b^2}{4}$ There are other possible solutions	*2 marks for correct solution* *1 mark for either equation 1 or equation 2.*
(a)iv	If $\sqrt{3}\ i$ is a double root then so to is $\quad -\sqrt{3}\ i$ using equation ② $b^2=4\alpha^4$ $b^2=4\times(\sqrt{3}\ i)^4$ $b^2=36$ $b=\pm 6$	*1 mark*
(a)v	As $\quad b=-2\alpha^2$ (*see part iii*) $\alpha^2=-\frac{b}{2}$ $\therefore$ for α to be a real root $b\leq 0$	*1 mark*

(b)i	$dv=2\pi xe^{-x^2}dx$	*1 mark*
(b)ii	$V=\lim\limits_{dx\to 0}\sum\limits_{x=0}^{x=a}2\pi xe^{-x^2}dx$ $V=2\pi\int_0^a xe^{-x^2}dx$ $V=-\pi\int_0^a -2xe^{-x^2}dx$ $V=-\pi\int_0^a e^{-x^2}d\left(-x^2\right)$ $V=-\pi\left[e^{-x^2}\right]_0^a$ $V=-\pi\left(e^{-a^2}-1\right)$ $V=\pi\left(1-\frac{1}{e^{a^2}}\right)$ *units*^{3}	*3 marks for correct solution* *2 marks for correct primitive* *1 mark for* $V=\lim\limits_{dx\to 0}\sum\limits_{x=0}^{x=a}2\pi xe^{-x^2}dx$
(b)iii	As $a\to\infty$, $\frac{1}{e^{a^2}}\to 0$ $\therefore\quad V\to\pi$ *units*^{3}	*1 mark*
(c)i	Horizontal resultant force $=m\,\omega^2r=m\,\omega^2a$	*1 mark*
(c)ii	Horizontally $T\sin\theta=m\omega^2a\ldots$ ① Vertically $T\cos\theta=mg\ldots$ ② ① ÷ ② gives $\tan\theta=\frac{\omega^2a}{g}$ but $\tan\theta=\frac{a}{h}$ $\therefore\quad \frac{a}{h}=\frac{\omega^2a}{g}$ $\omega^2=\frac{g}{h}$	*3 marks for correct proof* *2 marks for* $\tan\theta=\frac{\omega^2a}{g}$ *1 mark either horizontal or vertical equation.*

Question 15:

(a)i	*Vertically* $N\cos\theta + F\sin\theta = mg \ldots$ ① *Horizontally* $N\sin\theta - F\cos\theta = m\frac{v^2}{R} \ldots$ ② ② ÷ ① *gives* $\frac{v^2}{Rg} = \frac{N\sin\theta - F\cos\theta}{N\cos\theta + F\sin\theta}$	*3 marks for correct proof.* *2 marks for both vertical and horizontal equations.* *1 mark either vertical or horizontal equation.*
(a)ii	*When* $F = 0$ $\frac{v_0^{\ 2}}{Rg} = \frac{N\sin\theta}{N\cos\theta}$ $\frac{v_0^{\ 2}}{Rg} = \tan\theta$ $v_0^{\ 2} = Rg\tan\theta$ $v_0 = \sqrt{Rg\tan\theta}$	*2 marks for correct proof* *1 mark for* $\frac{v_0^{\ 2}}{Rg} = \frac{N\sin\theta}{N\cos\theta}$
(a)iii	*When* $F = \mu N$ $\frac{v_{min}^{\ 2}}{Rg} = \frac{N\sin\theta - \mu N\cos\theta}{N\cos\theta + \mu N\sin\theta}$ $\frac{v_{min}^{\ 2}}{Rg} = \frac{N(\sin\theta - \mu\cos\theta)}{N(\cos\theta + \mu\sin\theta)}$ $\frac{v_{min}^{\ 2}}{Rg} = \frac{\sin\theta - \mu\cos\theta}{\cos\theta + \mu\sin\theta}$ ÷ *by* cos θ *top and bottom* $\frac{v_{min}^{\ 2}}{Rg} = \frac{\tan\theta - \mu}{1 + \mu\tan\theta}$ $v_{min}^{\ 2} = \frac{Rg(\tan\theta - \mu)}{1 + \mu\tan\theta}$ $v_{min} = \sqrt{\frac{Rg(\tan\theta - \mu)}{1 + \mu\tan\theta}}$	*2 marks for correct solution* *1 mark for* $\frac{v_{min}^{\ 2}}{Rg} = \frac{\sin\theta - \mu\cos\theta}{\cos\theta + \mu\sin\theta}$
(a)iv	$\sqrt{Rg\tan\theta} - \sqrt{\frac{Rg(\tan\theta - \mu)}{1 + \mu\tan\theta}}$ $= \sqrt{500 \times 9\cdot8 \times \tan 15^\circ} - \sqrt{\frac{500 \times 9\cdot8(\tan 15^\circ - 0\cdot1)}{1 + 0\cdot1 \times \tan 15^\circ}}$ $= 7\cdot924322$ $\cong 8\ ms^{-1}$ *(nearest whole number)*	*1 mark*

(b)i	$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ $\frac{2x}{a^2} + \frac{2y}{b^2}\frac{dy}{dx} = 0$ $\frac{dy}{dx} = -\frac{2x}{a^2} \times \frac{b^2}{2y}$ $\frac{dy}{dx} = -\frac{b^2x}{a^2y}$ *At P,* $\frac{dy}{dx} = -\frac{b^2a\cos\theta}{a^2b\sin\theta}$ $\frac{dy}{dx} = -\frac{b\cos\theta}{a\sin\theta}$ $\therefore\ m_{normal} = \frac{a\sin\theta}{b\cos\theta}$ *equation of normal is* $y - b\sin\theta = \frac{a\sin\theta}{b\cos\theta}(x - a\cos\theta)$ $by\cos\theta - b^2\sin\theta\cos\theta = ax\sin\theta - a^2\sin\theta\cos\theta$ $a^2\sin\theta\cos\theta - b^2\sin\theta\cos\theta = ax\sin\theta - by\cos\theta$ $ax\sin\theta - by\cos\theta = (a^2 - b^2)\sin\theta\cos\theta$	*2 marks for correct proof* *1 mark for correct derivative.*

(b)ii	$ax\sin\theta - by\cos\theta = (a^2-b^2)\sin\theta\cos\theta$. When x=0, $y = \frac{(a^2-b^2)\sin\theta\cos\theta}{-b\cos\theta} = \frac{(a^2-b^2)\sin\theta}{-b}$ Note $a > b > 0 \therefore a^2 > b^2$ and $a^2 - b^2 > 0$ also $0 < \theta < \frac{\pi}{2} \therefore \sin\theta > 0$ $\therefore \frac{(a^2-b^2)\sin\theta}{-b} < 0$ distance $OB = \left\| \frac{-(a^2-b^2)\sin\theta}{b} \right\| = \frac{(a^2-b^2)\sin\theta}{b}$. When y=0, $x = \frac{(a^2-b^2)\sin\theta\cos\theta}{a\sin\theta}$ $\therefore$ distance $OA = \frac{(a^2-b^2)\cos\theta}{a}$. Area $\Delta OAB = \frac{1}{2} \times distanceOA \times distanceOB$ $= \frac{1}{2} \times \frac{(a^2-b^2)\cos\theta}{a} \times \frac{(a^2-b^2)\sin\theta}{b}$ $= \frac{(a^2-b^2)^2\sin\theta\cos\theta}{2ab}$	2 marks for correct proof 1 mark for abcissae of A or ordinate of B
(b)iii	$A = \frac{(a^2-b^2)^2\sin\theta\cos\theta}{2ab}$ $A = \frac{(a^2-b^2)^2\sin 2\theta}{4ab}$ $\frac{dA}{d\theta} = \frac{2(a^2-b^2)^2\cos 2\theta}{4ab}$ Stationary points occur when $\cos 2\theta = 0$ for $0 < 2\theta < \pi$ $2\theta = \frac{\pi}{2}$ $\theta = \frac{\pi}{4}$ for $0 < \theta < \frac{\pi}{2}$	3 marks for finding the correct area and demonstrating that it is the maximum area. 2 marks for determining nature of stationary point. 1 mark for stationary point at $\theta = \frac{\pi}{4}$

	$\frac{d^2A}{d\theta^2} = \frac{-4(a^2-b^2)^2\sin 2\theta}{4ab}$ At $\theta = \frac{\pi}{4}$, $\frac{d^2A}{d\theta^2} = \frac{(a^2-b^2)^2\sin\frac{\pi}{2}}{ab}$ $\frac{d^2A}{d\theta^2} = \frac{-(a^2-b^2)^2}{ab} < 0$ $\therefore$ Stationary point at $\theta = \frac{\pi}{4}$ is a maximum. the co-ordiantes of P are $\left(\frac{a}{\sqrt{2}}, \frac{b}{\sqrt{2}}\right)$ $\therefore Area_{max} = \frac{(a^2-b^2)^2\sin\left(2\times\frac{\pi}{4}\right)}{4ab}$ $= \frac{(a^2-b^2)^2}{4ab}$ Alternatively, the maximum area maybe derived as follows As a and b are constants, then $\frac{(a^2-b^2)^2}{4ab}$ is constant. So area depends only on value of $\sin 2\theta$ $0 \le \theta \le \frac{\pi}{2}$ $0 \le 2\theta \le \pi$ $0 \le \sin 2\theta \le 1$ draw a graph to confirm this. $\therefore$ maximum value of area occurs when $\sin 2\theta = 1$ $2\theta = \frac{\pi}{2}$ $\theta = \frac{\pi}{4}$ for $0 \le \theta \le \frac{\pi}{2}$ $\therefore Area_{max} = \frac{(a^2-b^2)^2\sin\left(2\times\frac{\pi}{4}\right)}{4ab}$ $= \frac{(a^2-b^2)^2}{4ab}$	

Question 16:

(a)ii	$I_0 = \int_0^1 \frac{1}{\sqrt{1+x}}\,dx$ $= 2\left[\sqrt{1+x}\right]_0^1$ $= 2\sqrt{2} - 2$	*1 mark – correct solution*
	$I_n = \int_0^1 \frac{x^n}{\sqrt{1+x}}\,dx \qquad u = x^n \qquad v' = (1+x)^{-\frac{1}{2}}$ $= \qquad u' = nx^{n-1} \qquad v = 2\sqrt{1+x}$ $I_n = \left[2x^n\sqrt{1+x}\right]_0^1 - 2n\int_0^1 x^{n-1}\sqrt{1+x}\,dx$ by parts $= 2\sqrt{2} - 2n(I_{n-1} + I_n)$ $(2n+1)I_n = 2\sqrt{2} - 2nI_{n-1}$ $I_n = \frac{2\sqrt{2} - 2nI_{n-1}}{2n+1}$	*2 marks – correct solution by correct process* *1 mark – correct integration*
	$I_1 = \frac{1}{3}(2\sqrt{2} - 2I_0)$ $= \frac{1}{3}(4 - 2\sqrt{2})$ $I_2 = \frac{1}{5}(2\sqrt{2} - 4I_1)$ $= \frac{1}{5}\left(2\sqrt{2} - \frac{16}{3} + \frac{8\sqrt{2}}{3}\right)$ $= \frac{1}{15}(14\sqrt{2} - 16)$	*Two marks - Correct for I_2* *One mark – Correct for I_1*
b(i)	$f(x) = x - \ln\left(1 + x + \frac{x^2}{2}\right)$ $f(0) = 0 - \ln(1+0+0) = 0$	*1 mark – correct answer.*

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(b) (ii)	$f(x) = x - \ln\left(1 + x + \frac{x^2}{2}\right)$ $f'(x) = 1 - \frac{1+x}{1 + x + \frac{x^2}{2}}$ $= \frac{\frac{x^2}{2} + x + 1 - x - 1}{\frac{2 + 2x + x^2}{2}}$ $= \frac{x^2}{x^2 + 2x + 2}$ $= \frac{x^2}{(x+1)^2 + 1} \geq 0$ *for all x as* $(f(x))^2 \geq 0$ $= \frac{x^2}{(x+1)^2 + 1} > 0 \qquad$ *for* $x < 0$ Therefore the curve is increasing for $x < 0$	*One mark – correct solution fully demonstrating why $f(x) > 0$*
B (iii)	As the curve $f(x)$ is increasing for $x < 0$ and $f(0) = 0$ Then $f(x) < 0$ for $x < 0$ $f(x) < 0$ $\therefore x - \ln\left(1 + x + \frac{x^2}{2}\right) < 0$ $x < \ln\left(1 + x + \frac{x^2}{2}\right)$ $e^x < 1 + x + \frac{x^2}{2}$ As required.	*Two marks – correct solution correct demonstrated.* *One mark – identification of why $f(x)<0$ for $x<0$*
c(i)	$(1 + ic)^5 = 1 + 5ic - 10c^2 - 10ic^3 + 5c^4 + ic^5$	*One mark – correct solution*
c (ii)	As $(1 + ic)^5$ is real. $Im\,(1 + \mathbf{i}c)^5 = 0$ $\therefore 5c - 10c^3 + c^5 = 0$ $c^4 - 10c^2 + 5 = 0 \quad$ *as* $c \neq 0$	*2 marks – correct solution* *1 mark – identify that as $(1 + \mathbf{i}c)^5$ is real therefore imaginary part =0*

	$c^4 - 10c^2 + 5 = 0$ $c^2 = \dfrac{10 + \sqrt{80}}{2}$ $\therefore \quad c = \pm\sqrt{5 - 2\sqrt{5}}\ ,\ \pm\sqrt{5 + 2\sqrt{5}}$	*One mark – correct solution*
c (iv)	$(r\text{cis}\theta)^5 = r^5\text{cis}\,5\theta$ *As this is real, then* $\sin 5\theta = 0$ $5\theta = n\pi$ $\theta = \dfrac{n\pi}{5}$ $\theta = \dfrac{\pi}{5}$ *for smallest possible value of* θ	*One mark – correct solution*
	This corresponds to the smallest positive value of c. $\tan\dfrac{\pi}{5} = \dfrac{c}{1}$ Thus $= \sqrt{5 - 2\sqrt{5}}$	*One mark – correct solution*